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On certain rational perfect numbers, II

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Abstract: We continue the study from [1], by studying equations of type $\psi(n) = \frac{k+1}{k} \cdot n + a$ $a \in \{0, 1, 2, 3\}$, and $\varphi(n) = \frac{k-1}{k} \cdot n - a$, $a \in \{0, 1, 2, 3\}$ for $k > 1$, where $\psi(n)$ and $\varphi(n)$ denote the Dedekind, respectively Euler's, arithmetical functions.

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1 Introduction

Let $\psi(n)$ denote Dedekind's arithmetical function, defined for the prime factorization of $n = \prod p^a > 1$ as

$$\psi(n) = n \cdot \prod_{p|n} \left(1 + \frac{1}{p}\right) = \prod_{p^a \| n} (p^a + p^{a-1}). \tag{1}$$

Euler's totient can be defined similarly, namely

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right) = \prod_{p^a \| n} (p^a - p^{a-1}). \tag{2}$$

In the first part ([1]) we have considered equations of type $\sigma(n) = \frac{k+1}{k} \cdot n + a$, where $a \in \{0, 1, 2, 3\}$, and k a positive integer, with $\sigma(n)$ denoting the sum of divisors of n.

The aim of this paper is to study the similar equations for Dedekind's arithmetical function $\psi(n)$, as well as the similar equations

$$\varphi(n) = \frac{k-1}{k} \cdot n - a$$

for Euler's totient function $\varphi(n)$.

2 Auxiliary results

Lemma 1. *For any positive integers A, B one has*

$$\psi(A \cdot B) \geq A \cdot \psi(B) \tag{3}$$

and

$$\varphi(A \cdot B) \leq A \cdot \psi(B). \tag{4}$$

There is equality in (3) and (4), iff $A = 1$ or for any prime divisor p of A, p is prime divisor of B, too. (i.e., $p \mid A \Rightarrow p \mid B$).

This lemma is well-known (see e.g. [3]), and follows from (1) and (2).

Lemma 2.

(i) $\psi(n) = n + 1$ *iff* $n =$ *prime;*

(ii) $\psi(n) = n + 2$ *iff* $n = 4$*;*

(iii) $\psi(n) = n + 3$ *iff* $n = 9$*;*

(iv) $\psi(n) = n + 4$ *iff* $n = 8$.

Proof. Let $n = \prod p^a > 1$. Then $\psi(n) = \prod p^{a-1} \cdot (p + 1) \geq \prod(p^a + 1) \geq \prod p^a + 1 = n + 1$, by $(x_1 + 1) \ldots (x_r + 1) \geq x_1 \ldots x_r + 1$, with equality only for $r = 1$, i.e., when $n = p^a$ and $a = 1$, i.e., when n is prime. So (i) follows.

Writing (ii) as $\prod p^{a-1} \cdot [\prod(p+1) - \prod p] = 2$, remark that if $a \geq 2$, as $\prod(p+1) - \prod(p) \geq 1$, we get that if $\omega(n) \geq 2$ and n is not squarefree, then the equality is impossible. If $n = p^a$, we get $n = 2^2 = 4$, which is a solution. If n is squarefree, the equation $\prod(p + 1) - \prod p = 2$ is impossible if all $p \geq 3$. If $p_1 = 2$, we get $3 \cdot (p_2 + 1) \ldots (p_r + 1) - 2p_2 \ldots p_r = 2$. If $p_2 \geq 3$, and $r = 2$, $3 \cdot (p_2 + 1) - 2p_2 = p_2 + 3 > 3$. If $r \geq 3$,

$$\begin{aligned} 3 \cdot (p_2 + 1) \ldots (p_r + 1) - 2p_2 \ldots p_r &> 3p_2 \ldots p_r + 3 - 2p_2 \ldots p_r \\ &= p_2 \ldots p_r + 3 \\ &> 2. \end{aligned}$$

For (iii) we proceed in the same manner: $\prod p^{a-1}[\prod(p + 1) - \prod(p)] = 3$. If $p_1 = 2$, this is impossible, so $p_1 \geq 3$, $p_2 \geq 5$. Now $3^{a-1} \geq 3$ if $a \geq 2$. For $r = 1$ we get $n = 3^2 = 9$, a solution. If $n = p^a$, $\psi(p^a) - p^a = \prod p^{a-1} = 3 \Leftrightarrow p = 3$, $n = p^2 = 9$. If $\omega(n) \geq 2$, then $p_1^{a-1} \cdot p_2^{a'-1} \ldots \geq 3^{a-1} \cdot 5^{a'-1} > 3$ if $a > 1$, $a' > 1$. If $a' = a = 1$, then $(p_1 + 1)(p_2 + 1) - p_1 p_2 = 3$ so $p_1 + p_2 = 2$, impossible.

Finally, for case (iv) one has $\prod p^{a-1} \cdot [\prod(p + 1) - \prod p] = 4$ and if all $a \geq 3$, we get $p^{a-1} \geq 2^2 = 4$. So $n = 2^3 = 8$ is a solution. Now $p_1^{a-1} \cdot p_2^{a'-1} \geq 2^{a-1} \cdot 3^{a'-1} > 6$ if $a \geq 2$, $a' \geq 2$. If $a = 1$, $3^{a'-1} \cdot (p_1 + p_2 + 1) \geq 3^{a'-1} \cdot 6 > 4$. Generally, $2^{a-1} \cdot 3^{a'-1} \cdot 6 = 2^a \cdot 3^{a'} \geq 6 > 4$. Finally, if $n = p^a$ with $a \in \{1, 2\}$ one has $3^{a-1} > 4$ if $a \geq 2$. If $n = 3, 2, 2^2$, then none is acceptable. Thus $n = 8$ is the single solution. □

Lemma 3.

(i) $\varphi(n) = n - 1$ *iff* $n =$ *prime;*

(ii) $\varphi(n) = n - 2$ *iff* $n = 4$*;*

(iii) $\varphi(n) = n - 2$ *iff* $n = 9$*;*

(iv) $\varphi(n) = n - 2$ *iff* $n = 6$.

Proof. (i) is well-known.

For (ii), remark that as $n \geq 3$, and since $\varphi(n)$ is even, one must have $n =$ even. Let $n = 2^r \cdot M$, where M is odd. As $2^{r-1} \cdot \varphi(M) = 2^r \cdot M - 2$, we get $2^{r-1} \mid 2$, so $r \in \{1, 2\}$. For $r = 1$ one has $k = 2M$, so $\varphi(M) = 2 \cdot (M - 1)$. As $\varphi(M) \leq M - 1$, this is possible only if $2(M - 1) \leq M - 1$, i.e., $M = 1$, impossible. If $r = 2$, one has $n = 4 \cdot M$, so $2\varphi(M) = 4M - 2$, i.e., $\varphi(M) = 2M - 1$ so $M = 1$ and we get $n = 4$.

For (iii), let $n = \prod p^a$, so we get $\prod p^{a-1} \cdot (p - 1) = \prod p^a - 3$, which implies that $p^{a-1} \mid 3$, so $a = 1$ or $a = 2$ and $p = 3$. Them $n = p$ or $n = 9$ and only $n = 9$ is a solution.

Finally, write equation (iv) as $n = 2^a \cdot \prod p^{a'}$, as n must be even. We get $2^{a-1} \cdot \prod p^{a'-1} \cdot (p-1) = 2^a \cdot \prod p^{a'} - 4$. As $p - 1$ is even, we get $2^a \mid 4$ so $a \in \{1, 2\}$.

For $a = 1$ we get $\prod p^{a'-1} \cdot (p - 1) = 2 \cdot \prod p^{a'} - 4$ which is divisible by 2, but not by 4. So, there is a single prime p, i.e., $n = 2 \cdot p^a$. If $p^{a-1} \cdot (p - 1) = 2p^a - 4$ if $a > 1$ $p^a \mid 4$ impossible. So $a = 1$. Then $n = 2p$, and $\varphi(2p) = 2p - 4 \Leftrightarrow p - 1 = 2p - 4 \Leftrightarrow p = 3$.

Thus $n = 6$ is a solution.

If $a = 2$, then the equation becomes $2 \cdot \prod p^{a-1} \cdot (p - 1) = 4 \cdot \prod p^a - 4$, i.e., $\prod p^{a-1} \cdot (p - 1) = 2 \cdot \prod p^a - 2$. If $a > 1$, this is impossible as $p^{a-1} \mid 2$. Thus $a = 1$ in which case $\prod(p-1) = 2 \prod p - 2$, impossible, as $\prod(p - 1) < \prod p = 2 \prod p - 2$ and $\prod p > 2$. □

3 Main results

Theorem 1. *The equation*

$$\psi(n) = \frac{k + 1}{k} \cdot n \tag{5}$$

is solvable for $k > 1$ only if $k = p =$ prime and $n = p^{s+1}$ ($s \geq 0$ arbitrary integer).

Proof. Let $n = k \cdot m$, with $m \geq 1$ integer. Then as $\psi(km) = (k + 1) \cdot m \geq m \cdot \psi(k)$ by (3). As $\psi(k) \geq k + 1$ for $k > 1$ one must have $\psi(k) = k + 1$, i.e., $k =$ prime by Lemma 2 (i). There is equality only if $q \mid m \Rightarrow q \mid k$ ($q =$ prime). As $k = p =$ prime, we get $q = p$ and $n = p \cdot p^s = p^{s+1}$. □

Theorem 2. *Let $k > 1$. Then the equation*

$$\varphi(n) = \frac{k - 1}{k} \cdot n \tag{6}$$

is solvable only if $k = p =$ prime and $n = p^{s+1}$ ($s \geq 0$ integer).

Proof. Since $(k-1,k)=1$, we get $k \mid n$, so let $n=k\cdot m$. Then $\varphi(km)=(k-1)m \le m\varphi(k)$ by (4).

As $\varphi(k) \le k-1$ for $k>1$, we get that $\varphi(k)=k-1$, i.e., $k=p=$ prime by Lemma 3 (i).

By Lemma 1 one must have $n=p\cdot p^s=p^{s+1}$, where $s \ge 0$ is arbitrary. □

Theorem 3. *The equation*

$$\psi(n)=\frac{k+1}{k}\cdot n+1 \tag{7}$$

is solvable for $k>1$ only if $n=k=4$.

Proof. Let $n=k\cdot m$; then we get the equation $\psi(km)=(k+1)m+1$. As $\psi(km)\ge m\psi(k)$, we get $(k+1)m+1 \ge m\psi(k)$, i.e.

$$m\cdot[\psi(k)-(k+1)] \le 1. \tag{8}$$

There are two possible situations:

a) $m=$ arbitrary, $\psi(k)-(k+1)=0$;

b) $m=1$, $\psi(k)-(k+1)=1$.

In case b) we get $\psi(k)=k+2$, so by Lemma 2 (ii) we get $k=4$. As $m=1$, we get $n=4$, i.e., $n=k=4$ is a solution.

In case a) as $\psi(k)=k+1$, $k=p=$ prime, and n can be written as $n=p^a\cdot N$, where $(p,N)=1$. As $\psi(n)=p^{a-1}\cdot(p+1)\psi(N)$, the equation (7) becomes

$$p^a\cdot(p+1)\psi(N)=(p+1)\cdot p^{a-1}\cdot N+p. \tag{9}$$

Let $\psi(N)=N+T$, with $T\ge 0$. Then (9) may be rewritten as

$$N\cdot(p^{a+1}-p^{a-1})+T\cdot(p^{a+1}+p^a)=p. \tag{10}$$

Here $p^{a+1}-p^{a-1}=p^{a-1}\cdot(p^2-1)>p$ as $p^{a-1}\ge 1$, $p^2-1>p$ by $p^2-p=p(p-1)\ge 2>1$. Thus (10) is impossible. □

Theorem 4. *The equation*

$$\varphi(n)=\frac{k-1}{k}\cdot n-1 \tag{11}$$

is solvable only if $k=4$ and $n=2q$, where $q\ge 3$ is a prime.

Proof. Let $n=k\cdot m$, then we get $\varphi(km)=(k-1)m-1\le m\varphi(k)$ by (4) of Lemma 1. Thus we get

$$m\cdot[k-1-\varphi(k)]\le 1. \tag{12}$$

Therefore, we can distinguish:

a) m arbitrary, $k-1-\varphi(k)=0$;

b) $m=1$, $k-1-\varphi(k)=1$.

In case b) one has $\varphi(k) = k - 2$, so by Lemma 3 (ii) we get $k = 4$. Now $n = 4$ and it can be verified that $\varphi(4) = \frac{3}{4} \cdot 4 - 1$; i.e., $\varphi(4) = 2$.

In case a) $\varphi(k) = k - 1$, so k is a prime; $k = p$. Let $n = p^a \cdot N$, with $(p, N) = 1$. Then equation (11) becomes

$$p^a \cdot (p - 1)\varphi(N) = (p - 1) \cdot p^a \cdot N - p. \tag{13}$$

Thus $p^a \mid p$ so $a = 1$; in which case we get

$$p(p - 1)\varphi(N) = (p - 1) \cdot p \cdot N - p,$$

so $(p-1)\varphi(N) = (p-1) \cdot N - 1$, implying that $p - 1 \mid 1$, i.e., $p = 2$. Then we get $\varphi(N) = N - 1$, i.e., $N = q =$ prime. Thus, finally, we get $n = 2 \cdot q$, where $q \geq 3$ is an odd prime. □

Theorem 5. *The equation*

$$\psi(n) = \frac{k+1}{k} \cdot n + 2 \tag{14}$$

is solvable for $k > 1$ only if $n = k = 9$ or $n = 8$, $k = 4$.

Proof. Let $n = k \cdot n$, and the equation becomes $\psi(km) = (k + 1)m + 2$. As $\psi(km) \geq m\psi(k)$, we get

$$m \cdot [\psi(k) - (k + 1)] \leq 2. \tag{15}$$

We have to consider three cases:

a) m arbitrary, $\psi(k) - (k + 1) = 0$;

b) $m = 1$, $\psi(k) = k + 3$;

c) $m = 2$, $\psi(k) = k + 2$.

Remark that in case b), by Lemma 2 (iii) one has $k = 9$. As $m = 1$, we get $n = 9$. One can verify that $\psi(9) = \frac{10}{9} \cdot 9 + 2 = 12$.

In case c) by Lemma 2 (ii) we get $k = 4$. As $m = 2$, we get the solution $n = 8$.

Finally, in case a) $k = p =$ prime. Let $n = p^a \cdot N$. We get the equation

$$N \cdot (p^{a+1} - p^{a-1}) + T \cdot (p^{a+1} + p^a) = 2p, \tag{16}$$

writing $\psi(N) = N + T$, with $T \geq 0$.

Remark that $p^{a+1} - p^{a-1} = p^{a-1} \cdot (p^2 - 1) > 2p$ if $a \geq 1$, $p \geq 3$. If $p = 2$ one has that $2^{a-1} \cdot 3 > 4$ for $a \geq 2$. Finally, for $a = 1$, $p = 2$ as $n = 2 \cdot N$, we get $3 \cdot N + 6T = 4$, impossible as N is odd. □

Theorem 6. *The equation*

$$\varphi(n) = \frac{k-1}{k} \cdot n - 2 \tag{17}$$

is solvable only if $n = 8$, $k = 4$ or $n = k = 9$ or $n = 4 \cdot q$ ($q \geq$ prime), $k = 2$ or $n = 3q$ ($q \geq 5$ prime), $k = 3$.

Proof. By letting $n = km$, now the relation similar to (12) will be

$$m \cdot [k - 1 - \varphi(k)] \leq 2. \tag{18}$$

The following cases showed be considered:

a) m arbitrary, $k - 1 - \varphi(k) = 0$;

b) $m = 1$, $k - 1 - \varphi(k) = 1$;

c) $m = 1$, $k - 1 - \varphi(k) = 2$;

d) $m = 2$, $k - 1 - \varphi(k) = 1$.

In case b) $\varphi(k) = k - 2$, so by Lemma 3 ii) we get $k = 4$. Then $n = 4$, which is not a solution, as $\varphi(4) \neq \frac{3}{4} \cdot 4 - 2$.

In case c) $m = 1$, $\varphi(k) = k - 3$, so $k = 9$ and we get $n = 9$, which can be verified to be a solution: $\varphi(9) = \frac{8}{9} \cdot 9 - 2 = 6$.

In case d) $m = 2$, and $\varphi(k) = k - 2$ so $k = 4$ and $n = 8$, which provide a solution, as $\varphi(8) = \frac{3}{4} \cdot 8 - 2 = 4$.

Finally, in case a) $k = \text{prime} = p$ and let $n = p^a \cdot N$ with $(1, N) = 1$. We get the equation

$$p^{a-1} \cdot (p-1)\varphi(N) \cdot p = (p-1) \cdot p^a \cdot N - 2p. \tag{19}$$

<u>Subcase 1.</u> $p = 2$. Then $2^a \cdot \varphi(N) = 2^a \cdot N - 4$, so $2^a \mid 4$, which means that $a \in \{1, 2\}$. For $a = 1$ we get $2\varphi(N) = 2N - 4$, i.e., $\varphi(N) = N - 2$ giving $N = 4$, impossible as N is odd.

For $a = 2$ we get $4\varphi(N) = 4N - 4$ or $\varphi(N) = N - 1$, so $N = q = \text{prime}$. Then $n = 2^2 \cdot q = 4q$, where q is an odd prime.

<u>Subcase 2.</u> $p \geq 3$. Then (19) implies $(p - q) \cdot p^a \mid 2p$. If $p - 1 \geq 2$ (i.e., $p \geq 3$) and $a \geq 2$, we get a contradiction. Thus $a = 1$, in which case $(p-1) \cdot p \mid 2p$, so $p - 1 \mid 2$, possible only for $p = 3$. Then we get the equation $3 \cdot 2 \cdot \varphi(N) = 3 \cdot 2 \cdot N - 6$, i.e., $\varphi(N) = N - 1$. Thus $N = q = \text{prime}$. Then $k = p = 3$ and so $n = 3q$, where $q \geq 5$ is a prime. This gives indeed a solution, as $\varphi(3q) = \frac{2}{3} \cdot 3q - 2$, i.e., $2\varphi(q) = 2q - 2$ which is true. So, this subcase is settled, too. □

Theorem 7. *The equation*

$$\psi(n) = \frac{k+1}{k} \cdot n + 3 \tag{20}$$

is solvable for $k > 1$ *only if* $n = k = 8$ *and* $k = 2$, $n = 18$.

Proof. Letting $n = k \cdot m$, we get similarly to (15):

$$m \cdot [\psi(k) - (k+1)] \leq 3. \tag{21}$$

Logically, six cases are possible:

a) m arbitrary, $\psi(k) - (k+1) = 0$;

b) $m = 1$, $\psi(k) - (k+1) = 1$;

c) $m = 2$, $\psi(k) - (k+1) = 1$;

d) $m = 3$, $\psi(k) - (k+1) = 1$;

e) $m = 1$, $\psi(k) - (k+1) = 2$;

f) $m = 1$, $\psi(k) - (k+1) = 3$.

In case b) $\psi(k) = k + 2$, so $k = 4$ and $n = 4$, which is not a solution, as $6 \neq \frac{5}{4} \cdot 4 + 3 = 8$.

In case c) $m = 2$, $k = 4$, $n = 8$ and again not a solution, as $12 \neq \frac{5}{4} \cdot 8 + 3$.

In case d) $m = 3$, $k = 4$ so $n = 12$ and as $\psi(12) = 24 \neq \frac{5}{4} \cdot 12 + 3 = 18$.

In case e) we get $m = 1$, $k = 9$ and $n = 9$ and as $\psi(9) = 12 \neq \frac{10}{9} \cdot 9 + 3$ we have no solution.

Finally, in case f) $m = 1$, $k = 8$, $n = 8$ and $18 = \frac{9}{8} \cdot 8 + 3$ a solution.

Let us now consider the most difficult case, i.e., a) when $k = p =$ prime. Let $n = p^a \cdot N$ when we get the similar equation to (16):

$$N \cdot (p^{a+1} - p^{a-1}) + T \cdot (p^{a+1} + p^a) = 3p. \tag{22}$$

Now, remark that $p^{a+1} - p^{a-1} = p^{a-1} \cdot (p^2 - 1) > 3p$, which is true for $p \geq 5$ as $p^2 - 1 > 3p$, i.e., $p(p-3) > 1$. If $p = 3$, then $3^{a-1} \cdot 8 > 9$ for $a \geq 2$. Thus we have to consider $p = 3$, $a = 1$. As $\psi(3) = 3 + T$, we must have $T = 1$, and then (22) is not satisfied, as $3 \cdot (3^2 - 1) + 3^2 + 3 \neq 9$.

If $p = 2$, then $2^{a-1} \cdot 3 \geq 3\,p = 6$ true, if $a \geq 2$. For $a = 2$, we get $T = 0$, impossible, and when $a = 1$, $n = 2$, $N = 2$, $T = 1$, and (22) again is not true, as $2 \cdot (2^2 - 1) + 2^2 + 2 \neq 6$. $\square$

Theorem 8. *The only solution to equation*

$$\varphi(n) = \frac{k-1}{k} \cdot n - 3 \tag{23}$$

is $n = k = 6$.

Proof. The analogue of (18) now is

$$m \cdot [k - 1 - \varphi(k)] \leq 3. \tag{24}$$

We have to consider six distinct cases, similar to the case of Theorem 7.

In case b) as $\varphi(k) = k - 2$ we get $k = 4$ and $n = 4$ and $\varphi(4) \neq \frac{3}{4} \cdot 4 - 3$.

In case c) $m = 2$, $\varphi(k) = k - 2$ so $k = 4$, $n = 8$ and $\varphi(8) \neq \frac{3}{4} \cdot 8 - 3 = 3$.

In case d) $m = 3$ and $k = 4$, so $n = 12$ and $\varphi(12) \neq \frac{3}{4} \cdot 12 - 3 = 6$ as $\varphi(12) = 4$.

In case e) $m = 1$, $\varphi(k) = k - 3$ so $k = 9$ and $n = 9$ and $\varphi(9) \neq \frac{8}{9} \cdot 9 - 3$.

In case f) $m = 1$, $\varphi(k) = k - 4$, so by Lemma 3 (iv) we get $k = 6$. Then $n = 6$ and as $\varphi(6) = \frac{5}{6} \cdot 6 - 3 = 2$, we get a solution.

Finally, in case a) when $k = p =$ prime, let $n = p^a \cdot N$. Then the equation similar to (19) will be

$$p^{a-1} \cdot (p-1)\varphi(N) \cdot p = (p-1) \cdot p^a \cdot N - 3 \cdot p. \tag{25}$$

If $p = 2$, we get $2^a \cdot \varphi(N) = 2^a \cdot N - 6$, so $2^a \mid 6$, which is possible only if $a = 1$. Then $2\varphi(N) = 2N - 6$, so $\varphi(N) = N - 3$ and by Lemma 3 (iii), $N = 9$. Then $n = 2^1 \cdot 9 = 18$. So $k = 2$ and $n = 18$ is a solution. If $p \geq 3$, then $(p-1) \cdot p^a \mid 3p$. For $a \geq 3$ this would imply $(p-1)p^2 \mid 3 \cdot p$, so $(p-1) \cdot p \mid 3$ which is impossible as $(p-1) \cdot p > 3$. If $a = 1$, then $(p-1)p \mid 3p$, so $p - 1 \mid 3$, thus $p - 1 = 1$ or 3 and $p = 2, 4$ are impossible. $\square$

Remark 1. *Other equations involving* ψ *and* φ *can be found in* [3] *and* [2].

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