Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 3, 517–524
DOI: 10.7546/nntdm.2022.28.3.517-524

On edge irregularity strength of line graph and line cut-vertex graph of comb graph

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Received: 4 March 2022 **Revised:** 2 August 2022
Accepted: 5 August 2022 **Online First:** 10 August 2022

Abstract: For a simple graph G, a vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called k-labeling. The weight of an edge xy in G, written $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e., $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f, $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, written $es(G)$. In this paper, we find the exact value of edge irregularity strength of line graph of comb graph $P_n \bigodot K_1$ for $n = 2, 3, 4$; and determine the bounds for $n \geq 5$. Also, the edge irregularity strength of line cut-vertex graph of $P_n \bigodot K_1$ for $n = 2$; and determine the bounds for $n \geq 3$.
Keywords: Irregular assignment, Irregularity strength, Irregular total k-labeling, Edge irregularity strength, Comb graph.
2020 Mathematics Subject Classification: 05C38, 05C78.

1 Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set (the edge-set), then the labeling is called *vertex labelings* (*edge labelings*).

If the domain is $V(G) \cup E(G)$, then the labeling is called *total labeling*. Thus, for an edge k-labeling $\delta : E(G) \to \{1, 2, \ldots, k\}$ the associated weight of a vertex $x \in V(G)$ is $w_\delta(x) = \sum \delta(xy)$, where the sum is over all vertices y adjacent to x.

Chartrand et al. [10] defined irregular labeling for a graph G as an assignment of labels from the set of natural numbers to the edges of G such that the sums of the labels assigned to the edges of each vertex are different. The minimum value of the largest label of an edge over all existing irregular labelings is known as the *irregularity strength* of G and it is denoted by $s(G)$. Finding the irregularity strength of a graph seems to be hard even for simple graphs [10, 11].

Motivated by this, Bača et al. [8] investigated two modifications of the irregularity strength of graphs, namely *total edge irregularity strength*, denoted by $tes(G)$; and *total vertex irregularity strength*, denoted by $tvs(G)$. Some results on the total edge irregularity strength and the total vertex irregularity strength can be found in [2, 3, 5, 7, 9].

Motivated by the work of Chartrand et al. [10], Ahmad et al. [4] introduced the concept of edge irregular k-labelings of graphs.

A vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called k-labeling. The weight of an edge xy in G, written $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e., $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling of a graph G is defined to be an *edge irregular k-labeling* of the graph G if for every two different edges e and f, $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, written $es(G)$.

Over the last years, $es(G)$ has been investigated for different families of graphs including trees with the help of algorithmic solutions, see [15, 17, 18]. The most complete recent survey of graph labelings is [12].

2 Preliminary results

In [4], the authors estimated the bounds of the edge irregularity strength and then determined its exact values for several families of graphs namely, paths, stars, double stars, and Cartesian product of two paths. Ahmad et al. [6] determined the edge irregularity strength of Toeplitz graphs. Tarawneh et al. [14] determined the exact value of edge irregularity strength of corona product of graphs with paths. Tarawneh et al. [16] determined the exact value of edge irregularity strength of corona product of graphs with cycle. Recently, Zhang et al. [19] determined the exact value of edge irregularity strength of certain families of comb graph.

The following theorem in [4] establishes the lower bound for the edge irregularity strength of a graph G.

Theorem 2.1. *Let $G = (V, E)$ be a simple graph with maximum degree $\Delta(G)$. Then*

$$es(G) \geq \max\{\lceil \tfrac{|E(G)|+1}{2} \rceil, \Delta(G)\}.$$

3 Edge irregularity strength of line graph of comb graph

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, line cut-vertex graphs; total graphs; middle graphs; and their generalizations.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.

Let P_n be a path on $n \geq 1$ vertices. Then the *comb graph* is defined by $P_n \bigodot K_1$. It has $2n$ vertices and $2n-1$ edges.

In the next theorem, we find the exact value of edge irregularity strength of line graph of comb graph $P_n \bigodot K_1$ for $n = 2, 3, 4$; and determine the bounds for $n \geq 5$.

Theorem 3.1. *Let $G = P_n \bigodot K_1$, $n \geq 2$, be a comb graph. Then*

$$es(L(G)) = \begin{cases} n, & \text{for } n = 2, 3 \\ 5, & \text{for } n = 4 \end{cases}$$

For $n \geq 5$,

$$\lceil \tfrac{3n-3}{2} \rceil \leq es(L(G)) \leq 2n - 3.$$

Proof. Let $G = P_n \bigodot K_1$, $n \geq 2$, be a comb graph. We consider the following cases.

Case 1: If $G = P_2 \bigodot K_1$, then $L(G)$ is P_3. Clearly, $es(L(G)) = 2$.

Case 2: If $G = P_3 \bigodot K_1$, then $L(G)$ is a triangle with two disjoint edges, i.e., the bull graph. Let us consider the vertex set and the edge set of $L(G)$:

$$V(L(G)) = \{x_i : 1 \leq i \leq 4\} \cup \{y_1\},$$
$$E(L(G)) = \{x_i x_{i+1}, x_2 y_1, x_3 y_1 : 1 \leq i \leq 3\}.$$

Clearly, $|V(L(G))| = 5$, $|E(L(G))| = 5$, and the maximum degree $\Delta = 3$. According to the Theorem 2.1, $es(L(G)) \geq \max\{\lceil \frac{5+1}{2} \rceil, 3\}$. Hence, $es(L(G)) \geq \lceil 3 \rceil = 3$.

To prove the equality, it is sufficient to prove the existence of an edge irregular 3-labeling. Define a labeling on vertex set of $L(G)$ as follows:

Let $\phi : V(L(G)) \to \{1, 2, 3\}$ such that $\phi(x_i) = 1$ for $1 \leq i \leq 2$; $\phi(x_i) = 3$ for $3 \leq i \leq 4$; and $\phi(y_1) = 2$.

The edge weights are as follows:

$$w_\phi(x_i x_{i+1}) = 2i \text{ for } 1 \leq i \leq 3; w_\phi(x_2 y_1) = 3, w_\phi(x_3 y_1) = 5.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, $es(L(G)) = 3$.

Figure 1. The comb graph $P_3 \odot K_1$ and its line graph

Case 3: Let $G = P_4 \odot K_1$ be the comb graph. Let us consider the vertex set and the edge set of $L(G)$:

$$V(L(G)) = \{x_i : 1 \leq i \leq 5\} \cup \{y_j : 1 \leq j \leq 2\},$$
$$E(L(G)) = \{x_i x_{i+1}, x_{j+1} y_j, x_{j+2} y_j : 1 \leq i \leq 4; 1 \leq j \leq 2\}.$$

Clearly, $|V(L(G))| = 7$, $|E(L(G))| = 8$, and the maximum degree $\Delta = 4$. According to the Theorem 2.1, $es(L(G)) \geq \max\{\lceil \frac{8+1}{2} \rceil, 4\}$. Hence, $es(L(G)) \geq \lceil \frac{8+1}{2} \rceil (= 5)$.

To prove the equality, it is sufficient to prove the existence of an edge irregular 5-labeling. Define a labeling on vertex set of $L(G)$ as follows: Let $\phi : V(L(G)) \to \{1, 2, 3, 4, 5\}$ such that $\phi(x_1) = \phi(x_2) = \phi(x_5) = 1$; $\phi(x_3) = 3$; $\phi(x_4) = 5$; $\phi(y_1) = 4$; and $\phi(y_2) = 2$.

The edge weights are as follows:

$$\begin{aligned} w_\phi(x_i x_{i+1}) &= 2i \quad &&\text{for } 1 \leq i \leq 2; w_\phi(x_3 x_4) = 8, w_\phi(x_4 x_5) = 6 \\ w_\phi(x_i y_{i-1}) &= 2i + 1 &&\text{for } 2 \leq i \leq 3; w_\phi(x_3 y_1) = 5, w_\phi(x_4 y_2) = 9 \end{aligned}$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, $es(L(G)) = 5$.

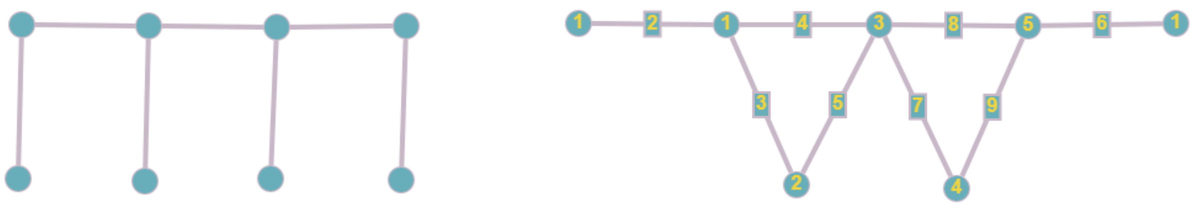

Figure 2. The comb graph $P_4 \odot K_1$ and its line graph

Finally, we have to prove that $\lceil \frac{3n-3}{2} \rceil \leq es(L(G)) \leq 2n - 3$, for $n \geq 5$.

Case 4: Let $G = P_n \odot K_1$, $n \geq 5$, be a comb graph. Let us consider the vertex set and the edge set of $L(G)$:

$$V(L(G)) = \{x\} \cup \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq n-2\},$$
$$E(L(G)) = \{x x_1, x_i x_{i+1}, x_j y_j, x_{j+1} y_j : 1 \leq i \leq n-1; 1 \leq j \leq n-2\}.$$

Clearly, $|V(L(G))| = 2n - 1$, $|E(L(G))| = 3n - 4$, and the maximum degree $\Delta = 4$. According to the Theorem 2.1, $es(L(G)) \geq \max\{\lceil \frac{3n-3}{2} \rceil, 4\}$. Since $\lceil \frac{3n-3}{2} \rceil > 4$ for $n \geq 5$, it follows that $es(L(G)) \geq \lceil \frac{3n-3}{2} \rceil$.

For the upper bound, we define a vertex labeling ϕ as follows:

$$\begin{aligned}\phi(x) &= \phi(x_1) = 1;\\ \phi(x_i) &= 2i-2 \text{ for } 2 \le i \le n-1;\\ \phi(x_n) &= 2i+4, i = n-4; \text{ and}\\ \phi(y_i) &= 2i+1 \text{ for } 1 \le i \le n-2.\end{aligned}$$

The edge weights are as follows:

$$\begin{aligned}w_\phi(xx_1) &= 2,\\ w_\phi(x_1x_2) &= 3,\\ w_\phi(x_1y_1) &= 4,\\ w_\phi(x_ix_{i+1}) &= 4i-2 \text{ for } 2 \le i \le n-2,\\ w_\phi(x_iy_i) &= 4i-1 \text{ for } 2 \le i \le n-2,\\ w_\phi(x_{i+1}y_i) &= 4i+1 \text{ for } 1 \le i \le n-2.\end{aligned}$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, the vertex labeling ϕ is an edge irregular $(2n-3)$-labeling, i.e., $es(L(G)) \le 2n-3$ for $n \ge 5$. This completes the proof. □

Open Problem 1. Determine the edge irregularity strength of the line graph of comb graph $P_n \bigodot K_1$ for $n \ge 5$.

4 Edge irregularity strength of line cut-vertex graph of a comb graph

The authors in [13] gave the following definition. The *line cut-vertex graph* of a graph G, written $L_c(G)$, is the graph whose vertices are the edges and cut-vertices of G, with two vertices of $L_c(G)$ adjacent whenever the corresponding edges of G have a vertex in common; or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G such that e_i is incident with c_j.

Clearly, $L_c(G)$ of a graph G exists only if the number of cut-vertices of G is at least one.

The *friendship graph F_n* is the graph obtained by joining n copies of the cycle graph C_3 with a vertex in common.

The authors in [1] defined the chain graph and the mK_3-path as follows. A *chain graph* is a graph with blocks $B_1, B_2, \ldots, B_n$ such that for every i, B_i and B_{i+1} have a vertex in common in such a way that the block-cut-vertex graph is a path.

A *mK_3-path* is a chain graph with m blocks where each block is identical and isomorphic to the complete graph k_n. Clearly, the $2K_3$-path is the friendship graph F_2.

Regarding the edge irregularity strength of the mK_3-path, we present the following theorem which is due to Ahmed et al. [1].

Theorem 4.1. *Let G be a mK_3-path. For any positive integer m, $\lceil \frac{3(m+1)}{2} \rceil \le es(G) \le 2m+1$.*

In the next theorem, we find the exact value of edge irregularity strength of line cut-vertex graph of comb graph $P_n \bigodot K_1$ for $n = 2$; and determine the bounds for $n \geq 3$.

Theorem 4.2. *Let $G = P_n \bigodot K_1$, $n \geq 2$, be a comb graph. For $n = 2$, $es(L_c(G)) = 5$. For $n \geq 3$,*

$$\lceil \frac{6n-5}{2} \rceil \leq es(L_c(G)) \leq 5n - 3.$$

Proof. Let $G = P_n \bigodot K_1, n \geq 2$ be a comb graph. We consider the following cases.

<u>Case 1:</u> If $G = P_2 \bigodot K_1$, then $L_c(G) = F_2(= 2K_3-\text{path})$. According to the upper bound of Theorem 4.1, $es(L_c(G)) = 5$.

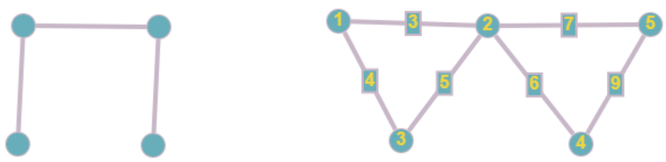

Figure 3. The comb graph $P_2 \bigodot K_1$ and its line cut-vertex graph

<u>Case 2:</u> Let $G = P_n \bigodot K_1$, $n \geq 3$, be a comb graph. Let us consider the vertex set and the edge set of $L_c(G)$:

$$\begin{aligned} V(L_c(G)) &= \{v_i : 1 \leq i \leq n+1\} \cup \{v_j : n+2 \leq j \leq 2n-1\} \cup \{v_k : 2n \leq k \leq 3n-1\}, \\ E(L_c(G)) &= \{v_i v_{i+1}, v_i v_{i+5}, v_j v_{j+4}, v_r v_{r+n}, v_s v_{s+n-1}, v_{t+n} v_{t+2n-1} : \\ &\qquad 1 \leq i \leq n; 2 \leq j \leq n+1; 2 \leq r \leq n-1; 3 \leq s \leq n; 2 \leq t \leq n-1\} \end{aligned}$$

Clearly, $|V(L_c(G))| = 3n - 1, |E(L_c(G))| = 6(n - 1)$, and the maximum degree $\Delta = 6$. According to the Theorem 2.1, $es(L_c(G)) \geq \max\{\lceil \frac{6n-5}{2} \rceil, 6\}$. Since $\lceil \frac{6n-5}{2} \rceil > 6$ for $n \geq 3$, it follows that $es(L_c(G)) \geq \lceil \frac{6n-5}{2} \rceil$.

For the upper bound, we define a vertex labeling ϕ as follows:

$$\begin{aligned} \phi(v_i) &= i \text{ for } 1 \leq i \leq n+1; \\ \phi(v_i) &= 2i + j \text{ for } n+2 \leq i \leq 2n-1, 0 \leq j \leq n-3; \\ \phi(v_{3n-1}) &= n + 2; \text{ and} \\ \phi(v_i) &= i + 2j + 1 \text{ for } 2n \leq i \leq 3n-2, 1 \leq j \leq n-1. \end{aligned}$$

The edge weights are as follows:

$$\begin{aligned} w_\phi(v_i v_{i+1}) &= 2i + 1 \text{ for } 1 \leq i \leq n; \\ w_\phi(v_1 v_{2n}) &= 2n + 4; \\ w_\phi(v_2 v_{2n}) &= 2n + 5; \\ w_\phi(v_n v_{3n-1}) &= 2n + 2; \\ w_\phi(v_{n+1} v_{3n-1}) &= 2n + 3; \end{aligned}$$

$$w_\phi(v_iv_{i+n}) = 3i + j + 2n \text{ for } 2 \le i \le n-1, 0 \le j \le n-3;$$
$$w_\phi(v_iv_{i+n-1}) = 3i + j + 2n - 2 \text{ for } 3 \le i \le n, 0 \le j \le n-3;$$
$$w_\phi(v_iv_{i+n-1}) = 2i + 4j + 2n + 2 \text{ for } n+2 \le i \le 2n-1, 1 \le j \le n-2;$$
$$w_\phi(v_iv_{2n+i-1}) = 2(n + i + j + 1) \text{ for } 2 \le i \le n-1, 1 \le j \le n-2;$$
$$w_\phi(v_{i+1}v_{2n+i-1}) = 2(n + i + j) + 3 \text{ for } 2 \le i \le n-1, 1 \le j \le n-2.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Thus the vertex labeling ϕ is an edge irregular $(5n-3)$-labeling, i.e., $es(L_c(G)) \le 5n-3$ for $n \ge 3$. This completes the proof. □

Open Problem 2. Determine the edge irregularity strength of the line cut-vertex graph of comb graph $P_n \bigodot K_1$ for $n \ge 3$.

5 Conclusion

In this paper, we investigated the edge irregularity strength, as a modification of the well-known irregularity strength, total edge irregularity strength, and total vertex irregularity strength. We obtained the exact value of edge irregularity strength of line graph of comb graph $P_n \bigodot K_1$ for $n = 2, 3, 4$; and determined the bounds for $n \ge 5$. Also, the edge irregularity strength of line cut-vertex graph of $P_n \bigodot K_1$ for $n = 2$; and the bounds for $n \ge 3$ are found. However, to find the exact values of edge irregularity strength of different graph operators still remains open.

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