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Some aspects of interchanging difference equation orders

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Abstract: This paper builds on Roettger and Williams' extensions of the primordial Lucas sequence to consider some relations among difference equations of different orders. This paper utilises some of their second and third order recurrence relations to provide an excursion through basic second order sequences and related third order recurrence relations with a variety of numerical illustrations which demonstrate that mathematical notation is a tool of thought.

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1 Introduction

Difference equations can be useful in applications which are matrix-oriented and in time located data where the functional behaviour between measurements is not known or may be subject to ill-conditioning [33]. This paper considers some aspects of connections between similar, but different, expressions of the same sequences of numbers. For example, Roettger and Williams have used cubic extensions of the Lucas sequence to develop simple, but not easy, tests for primality [20, 22, 24], Williams having long been a foremost authority on such tests [32].

The cubic extensions in question can be represented by the third order homogeneous linear recurrence relation

$$L_n^{(3)} = PL_{n-1}^{(3)} - QL_{n-2}^{(3)} + RL_{n-3}^{(3)}, \qquad n \geq 3, \tag{1.1}$$

where the coefficients are real numbers, and the three roots, α, β, γ, of the associated characteristic equation are assumed to be distinct, with

$$\delta = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha). \tag{1.2}$$

The recurrence relation (1.1) gives rise to two third order sequences $\{W_n^{(3)}\}$ and $\{C_n^{(3)}\}$ when the initial terms are given as in Table 1:

$\left\{\boldsymbol{X_n^{(3)}}\right\}$	***n* = 0**	***n* = 1**	***n* = 2**
$\left\{\boldsymbol{C_n^{(3)}}\right\}$	0	1	$PQ - R$
$\left\{\boldsymbol{W_n^{(3)}}\right\}$	6	$PQ - 3R$	$P^2Q^2 - 2Q^3 - 2P^3R + 4PQR - 3R^2$

Table 1. Initial terms of the sequences $\{W_n^{(3)}\}$ and $\{C_n^{(3)}\}$, [23]

Other authors have recently been studying Fibonacci and Tribonacci sequences [1, 6], but we are not here concerned with these particular accidental features. We shall use the material in a formal sense as we look at connections with three second order sequences. Of course, when $R = 0$ in (1.1) we are back to second order sequences anyway. It should be noted that Khomovsky [17] used Horadam [12] to generalize these to a higher order. We now define the three second order sequences in question by their general terms, $n \geq 0$. We now define the three second order sequences in question by their general terms, $n \geq 0$:

$$A_{1,n}^{(3)} = \frac{\gamma^n - \beta^n}{\gamma - \beta}, \qquad B_{1,n}^{(3)} = \frac{\alpha^n - \gamma^n}{\alpha - \gamma}, \qquad D_{1,n}^{(3)} = \frac{\beta^n - \alpha^n}{\beta - \alpha}, \tag{1.3}$$

and their complements are:

$$A_{2,n}^{(3)} = \gamma^n + \beta^n, \qquad B_{2,n}^{(3)} = \alpha^n + \gamma^n, \qquad D_{2,n}^{(3)} = \beta^n + \alpha^n, \tag{1.4}$$

with the relevant patterns to be examined in Section 3. Of course, it is well known that the Fibonacci numbers can be generated in a number of ways. For instance, the second order

$$F_n = F_{n-1} + F_{n-2}, n > 2,$$

and the third order

$$F_n = 2F_{n-2} + F_{n-3}, n > 3.$$

At one level, this is trivial, but the search for connections can generate some elegant number theory. There are pedagogic issues and pure mathematics connotations too but they are not directly the focus of this paper, although it should be pointed out that the demarcations between pure and applied mathematics are not necessarily obvious. For example, one of the current authors, in partnership with Krassimir Atanassov, developed ‘matrix tertions’ and ‘matrix noitrets’ as pure pedagogical objects between complex numbers and quaternions [1]. Their three dimensional extensions include ‘matrix rhotrices”, and they have since been utilized in a variety of applications, including cryptography [33].

2 Patterns

We can obtain from equations (1.1), (1.2) and (1.4) such results as

$$k = \gamma + \beta, \qquad l = \alpha + \gamma, \qquad m = \beta + \alpha,$$

so that

$$k + l + m = 2(\alpha + \beta + \gamma) = 2P$$

and

$$a + b + c = -(\alpha\gamma + \beta\alpha + \gamma\beta) = Q.$$

$\{X_{s,n}^{(2)}\}$	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$\left\{A_{1,n}^{(2)}\right\}$	0	1	k	$k^2 + a$
$\left\{B_{1,n}^{(2)}\right\}$	0	1	l	$l^2 + b$
$\left\{D_{1,n}^{(2)}\right\}$	0	1	m	$m^2 + c$
$\left\{A_{2,n}^{(2)}\right\}$	2	k	$k^2 + 2a$	$k^3 + 3ak$
$\left\{B_{2,n}^{(2)}\right\}$	2	l	$l^2 + 2b$	$l^3 + 3bl$
$\left\{D_{2,n}^{(2)}\right\}$	2	m	$m^2 + 2c$	$m^3 + 3cm$

Table 2. Pattern of first four terms in the second order sequences

Other well-known results are trivially preserved as

$$\left\{A_{1,n}^{(2)}\right\}\left\{A_{2,n}^{(2)}\right\} = \left\{A_{1,2n}^{(2)}\right\}$$

In turn these imply that

$$a = -\gamma\beta, \qquad b = -\alpha\gamma, \qquad c = -\beta\alpha.$$

Our second order recurrence relations can then be expressed in turn for $n \geq 2$, as

$$\begin{cases} A_{1,n}^{(2)} = kA_{1,n-1}^{(2)} + aA_{1,n-2,}^{(2)} \\ B_{1,n}^{(2)} = lB_{1,n-1}^{(2)} + bB_{1,n-2,}^{(2)} \\ D_{1,n}^{(2)} = mD_{1,n-1}^{(2)} + cD_{1,n-2,}^{(2)} \\ A_{2,n}^{(2)} = kA_{2,n-1}^{(2)} + aA_{2,n-2,}^{(2)} \end{cases} \tag{2.1}$$

and so on, so that we are in position to set up Table 3.

$\{X_{s,n}^{(2)}\}$	Coefficients of		Name of Sequence	Reference
	$\{X_{s,n-1}^{(2)}\}$	$\{X_{s,n-2}^{(2)}\}$		
$A_{1,n}^{(2)}$	1	1	Fibonacci	[29]
$B_{1,n}^{(2)}$	2	1	Pell	[10]
$D_{1,n}^{(2)}$	1	2	Jacobsthal	[9]
$A_{2,n}^{(2)}$	1	1	Lucas	[10]
$B_{2,n}^{(2)}$	2	1	Pell–Lucas	[11]

Table 3. Well-known 2nd order recurrences

At this stage it is useful to give examples for Equation (1.1) in a similar manner (Table 4).

$\{X_{s,n}^{(3)}\}$	***P***	***Q***	***R***	**Name**	**Reference**
$A_{1,n}^{(3)}$	1	1	1	Tribonacci	[5]
$B_{1,n}^{(3)}$	**0**	**1**	**1**	Padovan	[27]
$D_{1,n}^{(3)}$	**1**	**0**	**1**	Ocke	[21]
$A_{2,n}^{(3)}$	**1**	**1**	**0**	Fibonacci	[28]

Table 4. Known 3rd order recurrences

Thus, the expressions for $\{W_n^{(3)}\}$ and $\{C_n^{(3)}\}$ can be re-written respectively as:

$$W_n^{(3)} = \alpha^{2n}(\beta^n + \gamma^n) + \beta^{2n}(\gamma^n + \alpha^n) + \gamma^{2n}(\alpha^n + \beta^n) \tag{2.2}$$

and

$$\delta C_n^{(3)} = \alpha^{2n}(\gamma^n - \beta^n) + \beta^{2n}(\alpha^n - \gamma^n) + \gamma^{2n}(\beta^n - \alpha^n) \tag{2.3}$$

or in terms of the second order sequences, as

$$W_n^{(3)} = \alpha^{2n}A_{2,n} + \beta^{2n}B_{2,n} + \gamma^{2n}D_{2,n}, \tag{2.4}$$

and

$$C_n^{(3)} = \frac{\alpha^{2n}}{(\alpha-\beta)(\alpha-\gamma)}A_{1,n} + \frac{\beta^{2n}}{(\beta-\alpha)(\beta-\gamma)}B_{1,n} + \frac{\gamma^{2n}}{(\gamma-\alpha)(\gamma-\beta)}D_{1,n}. \tag{2.5}$$

This is one answer to the stated objective, if only formally. Another purely formal approach can be developed with the use of the shift operator $E: EW_n = W_{n+1}$. This was previously used to reduce the order of a recurrence relation in order to extend Leonard Carlitz' generating functions of powers from second order to third order recurrences [28]. Thus, Equation (1.1) can be rewritten in principle as

$$\begin{aligned} 0 &= (E-\alpha)(E-\beta)(E-\gamma)W_n \\ &= (E-\alpha)U_n \end{aligned} \tag{2.6}$$

in which

$$U_{n+2} = (\beta+\gamma)U_{n+1} - \beta\gamma U_n \tag{2.7}$$

can be set up as a second order recurrence [25].

3 Pertinent literature and notation

In order to put the notation used in this paper into a broader perspective, we outline some of the pertinent background literature in relation to arbitrary order, r, generalizations of the Fibonacci and Lucas sequences [26]. We define *r* "basic" sequences of order *r*, $\left\{U_{s,n}^{(r)}\right\}$, *s* = 1,2,…,*r*, by the recurrence relation

$$U_{s,n}^{(r)} = \sum_{j=1}^{r}(-1)^{j+1} P_{r,j} U_{s,n-j}^{(r)}, \ \ n > r, \tag{3.1}$$

and the initial terms when $n = 1, 2, \ldots, r$,

$$U_{s,n}^{(r)} = \delta_{s,n}, \tag{3.2}$$

where $\delta_{s,n}$ is the Kronecker delta, and $P_{r,j}$ are arbitrary integers. The adjective "basic" is used by analogy with the third order sequences of Bell [2]. To correspond with the second order "primordial" sequence of Lucas [19] we define the "primordial" sequence of order *r*, $\left\{U_{0,n}^{(r)}\right\}$. This sequence also satisfies the recurrence relation (3.1), but has initial terms given by

$$U_{0,n}^{(r)} = \begin{cases} 0, & n < 1, \\ \displaystyle\sum_{j=1}^{r} \alpha_{r,j}^{n-1}, & 1 \le n \le r, \end{cases} \tag{3.3}$$

where the $\alpha_{r,j}$ are the roots, assumed distinct, of the auxiliary equation associated with recurrence relation (3.1)

$$f_1(x) \equiv x^r - \sum_{j=1}^{r}(-1)^{j+1} P_{r,j}\, x^{r-j} = 0 \tag{3.4}$$

where

$$f_1(x) = \prod_{i=1}^{r}\left(x - \alpha_{r,i}\right).$$

We shall restrict ourselves to this non-degenerate case, but the essential arguments remain the same when the zeros are not distinct. One of the basic sequences is labelled "fundamental' by analogy with Lucas' second order fundamental sequence $\left\{U_{2,n}^{(2)}\right\}$. The fundamental sequence of order r is then $\left\{U_{r,n+r}^{(r)}\right\}$, which for notational convenience we can also represent as $\left\{u_n^{(r)}\right\}$. Similarly, we shall represent $\left\{U_{0,n+1}^{(r)}\right\}$ by $\left\{v_n^{(r)}\right\}$ to conform with the customary notations for the corresponding second order sequences.

Generally in the literature only one basic second order sequence is mentioned, namely the fundamental one, but Gootherts [8] has demonstrated the need for two *basic* second order sequences, as well as the primordial sequence. The fundamental nature of the sequence $\left\{U_{r,n+r}^{(r)}\right\}$ was studied by d'Ocagne [4] who showed in effect that any element $\left\{w_n^{(r)}\right\}$ of the set Ω of all sequences which satisfy (2.1) can be expressed in terms of the fundamental sequence and the initial terms of $\left\{w_{r,n}^{(r)}\right\}$ with $P_{r,0} = 1$ for notational convenience:

$$w_n^{(r)} = \sum_{j=0}^{r-1}\sum_{k=j}^{r-1}(-1)^{k-j}P_{r,k-j}u_{n-k}^{(r)}\,w_j^{(r)}, \qquad n \geq 0. \tag{3.5}$$

Because these 2nd order sequences are slightly different in notation from that commonly used, we display a few values in Table 5 to make the subsequent work easier to envisage.

n	1	2	3	4	5	6	7	8	9
$U_{0,n}^{(2)}$	1	3	4	7	11	18	29	47	76
$U_{1,n}^{(2)}$	1	0	1	1	2	3	5	8	13
$U_{2,n}^{(2)}$	0	1	1	2	3	5	8	13	21
$U_{1,n}^{(3)}$	1	0	0	1	1	2	4	7	13
$U_{2,n}^{(3)}$	0	1	0	1	2	3	6	11	20
$U_{3,n}^{(3)}$	0	0	1	1	2	4	7	13	24
$B_{1,n}^{(2)}$	0	1	2	5	12	29	70	169	408
$B_{2,n}^{(2)}$	1	1	3	7	17	41	99	239	577

Table 5. Basic sequences and Pell sequences

Some particular examples can be observed which can help to illuminate later work

$$U_{0,n}^{(2)} = U_{1,n+1}^{(2)} + U_{1,n+3}^{(2)} = U_{2,n}^{(2)} + U_{2,n+2}^{(2)},$$

$$U_{2,n}^{(3)} = U_{1,n}^{(3)} + U_{1,n-1}^{(3)} = U_{3,n-1}^{(3)} + U_{3,n-2}^{(3)},$$

$$B_{1,n}^{(2)} = U_{2,4}^{(2)}B_{1,n-1}^{(2)} + U_{1,4}^{(2)}B_{1,n-2}^{(2)},$$

$$B_{2,n}^{(2)} = B_{1,n}^{(2)} + B_{1,n-1}^{(2)},$$

and so on.

To return now to the themes of this paper, we point out that Williams [30] studied properties of generalized Lucas numbers defined in effect by

$$L_{s,n}^{(r)} = d^{-s} \sum_{j=1}^{r} \alpha_{r,j}^{n} \zeta_r^{s(j-1)}, \tag{3.6}$$

for $s = 0, 1, \ldots, r-1$, in which $\zeta_m = exp(2\pi i/m)$, $i^2 = -1$, is a modification of Carlitz, and d is some real number for $r > 2$; for $r = 2$, d is the difference between the roots of the auxiliary equation as usual. Thus, when $r = 2$, we get for instance,

$$L_{o,n}^{(2)} = \alpha_{2,1}^{n} + \alpha_{2,2}^{n} = v_n^{(2)}.$$

For $r = 2, 3$, Lucas and Williams restrict the coefficients in the recurrence relation to relatively prime numbers. On the other hand, Horadam in his studies of generalized second order recursive sequences, permits these coefficients to be arbitrary integers [12], and Williams allows these coefficients to be arbitrary numbers in his study of $\left\{H_{s,n}^{(r)}\right\}$ [31]:

$$H_{s,n}^{(r)} = \frac{1}{d}\sum_{j=1}^{r} c_{s,j}\varphi_{r,j}^{n-1} \tag{3.7}$$

for $s = 0, 1, \ldots, r-1$, in which for any r integers, a_i, $i = 1, 2, \ldots, r$, we define

$$\varphi_{r,i} = \sum_{j=0}^{r} a_j \alpha_{r,i}^{j}.$$

as a representation of $\varphi_{r,i}$ in the scale of $\alpha_{r,i}$ where the latter is the base of each scale [14], and $c_{s,j}$ is the cofactor of $\alpha_{r,j}^{s-1}$ in d, the Vandermonde determinant of the roots:

$$d = \begin{vmatrix} 1 & \alpha_{r,1} & \cdots & \alpha_{r,1}^{r-1} \\ 1 & \alpha_{r,2} & \cdots & \alpha_{r,2}^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{r,r} & \cdots & \alpha_{r,r}^{r-1} \end{vmatrix}, \tag{3.8}$$

Jarden established in effect that

$$dU_{s,n+1}^{(r)} = \sum_{j=1}^{r} d_j\, \alpha_{r,j}^{n}, \tag{3.9}$$

where d_j is the determinant of order r formed from d on replacing the j-th row by the r initial terms, so that

$$dU_{s,n+1}^{(r)} = \sum_{j=1}^{r} c_{s,j}\alpha_{r,j}^{n}.$$

Following Williams [31], we can then define a type of Horadam number sequence by:

$$H_{s,n}^{(r)} = \frac{1}{d}\sum_{j=1}^{r} c_{s,j}\varphi_{r,j}^{n-1}, \quad (s = 1,2,\ldots,r), \tag{3.10}$$

which is open to separate studies of its properties such as in [18] in relation to more conventional versions of the Horadam sequences. Here, for $s = 1,2,\ldots,r$

$$H_{s,n}^{(r)} = \frac{1}{d}\sum_{j=1}^{r} c_{s,j}\varphi_{r,j}^{n-1} = \frac{1}{d}\begin{vmatrix} 1 & \alpha_{r,1} & \cdots & \alpha_{r,1}^{s-2} & \varphi_{r,1}^{n-1} & \alpha_{r,1}^{s} & \cdots & \alpha_{r,1}^{r-1} \\ 1 & \alpha_{r,2} & \cdots & \alpha_{r,2}^{s-2} & \varphi_{r,2}^{n-1} & \alpha_{r,2}^{s} & \cdots & \alpha_{r,2}^{r-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{r,r} & \cdots & \alpha_{r,r}^{s-2} & \varphi_{r,r}^{n-1} & \alpha_{r,r}^{s} & \cdots & \alpha_{r,r}^{r-1} \end{vmatrix},$$

which can be expanded as

$$\varphi_{r,1}^{n-1} = H_{1,n}^{(r)} + \alpha_{r,1}H_{2,n}^{(r)} + \alpha_{r,1}^{2}H_{3,n}^{(r)} + \ldots + \alpha_{r,1}^{r-1}H_{r,n}^{(r)},$$

$$\varphi_{r,2}^{n-1} = H_{1,n}^{(r)} + \alpha_{r,2}H_{2,n}^{(r)} + \alpha_{r,2}^{2}H_{3,n}^{(r)} + \cdots + \alpha_{r,2}^{r-1}H_{r,n}^{(r)},$$

$$\ldots$$

$$\varphi_{r,r}^{n-1} = H_{1,n}^{(r)} + \alpha_{r,r}H_{2,n}^{(r)} + \alpha_{r,r}^{2}H_{3,n}^{(r)} + \cdots + \alpha_{r,r}^{r-1}H_{r,n}^{(r)},$$

or, expressed in a single equation, which yields

$$\varphi_{r,i}^{m-1} = \sum_{j=1}^{r} H_{j,m}^{(r)}\alpha_{r,i}^{j-1},$$

$i = 1, 2, \ldots, r$, so that

$$\frac{1}{d}\sum_{i=1}^{r} c_{s,i}\varphi_{r,i}^{m-1} = \sum_{j=1}^{r}\left(\frac{1}{d}\sum_{i=1}^{r} c_{s,i}\alpha_{r,i}^{j-1}\right)H_{j,m}^{(r)}. \tag{3.11}$$

4 Concluding comments

We can combine (3.11) with (3.9) and (3.10) to obtain results which are both elegant and illuminating, namely that

$$\frac{1}{d}\sum_{i=1}^{r} c_{s,i}\varphi_{r,i}^{m-1} = \sum_{j=1}^{r} U_{s,j}^{(r)}H_{j,m}^{(r)},$$

or with (3.10):

$$H_{s,m}^{(r)} = \sum_{j=1}^{r} U_{s,j}^{(r)}H_{j,m}^{(r)} \tag{4.1}$$

An example of this can be given by

$$H_{0,m}^{(2)} = U_{0,1}^{(2)}H_{1.m}^{(2)} + U_{0,2}^{(2)}H_{2,m}^{(2)}$$

or from Table 5, when allowing for variations in expressions for initial terms, as a well-known identity for relating the Lucas and Fibonacci numbers:

$$U_{0,n}^{(2)} = 2U_{1,n-1}^{(2)} + U_{2.n+1}^{(2)}$$

$$= U_{0,n-1}^{(2)} + U_{0,n-1}^{(2)},$$

or, as a symbolic difference without a numerical distinction,

$$U_{0,n}^{(3)} = 2U_{0,n-1}^{(3)} - U_{0,n-3}^{(3)}.$$

Thus, equation (4.1) in a sense illustrates both the basic nature of the $\left\{U_{r,n}^{(r)}\right\}$ sequences and as a means to switch between sequences of any two orders as in (2.6) and (2.7), which was the original motivation for the explorations in this paper. More fundamentally, the results show that mathematical notation is not only useful as *symbolic abbreviation*, but it can also be a *tool of thought* [15].

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