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Some congruences on the hyper-sums of powers of integers involving Fermat quotient and Bernoulli numbers

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Abstract: For a given prime $p \geq 5$, let $\mathbb{Z}_p$ denote the set of rational p-integers (those rational numbers whose denominator is not divisible by p). In this paper, we establish some congruences modulo a prime power p^5 on the hyper-sums of powers of integers in terms of Fermat quotient, Wolstenholme quotient, Bernoulli and Euler numbers.
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1 Introduction

For a given prime p and $k \in \{1, 2, \ldots, p-1\}$, the generalized harmonic numbers $H_{p-1}^{(k)}$ are the rational numbers defined as

$$H_{p-1}^{(k)} = \sum_{1 \leq i_1 < \cdots < i_k \leq p-1} \frac{1}{i_1 \cdots i_k}. \tag{1}$$

The harmonic numbers $h_{p-1}^{(k)}$ of order k are defined as

$$h_{p-1}^{(k)} = \sum_{i=1}^{p-1} \frac{1}{i^k}. \tag{2}$$

From Newton's formula [4, p. 140], we can express $H_{p-1}^{(k)}$ as

$$H_{p-1}^{(k)} = \frac{(-1)^{k-1}}{k} \left(h_{p-1}^{(k)} + \sum_{l=1}^{k-1} (-1)^l H_{p-1}^{(l)} h_{p-1}^{(k-l)} \right), \tag{3}$$

with $H_{p-1}^{(0)} = 1$.

The classical harmonic numbers h_{p-1} are defined by

$$h_{p-1} := H_{p-1}^{(1)} = h_{p-1}^{(1)} = 1 + \frac{1}{2} + \cdots + \frac{1}{p-1},$$

which correspond to the case $k = 1$ and $H_{p-1}^{(0)} = h_{p-1}^{(0)} = 1$ correspond to the case $k = 0$.

The rising factorial denoted by $x^{\overline{n}}$, is defined by $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ with $x^{\overline{0}} = 1$. The (unsigned) Stirling number $\left[{n \atop k}\right]$ of the first kind (see [2]) is defined by

$$x^{\bar{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k,$$

with

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 0 \text{ if } n > 0 \text{ and } \begin{bmatrix} n \\ k \end{bmatrix} = 0 \text{ if } k > n \text{ or } k < 0.$$

For a positive integer $n \geq k \geq 1$, the relation between the (unsigned) Stirling number and the generalized harmonic numbers is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)! H_{n-1}^{(k-1)}. \tag{4}$$

The Fermat's little theorem [3] states that if p is a prime and a is an integer not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$. This gives rise to the definition of the Fermat quotient of p to base a,

$$q_p(a) = \frac{a^{p-1} - 1}{p},$$

the first, second and cube powers of the Fermat quotient to base 2 is defined by

$$q_p(2)^n = \left(\frac{2^{p-1} - 1}{p} \right)^n, \text{ for } n = 1, 2, 3.$$

Note that $q_p(2) := q_p(2)^1$. The Bernoulli numbers B_n are defined recursively by

$$B_0 = 1 \text{ and } \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad \text{for } n \geq 2.$$

The Euler numbers E_n are integers defined by

$$E_0 = 1 \text{ and } \sum_{k=0}^{n} \binom{n}{k} E_{n-k} = 0, \text{ for } n \geq 1.$$

The hyper-sums of powers of integers $S_d^{(r)}(n)$ are defined recursively as

$$S_0^{(r)}(n) = \binom{n+r}{r+1}, r \geq 0,$$
$$S_d^{(0)}(n) = 1^d + 2^d + \cdots + n^d,$$
$$S_d^{(r)}(n) = \sum_{j=1}^{n} S_d^{(r-1)}(j), \quad n, r, d \geq 1.$$

In a recent paper, Bounebirat et al. [5] presented a new explicit formula for the hyper-sums of powers of integers $S_d^{(r)}(n)$ involving binomial coefficient

$$S_d^{(r)}(n) = \sum_{j=1}^{n} \binom{n+r-j}{r} j^d.$$

In the present paper, we show that

$$S_d^{(r)}(p) \equiv -1 + \frac{1}{(r+1)} \sum_{k=1}^{\alpha-1} H_r^{(k-1)} p^k \pmod{p^\alpha}.$$

2 Some basic congruences

In this section, we give some congruences modulo a prime power p^5 involving harmonic numbers and generalized harmonic numbers of second order and third order.

Lemma 1. *For any prime* $p > 5$. *Then*

$$p^2 h_{p-1} \equiv -\frac{1}{3} p^4 B_{p-3} \pmod{p^5}, \tag{5}$$

$$p^3 h_{p-1}^{(2)} \equiv \frac{2}{3} p^4 B_{p-3} \pmod{p^5} \tag{6}$$

and

$$p^2 h_{p-1}^{(3)} \equiv -\frac{6}{5} p^4 B_{p-5} \pmod{p^5}. \tag{7}$$

Proof. Multiplying the congruence (a) of Theorem 5.1 [7] by p^2 and taking $k=1$ we immediately obtain the congruence (5). By using the congruence of the Corollary 5.1 [7] with $k = 2$ and multiplying by p^3 we obtain the congruence (6). Finally using the congruence (a) of Theorem 5.1 [7] with $k = 3$ and multiplying by p^2 we can find the congruence (7). □

Lemma 2. *For any prime* $p > 5$. *Then*

$$p^2 h_{\frac{p-1}{2}} \equiv -2p^2 q_p(2) + p^3 q_p(2)^2 - \frac{2}{3} p^4 q_p(2)^3 - \frac{7}{12} p^4 B_{p-3} \pmod{p^5}, \tag{8}$$

$$p^3 h_{\frac{p-1}{2}}^{(2)} \equiv \frac{7}{3} p^4 B_{p-3} \pmod{p^5} \tag{9}$$

and

$$p^4 h_{\frac{p-1}{2}}^{(3)} \equiv -2p^4 B_{p-3} \pmod{p^5}. \tag{10}$$

Proof. Multiplying the congruence (c) of Theorem 5.2 [7] by p^2 we immediately obtain the congruence (8). By using the congruence (a) of Theorem 5.2 [7] with $k = 2$ and multiplying by p^3 we have the congruence (9). Finally, using the congruence (b) of Corollary 5.2 [7] with $k = 3$ and multiplying by p^4, we can find the congruence (10). □

Lemma 3. *For any prime $p > 5$. Then*

$$p^2 h_{[p/4]} \equiv -3p^2 q_p(2) + p^3 \left(\frac{3}{2} q_p(2)^2 + (-1)^{\frac{p-1}{2}} (E_{2p-4} - 2E_{p-3})\right) - p^4 \left(q_p(2)^3 + \frac{7}{12} B_{p-3}\right) \pmod{p^5}, \tag{11}$$

$$p^3 h^{(2)}_{[p/4]} \equiv p^3 (-1)^{\frac{p-1}{2}} (8E_{p-3} - 4E_{2p-4}) + \frac{14}{3} p^4 B_{p-3} \pmod{p^5} \tag{12}$$

and

$$p^4 h^{(3)}_{[p/4]} \equiv -9p^4 B_{p-3} \pmod{p^5}, \tag{13}$$

where $[p/4]$ the integral part of $p/4$.

Proof. Multiplying the congruence of Theorem 3.2 [8] by p^2 we immediately obtain the congruence (11). By using the congruence of Corollary 3.8 [8] and multiplying by p^3, we find the congruence (12). By using the congruence of Corollary 3.4 [8] and multiplying by p^4, we get the congruence (13). □

3 Main results

In 2017, Laissaoui, Bounebirat and Rahmani [5] proved, for any prime p and for $r \geq 0$ with $r + 1 | p$ and $p - 1 | d$, the congruence

$$S_d^{(r)}(p) \equiv -1 \pmod{p}.$$

This result, give to extend the congruence as follows:

Theorem 1. *For any prime p. Let $\alpha \in \mathbb{N}^*$ with $p \geq \alpha \geq 2$ then $d \in \mathbb{N}^*$ with $p^{\alpha-1}(p-1)|d$ and let $r \in \{\alpha - 2, \alpha - 1, \ldots, p - 2\}$, we have*

$$S_d^{(r)}(p) \equiv -1 + \frac{1}{(r+1)} \sum_{k=1}^{\alpha-1} H_r^{(k-1)} p^k \pmod{p^\alpha}. \tag{14}$$

Proof. By Euler's theorem for each $k \in \{1, 2, \ldots, p-1\}$ we have $k^{\varphi(p^\alpha)} \equiv 1 \pmod{p^\alpha}$ (where $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$ is the Euler's totient function)

$$S_d^{(r)}(p) \equiv -1 + \frac{1}{(r+1)!} \sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} p^k \pmod{p^\alpha}.$$

Since $r \in \{\alpha-2, \alpha-1, \ldots, p-2\}$, we have

$$\frac{1}{(r+1)!} \in \mathbb{Z}_p$$

and

$$\sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} p^k \equiv \begin{bmatrix} r+1 \\ 1 \end{bmatrix} p + \begin{bmatrix} r+1 \\ 2 \end{bmatrix} p^2 + \cdots + \begin{bmatrix} r+1 \\ \alpha-1 \end{bmatrix} p^{\alpha-1} \pmod{p^\alpha},$$

we find that

$$S_d^{(r)}(p) \equiv -1 + \frac{1}{(r+1)!} \sum_{k=1}^{\alpha-1} \begin{bmatrix} r+1 \\ k \end{bmatrix} p^k \pmod{p^\alpha},$$

using the identity (4), we get

$$S_d^{(r)}(p) \equiv -1 + \frac{1}{(r+1)} \sum_{k=1}^{\alpha-1} H_r^{(k-1)} p^k \pmod{p^\alpha}.$$

This completes the proof. □

Thus, for example, when $\alpha = 5$ in (14) and using (3), we obtain:

Corollary 2. *For any prime $p \geq 5$, with $p^4(p-1)|d$ and $r \in \{3, 4, \ldots, p-2\}$, we have*

$$S_d^{(r)}(p) \equiv -1 + \frac{1}{(r+1)} \left(pH_r^{(0)} + p^2 H_r^{(1)} + p^3 H_r^{(2)} + p^4 H_r^{(3)} \right) \pmod{p^5}$$

and

$$S_d^{(r)}(p) \equiv -1 + \frac{1}{(r+1)} \left(p + p^2 h_r + \frac{p^3}{2} \left(h_r^2 - h_r^{(2)} \right) + \frac{p^4}{6} \left(h_r^3 - 3h_r h_r^{(2)} + 2h_r^{(3)} \right) \right) \pmod{p^5}.$$

With the help of the congruence modulo a prime power p^5 on the hyper-sums of powers of integers of Corollary 2, we obtain the following new congruences.

For $r = p-2$ in the Corollary 2, give congruence $S_d^{(p-2)}(p) (\bmod p^5)$ involving Bernoulli numbers.

Corollary 3. *For any prime $p > 5$ with $p^4(p-1)|d$. Then*

$$S_d^{(p-2)}(p) \equiv -1 + \frac{p}{p-1} - \frac{p^2}{(p-1)^2} + \frac{p^3}{(p-1)^3} - \frac{p^4}{(p-1)^4} - \frac{2p^4}{3(p-1)} B_{p-3} \pmod{p^5}.$$

Proof. Taking $r = p-2$ in the Corollary 1, we find that

$$\begin{aligned} S_d^{(p-2)}(p) \equiv -1 + \frac{1}{p-1} \Bigg(& p + p^2 h_{p-2} + p^3 \frac{1}{2} \left(h_{p-2}^2 - h_{p-2}^{(2)} \right) \\ & + p^4 \frac{1}{6} \left(h_{p-2}^3 - 3h_{p-2} h_{p-2}^{(2)} + 2h_{p-2}^{(3)} \right) \Bigg) \pmod{p^5}, \end{aligned} \tag{15}$$

where $h_{p-2} = h_{p-1} - \frac{1}{p-1}$, $h_{p-2}^{(2)} = h_{p-1} - \frac{1}{(p-1)^2}, h_{p-2}^{(3)} = h_{p-1} - \frac{1}{(p-1)^3}$ and by using Lemma 1, we find that

$$p^2 H_{p-2}^{(1)} \equiv -p^4 \frac{1}{3} B_{p-3} - \frac{p^2}{(p-1)} \pmod{p^5}, \tag{16}$$

$$p^3H_{p-2}^{(2)} \equiv -p^4\frac{1}{3}B_{p-3} + \frac{p^3}{(p-1)^2} \pmod{p^5} \tag{17}$$

and

$$p^4H_{p-2}^{(3)} \equiv -\frac{p^4}{(p-1)^3} \pmod{p^5}. \tag{18}$$

Substituting the congruences (16), (17) and (18) into (15) and after some rearrangement, we obtain the desired result. □

The following result immediately follows from Corollary 3 give congruence in terms of Wolstenholme quotient.

Recall that a prime p is said to be a Wolstenholme prime [6] if

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}.$$

This gives rise to the definition of the Wolstenholme quotient

$$W_p = \frac{\binom{2p-1}{p-1} - 1}{p^3}, \text{ for } p \geq 5.$$

Corollary 4. *For Wolstenholme prime p. We have*

$$S_d^{(p-2)}(p) \equiv -\sum_{k=0}^{4}(-1)^k\frac{p^k}{(p-1)^k} + \frac{p^4}{p-1}2W_p \pmod{p^5}.$$

Proof. By using congruence $W_p \equiv -\frac{1}{3}B_{p-3} \pmod{p}$ (see [1]), we have

$$p^4B_{p-3} \equiv -3p^4W_p \pmod{p^5},$$

substituting this congruence into the congruence of Corollary 3, it immediately reduces to Corollary 4. □

The following result generalize the above Corollary 3 and Corollary 4, for $\alpha \in \{1,2,3,4,5\}$.

Corollary 5. *For any prime $p > 5$ with $p^{\alpha-1}(p-1)|d$, we have*

$$S_d^{(p-2)}(p) \equiv -\sum_{k=0}^{\alpha-1}\left((-1)^k\frac{p^k}{(p-1)^k} + \frac{2p^k}{3(p-1)}B_{p-3}\delta_{4,k}\right) \pmod{p^\alpha}$$

and for Wolstenholme prime p, we have

$$S_d^{(p-2)}(p) \equiv -\sum_{k=0}^{\alpha-1}\left((-1)^k\frac{p^k}{(p-1)^k} - \frac{2p^k}{p-1}W_p\delta_{4,k}\right) \pmod{p^\alpha},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

For $r = \frac{p-1}{2}$ in the Corollary 2, give congruence modulo $S_d^{\left(\frac{p-1}{2}\right)}(p) \equiv \pmod{p^5}$ involving the first, second and cube powers of the Fermat quotient to base 2, and Bernoulli numbers.

Corollary 6. *For any prime $p \geq 7$ with $p^4(p-1)|d$. Then*

$$S_d^{\left(\frac{p-1}{2}\right)}(p) \equiv -1 + \frac{2}{p+1}\left(p - 2p^2 q_2(p) + 3p^3 q_2(p)^2 - 4p^4 q_2(p)^3 - \frac{29}{12}p^4 B_{p-3}\right) \pmod{p^5}. \quad (19)$$

Proof. Taking $r = \frac{p-1}{2}$ in the Corollary 1 we find that

$$S_d^{\left(\frac{p-1}{2}\right)}(p) \equiv -1 + \frac{2}{p+1}\left(p + p^2 h_{\frac{p-1}{2}} + p^3 \frac{1}{2}\left(h_{\frac{p-1}{2}}^2 - h_{\frac{p-1}{2}}^{(2)}\right)\right.$$
$$\left. + p^4 \frac{1}{6}\left(h_{\frac{p-1}{2}}^3 - 3h_{\frac{p-1}{2}} h_{\frac{p-1}{2}}^{(2)} + 2h_{\frac{p-1}{2}}^{(3)}\right)\right) \pmod{p^5}. \quad (20)$$

By using Lemma 2, we get

$$p^2 H_{\frac{p-1}{2}}^{(1)} \equiv -2p^2 q_p(2) + p^3 q_p(2)^2 - \frac{2}{3}p^4 q_p(2)^3 - \frac{7}{12}p^4 B_{p-3} \pmod{p^5}, \quad (21)$$

$$p^3 H_{\frac{p-1}{2}}^{(2)} \equiv 2p^3 q_p(2)^2 - 2p^4 q_p(2)^3 - \frac{7}{6}p^4 B_{p-3} \pmod{p^5} \quad (22)$$

and

$$p^4 H_{\frac{p-1}{2}}^{(3)} \equiv -\frac{4}{3}p^4 q_p(2)^3 - \frac{2}{3}p^4 B_{p-3} \pmod{p^5}. \quad (23)$$

Substituting the congruences (21), (22) and (23) into (20) and after some rearrangement, we get

$$S_d^{\left(\frac{p-1}{2}\right)}(p) \equiv -1 + \frac{2}{p+1}\left(p - 2p^2 q_p(2) + 3p^3 q_p(2)^2 - 4p^4 q_p(2)^3 - \frac{29}{12}p^4 B_{p-3}\right)$$
$$= -1 + \frac{2}{p+1}\sum_{k=0}^{3}(-1)^k (k+1) p^{k+1} q_p(2)^k - \frac{29}{6}\frac{p^4}{p+1} B_{p-3} \pmod{p^5},$$

as desired. □

The following Corollary immediately follows from Theorem 6, give congruence in terms of the first, second and cube powers of the Fermat quotient to base 2, and Wolstenholme quotient.

Corollary 7. *For Wolstenholme prime p with $p^4(p-1)|d$. We have*

$$S_d^{\left(\frac{p-1}{2}\right)}(p) \equiv -1 + \frac{2}{p+1}\left(p - 2p^2 q_p(2) + 3p^3 q_p(2)^2 - 4p^4 q_p(2)^3 + \frac{29}{4}p^4 W_p\right) \pmod{p^5}.$$

Proof. This is immediate from Theorem 6 and congruence $W_p \equiv -\frac{1}{3}B_{p-3} \pmod{p}$. □

Now we give the following result more general than Corollary 6, and Corollary 7, for $\alpha \in \{1, 2, 3, 4, 5\}$.

Corollary 8. *For any prime $p \geq 7$ with $p^{\alpha-1}(p-1)|d$, we have*

$$S_d^{\left(\frac{p-1}{2}\right)}(p) \equiv -1 + \frac{2}{p+1}\sum_{k=0}^{\alpha-1}\left((k+1)p^{k+1}(-1)^k q_p(2)^k - \frac{29}{6}p^k B_{p-3}\delta_{4,k}\right) \pmod{p^\alpha},$$

and for Wolstenholme prime p, we have

$$S_d^{\left(\frac{p-1}{2}\right)}(p) \equiv -1 + \frac{2}{p+1}\sum_{k=0}^{\alpha-1}\left((k+1)p^{k+1}(-1)^k q_p(2)^k + \frac{29}{4}p^k W_p \delta_{4,k}\right) \pmod{p^\alpha},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

Now for the case $r = [p/4]$, in Corollary 2, give congruence $S_d^{([p/4])}(p) (\bmod p^5)$ involving the first, second and cube powers of the Fermat quotient to base 2, Bernoulli and Euler numbers.

Theorem 9. *Let $p \geq 13$ be a prime with $p^4(p-1)|d$. Then*

$$S_d^{([p/4])}(p) \equiv -1 + \frac{1}{[p/4]+1}\left(p - 3p^2 q_p(2) + 6p^3 q_p(2)^2 - 10p^4 q_p(2)^3\right) + \frac{p^3}{[p/4]+1}(3E_{2p-4} - 6E_{p-3}) + \frac{p^4}{[p/4]+1}\left(9q_p(2)E_{p-3} - \frac{71}{12}B_{p-3}\right) (\bmod\ ^5).$$

Proof. Taking $r = [p/4]$ in the Corollary 1, we get

$$S_d^{([p/4])}(p) \equiv \frac{1}{[p/4]+1}\left(p + +p^2 h_{[p/4]} + \frac{p^3}{2}\left(h_{[p/4]}^2 - h_{[p/4]}^{(2)}\right) + \frac{p^4}{6}\left(h_{[p/4]}^3 - 3h_{[p/4]}h_{[p/4]}^{(2)} + 2h_{[p/4]}^{(3)}\right)\right) (\bmod p^5). \tag{24}$$

By using Lemma 3, and $p^4 E_{2p-4} \equiv p^4 E_{p-3} (\bmod p^5)$, we have

$$p^2 H_{[p/4]}^{(1)} \equiv -3p^2 q_p(2) + p^3\left(\frac{3}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(E_{2p-4} - 2E_{p-3})\right) + p^4\left(-q_p(2)^3 - \frac{7}{12}B_{p-3}\right) \ (\bmod p^5), \tag{25}$$

$$p^3 H_{[p/4]}^{(2)} \equiv p^3\left(\frac{9}{2}q_p(2)^2 - (-1)^{\frac{p-1}{2}}(4E_{p-3} - 2E_{2p-4})\right) + p^4\left(-\frac{9}{2}q_p(2)^3 + 3(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3} - \frac{7}{3}B_{p-3}\right) \ (\bmod p^5) \tag{26}$$

and

$$p^4 H_{[p/4]}^{(3)} \equiv p^4\left(-\frac{9}{2}q_p(2)^3 + 6(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3} - 3B_{p-3}\right) \ (\bmod p^5). \tag{27}$$

Substituting the congruences (25), (26) and (27) into (24) and after some rearrangement, we obtain

$$S_d^{([p/4])}(p) \equiv -1 + \frac{1}{[p/4]+1}\left(p - 3p^2 q_p(2) + 6p^3 q_p(2)^2 - 10p^4 q_p(2)^3\right) + \frac{p^3}{[p/4]+1}(3E_{2p-4} - 6E_{p-3}) + \frac{p^4}{[p/4]+1}\left(9q_p(2)E_{p-3} - \frac{71}{12}B_{p-3}\right) \ (\bmod p^5).$$

This completes the proof. □

Other congruences for $S_d^{([p/4])}(p) (\bmod p^5)$ involving the first, second and cube powers of the Fermat quotient to base 2, Wolstenholme quotient and Euler number are given as follows.

Corollary 10. *For Wolstenholme prime p. We have*

$$S_d^{([p/4])}(p) \equiv -1 + \frac{1}{[p/4]+1}\left(p - 3p^2 q_p(2) + 6p^3 q_p(2)^2 - 10p^4 q_p(2)^3\right) + \frac{p^3}{[p/4]+1}(3E_{2p-4} - 6E_{p-3}) + \frac{p^4}{[p/4]+1}\left(9q_p(2)E_{p-3} + \frac{71}{4}W_p\right) \ (\bmod p^5).$$

The following result generalize the above Corollary 9 and Corollary 10 for $\alpha \in \{1, 2, 3, 4, 5\}$.

Corollary 11. *For any prime $p \geq 13$, we have*

$$S_d^{([p/4])}(p) \equiv -1 + \frac{1}{[p/4]+1} \sum_{k=0}^{\alpha-2} \binom{k+2}{2} \left((-1)^k p^{k+1} q_p(2)^k + p^{k+1} \left(\frac{1}{2} E_{2p-4} - E_{p-3} \right) \delta_{2,k} \right.$$
$$\left. + \, p^{k+1} \left(\frac{9}{10} q_p(2) E_{p-3} - \frac{71}{120} B_{p-3} \right) \delta_{3,k} \right) \pmod{p^\alpha}$$

and for Wolstenholme prime p, we have

$$S_d^{([p/4])}(p) \equiv -1 + \frac{1}{[p/4]+1} \sum_{k=1}^{\alpha-2} \binom{k+2}{2} \left((-1)^k p^{k+1} q_p(2)^k + p^{k+1} \left(\frac{1}{2} E_{2p-4} - E_{p-3} \right) \delta_{2,k} \right.$$
$$\left. + \, p^{k+1} \left(\frac{9}{10} q_p(2) E_{p-3} + \frac{71}{40} W_p \right) \delta_{3,k} \right) \pmod{p^\alpha},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

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