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Objects generated by an arbitrary natural number. Part 2: Modal aspect

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Abstract: The set $\underline{SET}(n)$, generated by an arbitrary natural number n, was defined in [2] and some arithmetic functions, defined over its elements are introduced in an algebraic aspect. Here, over the elements of $\underline{SET}(n)$, two arithmetic functions similar to the modal type of operators are defined and some of their basic properties are studied.

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1 Introduction

In 2020, in [2], the author introduced the object $\underline{SET}(n)$ and studies some fo its algebraic properties. Now, writing the present text, he realizes that the title of that paper should have contained the subtitle "Part 1: Algebraic aspect". So, in the present paper, the respective subtitle "Part 2: Modal aspect", is added, as two arithmetic functions similar to the modal type of operators are defined here over the elements of $\underline{SET}(n)$ and some of their basic properties are studied.

Similarly to [2], let everywhere below the number n have the canonical form:

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, $\ldots$, $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, \ldots, p_k$ are different prime numbers. In [1], the following notations related to n are introduced, and will be used below:

$$\underline{set}(n) = \{p_1, p_2, \ldots, p_k\},$$
$$\underline{mult}(n) = \prod_{i=1}^{k} p_i.$$

For the fixed $n \geq 2$, in [2] we defined the set

$$\underline{SET}(n) = \{m | m = \prod_{i=1}^{k} p_i^{\beta_i} \;\&\; \delta(n) \leq \beta_i \leq \Delta(n)\},$$

where

$$\delta(n) = \min(\alpha_1, \ldots, \alpha_k),$$
$$\Delta(n) = \max(\alpha_1, \ldots, \alpha_k).$$

We will remark here that other authors (see, e.g. [9]) denote the functions δ and Δ by h and H, respectively.

While in [2] it was shown that the new objects have properties specific to algebra, here we will discuss their modal properties, introducing two new arithmetic functions.

2 Main results

For the needs of the research, we give the following three definitions from [2], related to n:

$$\boxcircle\, n = (\underline{mult}(n))^{\delta(n)}, \tag{1}$$
$$\boxast\, n = (\underline{mult}(n))^{\Delta(n)}, \tag{2}$$

and for each $m \in \underline{SET}(n)$:

$$\neg m = \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\beta_i}.$$

Now, for each $m \in \underline{SET}(n)$, i.e., for $m = \prod_{i=1}^{k} p_i^{\beta_i}$, where $\delta(m) \leq \beta_i \leq \Delta(m)$, we define

$$\Box\, m = (\underline{mult}(n))^{\delta(m)}, \tag{3}$$

$$\Diamond m = (\underline{mult}(n))^{\Delta(m)}. \tag{4}$$

These artihmetic functions are analogues of the modal operators *"necessity"* and *"possibility"* (see, e.g. [4, 6]).

Obviously,

$$\delta(n) \leq \min_{1\leq i\leq k} \beta_i = \delta(m) \leq \Delta(m) = \max_{1\leq i\leq k} \beta_i \leq \Delta(n)$$

and from (1)–(4) it follows that

$$\Box n = ⊡ n,$$
$$\Diamond n = ⊞ n.$$

The following assertions are valid.

Theorem 1. *For a given* $m \in \underline{SET}(n)$

(a) $\Box\,\Box m = \Box m$,

(b) $\Box \Diamond m = \Diamond m$,

(c) $\Diamond \Box m = \Box m$,

(d) $\Diamond\Diamond m = \Diamond m$,

(e) $\Box m \leq m \leq \Diamond m$,

(f) $\neg \Box \neg m = \Diamond m$,

(g) $\neg\Diamond\neg m = \Box n$.

Proof. For *(a)*, from (3), we obtain:

$$\begin{aligned}\Box\,\Box m &= \Box\,(\underline{mult}(n))^{\delta(m)}\\ &= (\underline{mult}(n))^{\delta(\delta(m))}\\ &= (\underline{mult}(n))^{\delta(m)}\\ &= \Box m,\end{aligned}$$

because all powers of number $(\underline{mult}(n))^{\delta(m)}$ are equal to $\delta(m)$.

Statements *(b), (c), (d)* and *(e)* are proved in the same manner. The check of the validity of *(f)* is as follows:

$$\begin{aligned}\neg \Box \neg m &= \neg \Box \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\beta_i}\\ &= \neg \prod_{i=1}^{k} p_i^{\min(\Delta(n)+\delta(n)-\beta_i)}\\ &= \neg \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\Delta(m)}\\ &= \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-(\Delta(n)+\delta(n)-\Delta(m))}\\ &= \prod_{i=1}^{k} p_i^{\Delta(m)} = \Diamond m.\end{aligned}$$

Statement *(g)* is proved in the same manner. □

Equalities *(a)–(g)* are the axioms of the modal logic or properties of modal operators (see, e.g., [4,6,7]).

Corollary 1. *For a fixed number* n*,*

(a) $\boxcircle \boxcircle n = \boxcircle n$,

(b) $\boxcircle \boxast n = \boxast n$,

(c) $\boxast \boxcircle n = \boxcircle n$,

(d) $\boxast \boxast n = \boxast n$,

(e) $\boxcircle n \leq n \leq \boxast n$,

(f) $\neg \boxcircle \neg n = \boxast n$,

(g) $\neg \boxast \neg n = \boxcircle n$.

In [3], following and modifying the definitions from [5,8], the concepts of a second type of a cl-Feeble Topological Structure (cl-FTS2) and of a second type of an in-Feeble Topological Structure (in-FTS2) are introduced. The object

$$\langle \mathcal{X}, \mathcal{O}, \bullet, e_\bullet \rangle,$$

where $\mathcal{X}$ is a fixed universe with a minimal element O and maximal element E, $\mathcal{O} : \mathcal{X} \to \mathcal{X}$ is an operator, $\bullet : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is an operation, $e_\bullet \in \mathcal{X}$ is the unitary element with respect to operation $\bullet$, and for every two $A, B \in \mathcal{X}$:

C1 $\mathcal{O}(A \bullet B) \leq \mathcal{O}(A) \bullet \mathcal{O}(B)$,

C2 $A \leq \mathcal{O}(A)$,

C3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

C4 $\mathcal{O}(O) = O$;

is called a cl-FTS2 and in-FTS2 is the object for which the conditions are:

D1 $\mathcal{O}(A \bullet B) \geq \mathcal{O}(A) \bullet \mathcal{O}(B)$,

D2 $A \geq \mathcal{O}(A)$,

D3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

S4 $\mathcal{O}(E) = E$.

Below, we illustrate these definitions with examples, based on the introduced arithmetic functions.

Let the numbers $l, m \in \underline{SET}(n)$. Then $\delta(n) \leq \beta_i, \gamma_i \leq \Delta(n)$. Let

$$l = \prod_{i=1}^{k} p_i^{\beta_i}, \quad m = \prod_{i=1}^{k} p_i^{\gamma_i},$$

and let us define an operation multiplication by:

$$l \times m = \prod_{i=1}^{k} p_i^{\min(\beta_i + \gamma_i, \Delta(n))}.$$

Theorem 2. *$\langle \underline{SET}(n), \diamondsuit, \times, \boxplus n\rangle$ is a cl-FTS2.*

Proof. Let $l, m \in \underline{SET}(n)$ are the above defined numbers. Then, we obtain for condition C1:

$$\begin{aligned}
\diamondsuit(l \times m) &= \diamondsuit(\prod_{i=1}^{k} p_i^{\min(\beta_i+\gamma_i,\Delta(n))}) \\
&= (\underline{mult}(n))^{\Delta(\min(\beta_i+\gamma_i,\Delta(n)))} \\
&= (\underline{mult}(n))^{\max\limits_{1\le i\le k}(\min(\beta_i+\gamma_i,\Delta(n)))} \\
&= (\underline{mult}(n))^{\min(\max\limits_{1\le i\le k}(\beta_i+\gamma_i),\Delta(n))} \\
&\le (\underline{mult}(n))^{\min(\max\limits_{1\le i\le k}\beta_i+\max\limits_{1\le i\le k}\gamma_i,\Delta(n))} \\
&= (\underline{mult}(n))^{\min(\Delta(l)+\Delta(m),\Delta(n))} \\
&= (\underline{mult}(n))^{\Delta(l)} \times (\underline{mult}(n))^{\Delta(l)} \\
&= \diamondsuit(l) \times \diamondsuit(m).
\end{aligned}$$

Conditions C2–C4 are obvious, because

$$m = \prod_{i=1}^{k} p_i^{\gamma_i} \le \prod_{i=1}^{k} p_i^{\Delta(n)} = (\underline{mult}(n))^{\Delta(n)} = \boxplus n,$$

$$\diamondsuit\diamondsuit m = \diamondsuit(\underline{mult}(n))^{\Delta(m)} = (\underline{mult}(n))^{\Delta(\Delta(m))} = (\underline{mult}(n))^{\Delta(m)} = \diamondsuit m,$$

and

$$\diamondsuit \boxplus n = \diamondsuit(\underline{mult}(n))^{\Delta(n)} = (\underline{mult}(n))^{\Delta(\Delta(n))} = (\underline{mult}(n))^{\Delta(n)} = \boxplus n. \qquad \square$$

Theorem 3. *$\langle \underline{SET}(n), \square, \times, \boxdot n\rangle$ is an in-FTS2.*

The proof is similar to the above one.

3 Conclusion

In the future, other properties of the set $\underline{SET}(n)$ and the already defined over it operations and operators will be studied. Also, new operators that are analogues of the logical quantifiers will be introduced.

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