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An introduction to harmonic complex numbers and harmonic hybrid Fibonacci numbers: A unified approach

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Abstract: The purpose of this paper is to define and construct new number systems, called the harmonic complex Fibonacci sequences (HCF) and the harmonic hybrid Fibonacci (HHF) sequences. These sequences are defined by inspiring the well-known harmonic and hybrid numbers in literature. We give some fundamental definitions and theorems about these sequences in detail. Moreover, we examine some algebraic properties such as Binet-like-formula, partial sums related to these sequences. Finally, we provide a Maple 13 source code to verify the sequences easily.
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1 Introduction

The concept of hybrid numbers was firstly introduced by Özdemir [16], as a combination of complex, hyperbolic and dual numbers, defined as

$$Z = a + bi + c\epsilon + dh,$$

where $a, b, c \in \mathbb{R}$ and i, ϵ, h are operators such that

$$i^2 = -1 \ , \ \epsilon^2 = 0, \ h^2 = 1 \ \ ih = -hi = \ \epsilon + i.$$

The hybrid numbers can be considered as a mixture of real numbers, complex numbers, dual numbers, and hyperbolic numbers. It is known that the complex numbers are constructed by combining a real number and an imaginary unit denoted by i. They are defined in the geometry of the Euclidean plane. However, the hyperbolic numbers are defined in the Minkowski and Galilean planes, where the hyperbolic unit is denoted by h. Additionally, there is a one-to-one mapping between dual numbers and the Euclidean space, that is, the dual points of the unit dual sphere correspond to directional lines in the Euclidean space, proved by E. Study, in [17]. The dual numbers are introduced by William Clifford, in 1873, to solve some algebraic problems and were developed over time. Here we want to take your attention that the hyperbolic and dual numbers have the same structure with complex numbers But the hyperbolic unit satisfies $h^2 = 1$, and dual unit satisfies $\epsilon^2 = 0$. Recently, there has been many works devoted to the study of hybrid numbers and their some interesting identities (see the references [4, 14, 18–20] and therein). Furthermore, one can see some applications of these numbers in quantum and classic mechanics, see [8–10, 12].

The n-th harmonic number, denoted by H_n, is defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

where $H_0 = 0$. They are required in many areas of science such as in calculations of high energy physics, in computer science in the efficiency analysis of algorithms. The evaluation of harmonic number sums has been useful in analytic number theory. Moreover, especially in the last years, many researchers deal with the geometric and physical applications of complex, hyperbolic and dual numbers which are well known two dimensional number systems. More details are given at the references [2], [3], [5] and therein.

The Fibonacci sequence is defined by the following recurrence relation, for $n \geq 0$:

$$F_{n+2} = F_{n+1} + F_n$$

with $F_0 = 0$, $F_1 = 1$.

There are many authors who have investigated a lot of beneficial properties and theorems on combining with quaternions and the Fibonacci sequences. Moreover, bi-periodic conditions of these sequences are examined in detail, see [1, 11, 15]. In the light of these studies, the authors define the complex-type k-Fibonacci numbers and then give the correspondence between the k-step Fibonacci numbers and the complex-type k-Fibonacci numbers in [7]. The authors define the co-complex-type k-Fibonacci numbers and then denote the relation between the k-step Fibonacci numbers and the co-complex-type k-Fibonacci numbers in [6].

There is a fact that the complex numbers are quite easy to describe in terms of real numbers. In other words, every complex number has the form $a + bi$ where a and b are real numbers. If a complex number is at the denominator of a fraction, then it can be rewritten as below:

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

In this study, we give a new approach to generalization of the complex numbers. Motivated by such researches and inspired by the definitions of hybrid and harmonic numbers, we defined harmonic complex Fibonacci sequences (HCF), in [13], and extended this definition by defining harmonic hybrid Fibonacci sequences (HHF). Then we obtain some properties such as generating function, Binet formula, Cassini identity, etc. Moreover we give some differences with the results given in the literature. In other words, we make use of the method of hybrid numbers in a combined way to Fibonacci numbers, hybrid numbers and harmonic numbers. Finally, we provide a Maple 13 source code.

2 Harmonic complex Fibonacci sequences

At this section, we consider the complex numbers by inspiring harmonic numbers defined in [13]. In this approach, it is combined harmonic numbers with complex numbers. In other words, the authors defined the harmonic complex Fibonacci numbers as below:

$$\mathbb{H}_n^{\mathbb{C}} = \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1}}.$$

It is denoted the set of the harmonic hybrid Fibonacci numbers as follows:

$$K_1 = \left\{ \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1}} : F_k \text{ is the } k\text{-th Fibonacci number, } i^2 = -1 \right\}.$$

Any harmonic complex number can be rewritten as a combination of vector and scalar parts. In other words,

$$\mathbb{H}_n^{\mathbb{C}} = \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1}} = \sum_{k=1}^{n} \frac{F_k - iF_{k+1}}{F_k^2 + F_{k+1}^2}.$$

Example 2.1. *Let us compute the harmonic complex number for $n = 3$, i.e.:*

$$\begin{aligned}
\mathbb{H}_3^{\mathbb{C}} &= \sum_{k=1}^{3} \frac{1}{F_k + iF_{k+1}} = \frac{1}{F_1 + iF_2} + \frac{1}{F_2 + iF_3} + \frac{1}{F_3 + iF_4} \\
&= \frac{1}{1+i} + \frac{1}{1+2i} + \frac{1}{1+3i} \\
&= \frac{111}{130} - \frac{147}{130} i.
\end{aligned}$$

The harmonic complex Fibonacci numbers can be calculated by the following Maple 13 code:

```
Fib:=proc(n::nonnegint)
option remember;
if n>=2 then
RETURN(Fib(n-1)+Fib(n-2))
else
RETURN(n)
fi
end:SUM:=0;
```

```
for N from 1 to n by 1 do
SUM:=SUM+(1)/(Fib(N)+i*Fib(N+1))
od;
```

In [13], for all $\mathbb{H}_n^{\mathbb{C}} = \sum_{k=1}^{n} \frac{1}{F_k+iF_{k+1}}, \mathbb{H}_m^{\mathbb{C}} = \sum_{k=1}^{m} \frac{1}{F_k+iF_{k+1}} \in K_1$, the fundamental operators are defined as below:

(a) Addition:

(i) If $m = n$, $\mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}} = 2\sum_{k=1}^{n} \frac{1}{F_k+iF_{k+1}}$.

(ii) If $m < n$, $\mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}} = 2\sum_{k=1}^{m} \frac{1}{F_k+iF_{k+1}} + \sum_{k=m+1}^{n} \frac{1}{F_k+iF_{k+1}}$.

(iii) If $n < m$, $\mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}} = 2\sum_{k=1}^{n} \frac{1}{F_k+iF_{k+1}} + \sum_{k=n+1}^{m} \frac{1}{F_k+iF_{k+1}}$.

(b) Multiplication:

$$\mathbb{H}_n^{\mathbb{C}}.\mathbb{H}_m^{\mathbb{C}} = (\sum_{k=1}^{n} \frac{1}{F_k+iF_{k+1}}).(\sum_{k=1}^{m} \frac{1}{F_k+iF_{k+1}})$$

is calculated as distributing terms on the right by exploitting the each product of unit.

(c) Complex conjugate:

$$\overline{\mathbb{H}_n^{\mathbb{C}}} = \sum_{k=1}^{n} \frac{F_k+iF_{k+1}}{F_k^2+F_{k+1}^2}.$$

The complex conjugate operation can also verified by the following Maple 13 code:

```
Fib:=proc(n::nonnegint)
option remember;
if n>=2 then
RETURN(Fib(n-1)+Fib(n-2))
else
RETURN(n)
fi
end:SUM:=0;
for N from 1 to n by 1 do
SUM:=SUM+(Fib(N)+iFib(N+1))/(Fib(N)^2+Fib(N+1)^2)
od;
```

From the definition of complex conjugate, the norm is calculated as

$$N(\mathbb{H}_n^{\mathbb{C}}) = \parallel \mathbb{H}_n^{\mathbb{C}} \parallel = \sqrt{\mathbb{H}_n^{\mathbb{C}}\overline{\mathbb{H}_n^{\mathbb{C}}}}.$$

Moreover, some significiant properties of harmonic complex Fibonacci numbers are given as follows:

(i) $\mathrm{Re}(\mathbb{H}_n^{\mathbb{C}}) = \frac{\mathbb{H}_n^{\mathbb{C}}+\overline{\mathbb{H}_n^{\mathbb{C}}}}{2}, \mathrm{Im}(\mathbb{H}_n^{\mathbb{C}}) = \frac{\mathbb{H}_n^{\mathbb{C}}+\overline{\mathbb{H}_n^{\mathbb{C}}}}{2i}$,

(ii) $\overline{\mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}}} = \overline{\mathbb{H}_n^{\mathbb{C}}} + \overline{\mathbb{H}_m^{\mathbb{C}}}$,

(iii) $\overline{\overline{\mathbb{H}_n^{\mathbb{C}}}} = \mathbb{H}_n^{\mathbb{C}}$,

(iv) $\overline{\mathbb{H}_n^{\mathbb{C}}.\mathbb{H}_m^{\mathbb{C}}} = \overline{\mathbb{H}_n^{\mathbb{C}}}.\overline{\mathbb{H}_m^{\mathbb{C}}}$,

(v) $\| \overline{\mathbb{H}_n^{\mathbb{C}}} \| = \| \mathbb{H}_n^{\mathbb{C}} \|$,

(vi) $\| \mathbb{H}_n^{\mathbb{C}}.\mathbb{H}_m^{\mathbb{C}} \| = \| \mathbb{H}_n^{\mathbb{C}} \| . \| \mathbb{H}_m^{\mathbb{C}} \|$,

(vii) $\| \ \mathrm{Re}(\mathbb{H}_n^{\mathbb{C}}) \ \| \leq \| \ \mathbb{H}_n^{\mathbb{C}} \ \|$ and $\| \ \mathrm{Im}(\mathbb{H}_n^{\mathbb{C}}) \ \| \leq \| \ \mathbb{H}_n^{\mathbb{C}} \ \|$ inequalities hold if and only if $\mathrm{Re}(\mathbb{H}_n^{\mathbb{C}}) = \mathrm{Im}(\mathbb{H}_n^{\mathbb{C}}) = 0$.

Additionally, cosine and parallellogram laws are

$$\begin{aligned} \| \mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}} \|^2 &= \| \mathbb{H}_n^{\mathbb{C}} \|^2 + \| \mathbb{H}_m^{\mathbb{C}} \|^2 + 2Re(\mathbb{H}_n^{\mathbb{C}}\mathbb{H}_m^{\mathbb{C}}), \\ \| \mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}} \|^2 + \| \mathbb{H}_n^{\mathbb{C}} - \mathbb{H}_m^{\mathbb{C}} \|^2 &= 2(\| \mathbb{H}_n^{\mathbb{C}} \|^2 + \| \mathbb{H}_m^{\mathbb{C}} \|^2). \end{aligned}$$

Triangle inequality and Cauchy equality are represented as follows, respectively:

$$\begin{aligned} \| \mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}} \| &\leq \| \mathbb{H}_n^{\mathbb{C}} \| + \| \mathbb{H}_m^{\mathbb{C}} \|, \\ \| \mathbb{H}_n^{\mathbb{C}} + \mathbb{H}_m^{\mathbb{C}} \|^2 + \| \mathbb{H}_n^{\mathbb{C}} - \mathbb{H}_m^{\mathbb{C}} \|^2 &= 2(\| \mathbb{H}_n^{\mathbb{C}} \|^2 + \| \mathbb{H}_m^{\mathbb{C}} \|^2). \end{aligned}$$

For all $\mathbb{H}_n^{\mathbb{C}} = \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1}} \in K_1$, the polar form of $\mathbb{H}_n^{\mathbb{C}}$ is denoted as

$$\mathbb{H}_n^{\mathbb{C}} = r_1(\cos\theta_1 + i\sin\theta_1),$$

where $\cos\theta_1 = \frac{\sum_{k=1}^{n} \frac{F_k}{F_k^2 + F_{k+1}^2}}{r_1}, \sin\theta_1 = \frac{\sum_{k=1}^{n} \frac{-F_{k+1}}{F_k^2 + F_{k+1}^2}}{r_1}$ and $r_1 = \sqrt{\sum_{k=1}^{n} \frac{1}{F_k^2 + F_{k+1}^2}}$.

For any harmonic complex Fibonacci numbers

$$\begin{aligned} \mathbb{H}_n^{\mathbb{C}} &= r_1(\cos\theta_1 + i\sin\theta_1) \\ \mathbb{H}_m^{\mathbb{C}} &= r_2(\cos\theta_2 + i\sin\theta_2), \end{aligned}$$

we can write

$$\mathbb{H}_n^{\mathbb{C}}\mathbb{H}_m^{\mathbb{C}} = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

by using trigonometric identities. Also, the exponential form of $\mathbb{H}_n^{\mathbb{C}}$ is expressed as

$$\exp(\mathbb{H}_n^{\mathbb{C}}) = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

where $a = \sum_{k=1}^{m} (\frac{1}{F_k + iF_{k+1}})$. By using this exponential form, Euler's formula can be written as

$$\exp^{\sum_{k=1}^{\infty} (\frac{F_k - iF_{k+1}}{F_k^2 + F_{k+1}^2})} = \exp^{\sum_{k=1}^{\infty} (\frac{F_k}{F_k^2 + F_{k+1}^2})} \left(\cos\left(\sum_{k=1}^{\infty} \left(\frac{F_{k+1}}{F_k^2 + F_{k+1}^2} \right) \right) + i\sin\left(\sum_{k=1}^{\infty} \left(\frac{F_{k+1}}{F_k^2 + F_{k+1}^2} \right) \right) \right),$$

where

$$\exp^{i\sum_{k=1}^{\infty} (\frac{-F_{k+1}}{F_k^2 + F_{k+1}^2})} = \cos\left(\sum_{k=1}^{\infty} \left(\frac{F_{k+1}}{F_k^2 + F_{k+1}^2} \right) \right) + i\sin\left(\sum_{k=1}^{\infty} \left(\frac{F_{k+1}}{F_k^2 + F_{k+1}^2} \right) \right).$$

Let us consider complex logarithm for $\mathbb{H}_n^{\mathbb{C}}$. We know that $\mathbb{H}_n^{\mathbb{C}} = r_1(\cos\theta_1 + i\sin\theta_1)$, where $r_1 = \sqrt{\sum_{k=1}^{n} \frac{1}{F_k^2 + F_{k+1}^2}}$. Thus, we get

$$\ln(\mathbb{H}_n^{\mathbb{C}}) = \ln\left(\sum_{k=1}^{\infty}\left(\frac{1}{F_k^2+F_{k+1}^2}\right)\right) + i\theta_1.$$

If the exponential form of both sides of above equation is calculated, we have easily seen that $\exp(\ln(\mathbb{H}_n^{\mathbb{C}})) = \mathbb{H}_n^{\mathbb{C}}$. In a general statement, it can be written as below:

$$\ln(\mathbb{H}_n^{\mathbb{C}}) = \{\ln\left(\sum_{k=1}^{\infty}\left(\frac{1}{F_k^2+F_{k+1}^2}\right)\right) + i(\theta_1 + 2k\pi) : k \in Z\}.$$

Thus, if the exponent p is an integer, then $\mathbb{H}_n^{\mathbb{C}}$ is well-defined and the exponential formula simplifies to De Moivre's formula:

$$(\mathbb{H}_n^{\mathbb{C}})^p = r_1^n(\cos p\theta_1 + i\sin p\theta_1).$$

The authors, in [13], give an isomorphism with the map $\phi : K \longrightarrow M_{2\times 2}$, where K is a ring. Considering this isomorphism, they denote the matrix representation of $\mathbb{H}_n^{\mathbb{C}}$ as follows:

$$M = \begin{pmatrix} \sum_{k=1}^{n}\frac{F_k}{F_k^2+F_{k+1}^2} & \sum_{k=1}^{n}\frac{-F_{k+1}}{F_k^2+F_{k+1}^2} \\ \sum_{k=1}^{n}\frac{F_{k+1}}{F_k^2+F_{k+1}^2} & \sum_{k=1}^{n}\frac{F_k}{F_k^2+F_{k+1}^2} \end{pmatrix},$$

where the matrix representations of units are $1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let M be the matrix representation of $\phi(\mathbb{H}_n^{\mathbb{C}})$ is called harmonic complex Fibonacci matrix corresponding to $\mathbb{H}_n^{\mathbb{C}}$, where $\phi(\mathbb{H}_n^{\mathbb{C}}) = M$ and ϕ is a ring isomorphism.

Some determinantal properties are given in [13]. Here we prove them with the following theorem in detail.

Theorem 2.1. *Let M and N be the matrix representations of any non-zero $\mathbb{H}_n^{\mathbb{C}}$ and $\mathbb{H}_m^{\mathbb{C}}$, respectively. The following equalities are satisfied:*

(i) $\det(M) = \det(M^T) = \det(\bar{M})$,

(ii) *For any* $\lambda \in C$, $\det(\lambda M) = \lambda^2 \det(M)$,

(iii) $\det(MN) = \det M \det N$,

(iv) $\det(M)^2 = \det(M^T\bar{M})$.

Proof. (i) Exploiting the definitions of conjugate and transpose for matrices given above, we write the transpose of M

$$M^T = \begin{pmatrix} \sum_{k=1}^{n} \frac{F_k}{F_k^2+F_{k+1}^2} & \sum_{k=1}^{n} \frac{F_{k+1}}{F_k^2+F_{k+1}^2} \\ \sum_{k=1}^{n} \frac{-F_{k+1}}{F_k^2+F_{k+1}^2} & \sum_{k=1}^{n} \frac{F_k}{F_k^2+F_{k+1}^2} \end{pmatrix},$$

and the conjugate of M

$$\bar{M} = \begin{pmatrix} \sum_{k=1}^{n} \frac{F_k}{F_k^2+F_{k+1}^2} & \sum_{k=1}^{n} \frac{F_{k+1}}{F_k^2+F_{k+1}^2} \\ \sum_{k=1}^{n} \frac{-F_{k+1}}{F_k^2+F_{k+1}^2} & \sum_{k=1}^{n} \frac{F_k}{F_k^2+F_{k+1}^2} \end{pmatrix}.$$

If we calculate the determinant of these matrices, we obtain $\det(M) = \det(M^T) = \det(\bar{M})$.

(ii) By using the properties of determinant, the proof can simply seen for any $\lambda \in C$.

(iii) Let M and N be the matrix representations of any non-zero $\mathbb{H}_n^{\mathbb{C}}$ and $\mathbb{H}_m^{\mathbb{C}}$, respectively. From the multiplication of matrices, it can be easily shown this equality.

(iv) The proof can be written by using the definitions and properties of determinant. □

Example 2.2. *For $n = 1$ and $m = 2$, let us consider two matrices M and N corresponding to $\mathbb{H}_n^{\mathbb{C}}$ and $\mathbb{H}_m^{\mathbb{C}}$:*

$$M = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$N = \begin{pmatrix} \frac{7}{10} & -\frac{9}{10} \\ \frac{9}{10} & \frac{7}{10} \end{pmatrix}.$$

We can simply calculate that $\det(MN) = \det M. \det N$. *Moreover, the transpose of M is*

$$M^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The conjugate of M is equal to M. Therefore, it is seen that $\det(M)^2 = \det(M^T\bar{M})$. *Additionally, we verify the condition (iii) given Theorem 2.2 as calculating*

$$\det(MN) = \det(M). \det(N) = \tfrac{13}{20}.$$

If $\det M \neq 0$, the inverse of $\mathbb{H}_n^{\mathbb{C}}$ is obtained as

$$\begin{aligned} (\mathbb{H}_n^{\mathbb{C}})^{-1} &= \frac{\overline{(\mathbb{H}_n^{\mathbb{C}})}}{\det(\mathbb{H}_n^{\mathbb{C}})} \\ &= \frac{\sum_{k=1}^{n} \frac{F_k+iF_{k+1}}{F_k^2+F_{k+1}^2}}{\sum_{k=1}^{n} \frac{1}{F_k^2+F_{k+1}^2}} \\ &= \sum_{k=1}^{n} (F_k + iF_{k+1}). \end{aligned}$$

Example 2.3. *For $n = 1$, we have $\mathbb{H}_n^{\mathbb{C}} = \dfrac{1}{F_1 + iF_2} = \dfrac{1}{1+i}$. If we calculate the inverse of $\mathbb{H}_n^{\mathbb{C}}$ by using definition, we get*

$$(\mathbb{H}_n^{\mathbb{C}})^{-1} = \frac{\frac{1+i}{2}}{\frac{1}{2}} = 1 + i.$$

In [13], the authors obtain the generating function, Binet formula and Cassini identity for HCF:

- **Generating Function**: For $n \geq 0$, the generating function of $\mathbb{H}_n^{\mathbb{C}}$ is

$$G(t) = \frac{\mathbb{H}_0^{\mathbb{C}} + (\mathbb{H}_1^{\mathbb{C}} - \mathbb{H}_0^{\mathbb{C}})t}{1 - t - t^2}.$$

- **Binet Formula**: For $n \geq 1$, the Binet formula is

$$\mathbb{H}_n^{\mathbb{C}} = \sum_{k=1}^{n} \frac{\alpha - \beta}{(A_* \alpha^k - B_* \beta^k)}$$

  where $A_* = 1 + i\alpha$, $B_* = 1 - i\beta$ and $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

- **Cassini Identity**: Let $\mathbb{H}_n^{\mathbb{C}}$ be the sequence of harmonic complex Fibonacci sequence. Then for $n \geq 1$:

$$\mathbb{H}_{n+1}^{\mathbb{C}} \mathbb{H}_{n-1}^{\mathbb{C}} - (\mathbb{H}_n^{\mathbb{C}})^2 = 5(-1)^{n-1} AB,$$

  where $A = \frac{-2\sqrt{5}+7-i(2\sqrt{5}+3)}{10\sqrt{5}}$ and $B = \frac{-2\sqrt{5}-7-i(2\sqrt{5}-3)}{10\sqrt{5}}$.

Taking into account the use of a large area of complex numbers, it is notable to generalize harmonic complex numbers. At this content, at the following section, we consider harmonic hybrid Fibonacci numbers and obtain their considerable identities.

3 Harmonic hybrid Fibonacci sequences

In this section, a new number sequence called harmonic hybrid Fibonacci sequence is defined. Furthermore, some basic operations and algebraic properties are examined.

Definition 3.1. *The harmonic hybrid Fibonacci sequence, denoted by HHF, is defined as*

$$\mathbb{H}_n = \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}},$$

where F_k is the k-th Fibonacci number.

The set of HHF, denoted by K, is

$$K = \{\sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}} : i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i\}.$$

Example 3.1. *Let us compute the harmonic hybrid Fibonacci number for* $n = 2$,

$$\begin{aligned}
\mathbb{H}_2 &= \sum_{k=1}^{2} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}} \\
&= \frac{1}{F_1 + iF_2 + \epsilon F_3 + hF_4} + \frac{1}{F_2 + iF_3 + \epsilon F_4 + hF_5} \\
&= \frac{1}{1 + i + 2\epsilon + 3h} + \frac{1}{1 + 2i + 3\epsilon + 5h} \\
&= \frac{43 - 54i - 97\epsilon - 151h}{-352}.
\end{aligned}$$

The harmonic hybrid Fibonacci numbers can be calculated by the following Maple 13 code:

```
Fib:=proc(n::nonnegint)
option remember;
if n>=2 then
RETURN(Fib(n-1)+Fib(n-2))
else
RETURN(n)
fi
end:SUM:=0; epsilon^2=0:h^2:=1:I*h=-h*I:=epsilon+I:
for N from 1 to n by 1 do
SUM:=SUM+(Fib(N)-iFib(N+1)-epsilon*Fib(N+2)-h*Fib(N+3))/
 (Fib(N)^2+(Fib(N+1)-Fib(N+2))^2-Fib(N+2)^2-Fib(N+3)^2)
od;
```

The real, complex, dual and hyperbolic units are represented as

$$1 \longleftrightarrow \{1,0,0,0\}, \quad i \longleftrightarrow \{0,1,0,0\}, \quad \epsilon \longleftrightarrow \{0,0,1,0\}, \quad h \longleftrightarrow \{0,0,0,1\}$$

respectively. Here, these units are called hybrid units. Furthermore, the following four matrices are a base of a 2×2 matrix set associated to hybrid units, [16]:

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \epsilon \leftrightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, h \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Any HHF can be rewritten as a combination of vector and scalar parts. In other words,

$$\sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}},$$

can be considered as combination of

$$S(\mathbb{H}_n) = \sum_{k=1}^{n} \frac{F_k}{F_k^2 + (F_{k+1} - F_{k+2})^2 - F_{k+2}^2 - F_{k+3}^2}$$

and

$$V(\mathbb{H}_n) = \sum_{k=1}^{n} \frac{-iF_{k+1} - \epsilon F_{k+2} - hF_{k+3}}{F_k^2 + (F_{k+1} - F_{k+2})^2 - F_{k+2}^2 - F_{k+3}^2}.$$

Here the scalar part is denoted by S($\mathbb{H}_n$), and the vector part is denoted by V($\mathbb{H}_n$), respectively.

The harmonic hybrid Fibonacci numbers are equal with each other if all their components are equal. Zero is called the null element. Also, the sum of any harmonic hybrid Fibonacci numbers is defined by summing their components. Addition operation in HHF is associative and commutative. The inverse of $\mathbb{H}_n$ is represented as $-\mathbb{H}_n$. Hence, $(\mathbb{H}_n, +)$ is an Abelian group.

For any harmonic hybrid Fibonacci numbers

$$\mathbb{H}_n = \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}$$

and

$$\mathbb{H}_l = \sum_{k=1}^{l} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}},$$

the inner product

$$\left(\sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}\right)\left(\sum_{k=1}^{l} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}\right)$$

is obtained as distributing the terms on the right by using the each product of units which satisfies the following multiplication table:

.	1	i	ϵ	h
1	1	i	ϵ	h
i	i	-1	$1-h$	$\epsilon+i$
ϵ	ϵ	$h+1$	0	$-\epsilon$
h	h	$-\epsilon-i$	ϵ	1

This table shows that the multiplication operation in HHF is not commutative. However, it satisfies the property of associativity.

Definition 3.2. *The conjugate of HHF, denoted by* $\bar{\mathbb{H}}_n$, *is defined as*

$$\bar{\mathbb{H}}_n = (\sum_{k=1}^{n} \frac{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}{F_k^2 + (F_{k+1} - F_{k+2})^2 - F_{k+2}^2 - F_{k+3}^2}).$$

Example 3.2. *For* $n = 2$, *the conjugate of* $\mathbb{H}_2$ *is*

$$\bar{\mathbb{H}}_2 = \frac{-43 + 54i + 97\epsilon + 151h}{352}.$$

Moreover, according to inner product, we have $\mathbb{H}_n\bar{\mathbb{H}}_n = \bar{\mathbb{H}}_n\mathbb{H}_n$. The real number

$$\begin{aligned} C(H_n) &= \mathbb{H}_n\bar{\mathbb{H}}_n = \bar{\mathbb{H}}_n\mathbb{H}_n \\ &= -\langle\sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}, \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}\rangle \\ &= -\langle A_n, A_n, \rangle \end{aligned}$$

where $A_n = \sum_{k=1}^{n} \frac{1}{F_k+iF_{k+1}+\epsilon F_{k+2}+hF_{k+3}}$. Furthermore, the real number $\sqrt{C(\mathbb{H}_n)}$ denotes the norm of the HHF and is also represented as $\|\mathbb{H}_n\|$.

Remark 3.1. *Let* $\mathbb{H}_n = \sum_{k=1}^{n} \frac{1}{F_k+iF_{k+1}+\epsilon F_{k+2}+hF_{k+3}}$ *be a HHF. Using the definition of the norm of HHF, we write*

$$\|\mathbb{H}_n\| = \sqrt{|\mathbb{H}_n\bar{\mathbb{H}}_n|} = \sqrt{C(\mathbb{H}_n)} = \sqrt{|-\langle A_n, A_n\rangle|}.$$

This definition of the norm is generalized as follows:

1. If $\mathbb{H}_n$ is a harmonic hybrid complex Fibonacci number,

$$\|\mathbb{H}_n\| = \sqrt{\frac{\sum_{k=1}^{n}(F_k^2 + F_{k+1}^2)}{\sum_{k=1}^{n}(F_k^2 + (F_{k+1} - F_{k+2})^2 - F_{k+2}^2 - F_{k+3}^2)}}.$$

2. If $\mathbb{H}_n$ is a harmonic hybrid hyperbolic Fibonacci number,

$$\|\mathbb{H}_n\| = \sqrt{\frac{|\sum_{k=1}^{n}(F_k^2 - F_{k+3}^2)|}{\sum_{k=1}^{n}(F_k^2 + (F_{k+1} - F_{k+2})^2 - F_{k+2}^2 - F_{k+3}^2)}}.$$

3. If $\mathbb{H}_n$ is a harmonic hybrid dual Fibonacci number,

$$\|\mathbb{H}_n\| = \sqrt{\frac{|\sum_{k=1}^{n} F_k|}{\sum_{k=1}^{n}(F_k^2 + (F_{k+1} - F_{k+2})^2 - F_{k+2}^2 - F_{k+3}^2)}}.$$

Definition 3.3. *The inverse of HHF,* $\|\mathbb{H}_n\| \neq 0$ *is defined as*

$$\begin{aligned}
{\mathbb{H}_n}^{-1} &= \frac{\bar{\mathbb{H}}_n}{C(\mathbb{H}_n)} \\
&= \frac{\sum_{k=1}^{n}\left(\frac{F_k+iF_{k+1}+\epsilon F_{k+2}+hF_{k+3}}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2}\right)}{\left(\sum_{k=1}^{n}\frac{1}{F_k+iF_{k+1}+\epsilon F_{k+2}+hF_{k+3}}\right)^2}.
\end{aligned}$$

Consequently, it can be seen that the set of HHF is a non-commutative ring with respect to the addition and multiplication operations.

Example 3.3. *For* $n = 2$, *the inverse of* $\mathbb{H}_2 = \sum_{k=1}^{2} \frac{1}{F_k+iF_{k+1}+\epsilon F_{k+2}+hF_{k+3}}$, $\|\mathbb{H}_2\| \neq 0$ *is*

$$\begin{aligned}
{\mathbb{H}_2}^{-1} &= \frac{\bar{\mathbb{H}}_2}{C(\mathbb{H}_2)} \\
&= \frac{\frac{-43-54i-97\epsilon-151h}{352}}{\left(\frac{-43-54i-97\epsilon-151h}{352}\right)\left(\frac{-43+54i+97\epsilon+151h}{352}\right)} \\
&= \frac{352}{-43+54i+97\epsilon+151h}.
\end{aligned}$$

The scalar product of $\mathbb{H}_n$ and $\mathbb{H}_l$ is defined as follows:

$$\begin{aligned}
g &: K \times K \longrightarrow \mathbb{R}, \\
&(H_n, H_l) \mapsto g(H_n, H_l) = \frac{\mathbb{H}_n\bar{\mathbb{H}}_l + \mathbb{H}_l\bar{\mathbb{H}}_n}{2},
\end{aligned}$$

where $\mathbb{H}_n = \sum_{k=1}^{n} \frac{1}{F_k+iF_{k+1}+\epsilon F_{k+2}+hF_{k+3}}$ and $\mathbb{H}_l = \sum_{k=1}^{l} \frac{1}{F_k+iF_{k+1}+\epsilon F_{k+2}+hF_{k+3}}$.
Additionally, the vector product of $\mathbb{H}_n$ and $\mathbb{H}_l$ is

$$\begin{aligned}
\times &: K \times K \longrightarrow K, \\
H_n \times H_l &= \frac{\mathbb{H}_n\bar{\mathbb{H}}_l - \mathbb{H}_l\bar{\mathbb{H}}_n}{2}.
\end{aligned}$$

For the vector product, the rules of following table are satisfied:

$\times$	1	i	ϵ	h
1	0	$-i$	$-\epsilon$	$-h$
i	i	0	h	$-\epsilon - i$
ϵ	ϵ	$-h$	0	ϵ
h	h	$\epsilon + i$	$-\epsilon$	0

A matrix representation of HHF facilitates multiplication of HHF. By showing an isomorphism between 2×2 matrices and HHF, we can simply multiply the HHF and define many properties of them. Additionaly, we can write the matrix representation of HHF.

Theorem 3.1. *There exists an isomorphism between the HHF ring K and the 2×2 matrices $M_{2\times2}$.*

Proof. For the map $\phi : K \longrightarrow M_{2\times2}$,

$$\phi(\sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}})$$

is associated with the matrix representation:

$$\frac{1}{W}\begin{pmatrix} -\sum_{k=1}^{n} F_{k+1} & -2\sum_{k=1}^{n} F_{k+1} \\ -2\sum_{k=1}^{n} F_{k+2} & \sum_{k=1}^{n}(F_k + F_{k+2}) \end{pmatrix},$$

where $W = \sum_{k=1}^{n}(F_k + F_{k+2}) - 4\sum_{k=1}^{n}(F_{k+1}F_{k+2})$. This map satisfies a ring isomorphism. It can be simply shown that the equalities satisfy

$$\begin{aligned} \phi(\mathbb{H}_n\mathbb{H}_l) &= \phi(\mathbb{H}_n)\phi(\mathbb{H}_l) \\ \phi(\mathbb{H}_n + \mathbb{H}_l) &= \phi(\mathbb{H}_n) + \phi(\mathbb{H}_l) \end{aligned}$$

where $\mathbb{H}_n$ and $\mathbb{H}_l$ are HHF. Additionally, this map satisfies the properties of bijective and injective. On the other hand, for any 2×2 real matrix

$$A = \begin{pmatrix} \sum_{k=1}^{n} \frac{F_k}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2} & \sum_{k=1}^{n} \frac{-F_{k+1}}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2} \\ \sum_{k=1}^{n} \frac{-F_{k+2}}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2} & \sum_{k=1}^{n} \frac{-F_{k+3}}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2} \end{pmatrix},$$

there is a HHF

$$\begin{aligned} \mathbb{H}_n = &\left(\frac{\sum_{k=1}^{n}(\frac{2F_k}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2})}{2}\right) + \left(\frac{\sum_{k=1}^{n}(\frac{-F_{k+1}+F_{k+2}-F_{k+3}}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2})}{2}\right) i \\ &+ \left(\frac{\sum_{k=1}^{n}(\frac{-F_k-F_{k+3}}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2})}{2}\right) \epsilon + \left(\frac{\sum_{k=1}^{n}(\frac{-2F_{k+1}-2F_{k+2}}{F_k^2+(F_{k+1}-F_{k+2})^2-F_{k+2}^2-F_{k+3}^2})}{2}\right) h, \end{aligned}$$

where $\phi(\mathbb{H}_n) = A$. Hence, ϕ is a ring isomorphism. □

Definition 3.4. *The matrix $\phi(\mathbb{H}_n) \in M_{2\times2}(\mathbb{R})$ is called HHF matrix corresponding to $\mathbb{H}_n$.*

Theorem 3.2. *Assume that A is a 2×2 real matrix corresponding to $\mathbb{H}_n$. Then, there are the following equalities:*

$$C(\mathbb{H}_n) \neq \det A, \qquad \|\mathbb{H}_n\| \neq \sqrt{\det A}.$$

Proof. Let $\mathbb{H}_n = \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}$ be a HHF. Then, we get

$$|\det A| = \frac{1}{W} \det \begin{pmatrix} -\sum_{k=1}^{n} F_{k+1} & -2\sum_{k=1}^{n} F_{k+1} \\ -2\sum_{k=1}^{n} F_{k+2} & \sum_{k=1}^{n} (F_k + F_{k+2}) \end{pmatrix}.$$

However, $C(\mathbb{H}_n) = (\sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}})^2$. Thus, we write that $C(\mathbb{H}_n) \neq \det A$.
Since

$$\|\mathbb{H}_n\| = \sum_{k=1}^{n} \frac{1}{F_k + iF_{k+1} + \epsilon F_{k+2} + hF_{k+3}}$$

and

$$\sqrt{\det A} = \sqrt{\sum_{k=1}^{n} (\frac{F_{k+1}(F_k + 5F_{k+2})}{F_{k+2} + 4F_{k+1}})},$$

it can be simply seen that $\|\mathbb{H}_n\| \neq \sqrt{\det A}$. □

Example 3.4. *For $n = 1$, $\mathbb{H}_1 = \frac{1}{1+i+2\epsilon+3h}$. Additionally, we can write the matrix representation as follows:*

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

We conclude that $\det A = 1$. *Moreover,*

$$C(\mathbb{H}_1) = -1.$$

Hence, $C(\mathbb{H}_1) \neq \det A$.

Corollary 3.1. *The inverse of* $\mathbb{H}_n \in K$ *exists if and only if* $\det(\phi(\mathbb{H}_n)) \neq 0$.

Now, we will prove some algebraic properties for any HHF. The Binet formula is the explicit formula to obtain the nth term of the sequence. Moreover, Binet formula can be employed to drive many Fibonacci properties. For $n \geq 0$, the Binet formula is obtained for Fibonacci quaternions as below:

$$Q_n = \frac{1}{\sqrt{5}}(\frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta})$$

where $\bar{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\bar{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$, in [11].

Theorem 3.3 (Binet Formula)**.** *For $n \geq 1$, the Binet formula for the HHF is given as follows:*

$$\mathbb{H}_n = \sum_{k=1}^{n} \frac{(\alpha - \beta)[\alpha^k \alpha^* - \beta^k \beta^*]}{\alpha^{2k}\alpha_* + \beta^{2k}\beta_*}$$

where $\alpha^* = 1 - i\alpha - \varepsilon\alpha^2 - h\alpha^3$, $\beta^* = 1 - i\beta - \varepsilon\beta^2 - h\beta^3$ *and* $\alpha_* = 1 - \alpha^2 + 2\alpha^3 - 2\alpha^4 - \alpha^6$, $\beta_* = 1 - \beta^2 + 2\beta^3 - 2\beta^4 - \beta^6$.

Proof. By exploiting the Binet's formula for the Fibonacci numbers, the proof is obtained. □

Theorem 3.4 (Cassini Identity). *For $n \geq 1$, there exists the following formula:*

$$\mathbb{H}_{n+1}\mathbb{H}_{n-1} - \mathbb{H}_n^2 = 5(-1)^{n-1}AB,$$

where

$$A = \frac{(-430 + 214\sqrt{5}) + i(540 - 236\sqrt{5}) + \epsilon(970 - 450\sqrt{5}) + h(1510 - 686\sqrt{5})}{7040}$$

and

$$B = -\frac{(1720 + 856\sqrt{5}) + i(400 + 1616\sqrt{5}) + \epsilon(1240 + 3320\sqrt{5}) + h(1640 + 4936\sqrt{5})}{28160}.$$

Proof. Let us consider the equation

$$\mathbb{H}_n = A\alpha^n + B\beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Since $\alpha\beta = -1$ and $\alpha - \beta = \sqrt{5}$, we calculate

$$\begin{aligned}
\mathbb{H}_{n+1}\mathbb{H}_{n-1} - \mathbb{H}_n^2 &= (A\alpha^{n+1} + B\beta^{n+1})(A\alpha^{n-1} + B\beta^{n-1}) - (A\alpha^n + B\beta^n)^2 \\
&= AB(\alpha.\beta)^{n-1}(\alpha^2 - 2\alpha.\beta + \beta^2) \\
&= AB(-1)^{n-1}(\alpha.\beta + \beta)^{-1}(\alpha - \beta)^2 \\
&= 5(-1)^{n-1}AB.
\end{aligned}$$

The proof is completed. □

Note that the Cassini identity can be obtained by exploiting the Binet formula.

4 Conclusion

In this study, we present a systematic investigation to new combined approach to harmonic numbers with Fibonacci coefficients. We named these number sets as harmonic complex Fibonacci (HCF) numbers and harmonic hybrid Fibonacci (HHF) numbers. Also, we get various results including Binet's formulas, generating functions, some basic (algebraic) operations, summation formulas, etc., for these classes of harmonic numbers. Moreover, we verified a Maple 13 source code to get the new family of numbers. We can briefly summarize the results obtained in this study as follows:

1) It is known that hybrid numbers are a combination of the dual, complex and the hyperbolic numbers. Here, we associate this combination with harmonic numbers. In this content, these relations can be given with matrix representations with 2×2 matrix sets.
2) With the classifications of the defined number sets, they can be expressed via the polar representations. Therefore, we proved the De Moivre formula for HCF, considering the classifications.
3) We give some Maple 13 procedures for the defined number sets.

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