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Average value of some certain types of arithmetic functions with Piatetski-Shapiro sequences

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Abstract: In this paper, we study asymptotic behaviour of the sum $\sum_{n\le N} f\Big(\lfloor n^c\rfloor\Big)$, where $f(n)=\sum_{d^2|n} g(d)$ under three different types of assumptions on g and $1<c<2$.
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1 Introduction and results

The research on arithmetic functions with Piatetski-Shapiro sequences $\{\lfloor n^c\rfloor\}$, with real c, is an extensive topic in the number theory, see, for example, [1–4, 6, 7, 9, 11–16, 19]. Very recently, Srichan [14] studied asymptotic behaviour of the sum:

$$T^c_{f_k}(N) := \sum_{n\le N} f_k\Big(\lfloor n^c\rfloor\Big), \quad 1<c<2,$$

and established some asymptotic formulas for $T^c_{f_k}(N)$ under assumptions on

$$f_m(n) = \sum_{d^m|n} g(d),$$

where $g(d)$ is a multiplicative function with $g(d) = O(d^\varepsilon)$, $\varepsilon > 0$. He proved that for $1 < c < 3/2$, we have

$$T^c_{f_2}(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(N^{c/2+1/4+\varepsilon}\Big), \tag{1}$$

and for $1 < c < 2$, and $m \ge 3$,

$$T^c_{f_m}(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^m} + O\Big(N^{c/3+1/3+\varepsilon}\Big). \tag{2}$$

If $g(n) = \mu(n)$, (1) and (2) generalize many works involving square-free and m-free integers in Piatetski-Shapiro sequences, see, for example, [6, 7, 12–16, 19].

In this paper, we shall extend this problem with other classes of arithmetic functions. In 2019, Bordellés, Dai, Heyman, Pan and Shparlinski [5] introduced the following three different types of assumptions on g. Namely,

$$\textit{Type I:} \qquad |g(n)| \ll \tau_k(n) \quad (n\in\mathbb{N}), \tag{3}$$

where $\tau_k(n) := \sum_{n_1\dots n_k=n} 1$ is the generalized divisor function,

$$\textit{Type II:} \qquad |g(n)| \ll n^{\phi-1}(\log(en))^{-A} \quad (n\in\mathbb{N}), \tag{4}$$

where $A > 0$ and $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the Golden Ratio, and

$$\textit{Type III:} \qquad \sum_{n\le x}|g(n)|^2 \ll x^\theta \quad (x\ge 1, n\in\mathbb{N}), \tag{5}$$

where $0 < \theta < 2$. In [5], they studied the asymptotic behaviour of the quantity

$$S_g(x) := \sum_{n\le x} g\left(\left\lfloor\frac{x}{n}\right\rfloor\right),$$

where $\lfloor t\rfloor$ is the integral part of the real number t. Later, many authors have studied the functions in types (3)–(5) with integer parts of real-valued function, see, for example, [10, 17, 18, 20]. Note that the function $g(n)$ in [14] belongs to the class of the functions in (3). Thus, it would be interesting to study the same problem in [14] under the three different types of assumptions on g in (3)–(5).

In this paper, we shall study the asymptotic behaviour of multiplicative functions

$$f(n) := \sum_{d^2|n} g(d),$$

where $g(d)$ is a multiplicative function satisfying (3)–(5) on Piatetski-Shapiro sequences.

We study asymptotic behaviour of the sum

$$T_f^c(N) := \sum_{n\le N} f\Big(\lfloor n^c\rfloor\Big),$$

and prove the following theorems.

Theorem 1.1. *For $1 < c < 3/2$, and g is a multiplicative function satisfying (3), we have*

$$T_f^c(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(N^{c/2+1/4}(\log N)^{k-1}\Big).$$

Remark 1.2. *Theorem 1.1 covers Srichan's result (1) in [14].*

Theorem 1.3. *For $1 < c < (2+\phi)/2\phi$, ϕ the Golden Ratio and g is a multiplicative function satisfying (4), we have*

$$T_f^c(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(N^{1/2+c\phi/2-\phi/4}\Big).$$

Theorem 1.4. *For $1 < c < (5+\theta)/(2+2\theta)$, and g a multiplicative function satisfying (5), we have*

$$T_f^c(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(N^{(3-\theta+2c+2c\theta)/8}\Big).$$

This can easily be generalized to

$$f_m(n) = \sum_{d^m|n} g(d),\ \ m\ge 3,$$

instead of the case $m = 2$. We obtain the following results.

Theorem 1.5. *For $1 < c < 2$, $m \ge 3$ and g a multiplicative function satisfying (3), we have*

$$T_{f_m}^c(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^m} + O\Big(N^{(c+1)/3}(\log N)^{k-1}\Big).$$

Theorem 1.6. *Let ϕ be the Golden Ratio and let g be a multiplicative function satisfying (4). For $1 < c < \frac{2+\phi}{2\phi}$ and $2 \le m \le 4$, we have*

$$T_{f_m}^c(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^m} + O\Big(N^{(2c\phi-\phi+m)/2m}\Big).$$

For $1 < c < 2$ and $m \ge 5$, we have

$$T_{f_m}^c(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^m} + O\Big(N^{(c+1)/3}\Big).$$

Theorem 1.7. *Let* $1 < \theta < 2$ *and let* g *be a multiplicative function satisfying (5). For* $1 < c < (5+\theta)/(2+2\theta)$, *and* $2 \le m \le 4$, *we have*

$$T^c_{f_m}(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^m} + O\Big(N^{(2m-1-\theta+2c+2c\theta)/4m}\Big).$$

For $1 < c < 2$ *and* $m \ge 5$, *we have*

$$T^c_{f_m}(N) = N\sum_{d=1}^{\infty}\frac{g(d)}{d^m} + O\Big(N^{(c+1)/3}\Big).$$

The main ingredient in the proof of Theorems is a good estimate for the number of integers n up to x such that $\lfloor n^c \rfloor$ belongs to an arithmetic progression in [7].

Lemma 1.8. [Theorem 2, [7]] *For* $1 < c < 2$, x *being a real number and* q *and* a *being two integers such that* $0 \le a < q \le x^c$, *we have*

$$\sum_{\substack{n\le x\\ \lfloor n^c\rfloor \equiv a \pmod q}} 1 = \frac{x}{q} + O\Big(\min\Big(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}}\Big)\Big).$$

2 Proof of Theorems

Proof of Theorem 1.1. For $1 < c < 3/2$, we have

$$T^c_f(N) = \sum_{n\le N}\sum_{d^2 | \lfloor n^c\rfloor} g(d) = \sum_{d\le N^{c/2}} g(d) \sum_{\substack{n\le N\\ \lfloor n^c\rfloor \equiv 0 \pmod{d^2}}} 1.$$

Using Lemma 1.8, we get

$$T^c_f(N) = N\sum_{d\le N^{c/2}}\frac{g(d)}{d^2} + O\Big(\sum_{d\le N^{c/2}} |g(d)| \min\Big(\frac{N^c}{d^2}, \frac{N^{(c+1)/3}}{d^{2/3}}\Big)\Big). \tag{6}$$

In view of the hypothesis (3) and the well-known formula

$$\sum_{n\le N}\tau_k(n) \ll N(\log N)^{k-1}, \tag{7}$$

see [8, eq. 13.2] and the partial summation, we have

$$\sum_{d>N^{c/2}}\frac{|g(d)|}{d^2} \ll \sum_{d>N^{c/2}}\frac{\tau_k(d)}{d^2} \ll \frac{(\log N)^{k-1}}{N^{c/2}}.$$

Thus,

$$N\sum_{d\le N^{c/2}}\frac{g(d)}{d^2} = N\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(N^{1-c/2}(\log N)^{k-1}\Big). \tag{8}$$

To bound the error term in (6), we write

$$\sum_{d\le N^{c/2}} |g(d)| \min\Big(\frac{N^c}{d^2}, \frac{N^{(c+1)/3}}{d^{2/3}}\Big) = \sum_{d\le N^{c/2-1/4}}\frac{N^{(c+1)/3}|g(d)|}{d^{2/3}} + \sum_{N^{c/2-1/4}<d\le N^{c/2}}\frac{N^c|g(d)|}{d^2}.$$

Using again (7) and the partial summation, we have

$$\sum_{d\le N^{c/2-1/4}} \frac{N^{(c+1)/3}|g(d)|}{d^{2/3}} \ll N^{(c+1)/3} \sum_{d\le N^{c/2-1/4}} \frac{\tau_k(d)}{d^{2/3}} \ll N^{c/2+1/4}(\log N)^{k-1} \tag{9}$$

and

$$\sum_{N^{c/2-1/4}<d\le N^{c/2}} \frac{N^{c}|g(d)|}{d^{2}} \ll N^{c} \sum_{N^{c/2-1/4}<d\le N^{c/2}} \frac{\tau_k(d)}{d^{2}} \ll N^{c/2+1/4}(\log N)^{k-1}. \tag{10}$$

Inserting (8)–(10) in (6), Theorem 1.1 follows. □

Proof of Theorem 1.3. Let ϕ be the Golden Ratio and $1 < c < (2+\phi)/2\phi$. The proof follows closely that of Theorem 1.1 in the case $k = 1$ above. Since $|g(n)| \ll n^{\phi-1}(\log(en))^{-A}$, we have

$$\sum_{d\le x} |g(d)| \ll x^{\phi}, \ \sum_{d>x} \frac{|g(d)|}{d^2} \ll x^{\phi-2}.$$

Thus,

$$N \sum_{d\le N^{c/2}} \frac{g(d)}{d^2} = N \sum_{d=1}^{\infty} \frac{g(d)}{d^2} + O\Big(N^{1-c+c\phi/2}\Big). \tag{11}$$

Using again the partial summation with the hypothesis (4), we have

$$\sum_{d\le N^{c/2-1/4}} \frac{N^{(c+1)/3}|g(d)|}{d^{2/3}} \ll N^{(c+1)/3} \sum_{d\le N^{c/2-1/4}} \frac{d^{\phi-1}}{d^{2/3}} \ll N^{c\phi/2+1/2-\phi/4} \tag{12}$$

and

$$\sum_{N^{c/2-1/4}<d\le N^{c/2}} \frac{N^{c}|g(d)|}{d^{2}} \ll N^{c} \sum_{N^{c/2-1/4}<d\le N^{c/2}} \frac{d^{\phi-1}(\log N)^{-A}}{d^{2}} \ll N^{c\phi/2+1/2-\phi/4}(\log N)^{-A}. \tag{13}$$

Then, Theorem 1.3 follows from (11)–(13). □

Proof of Theorem 1.4. Let $1 < c < (5+\theta)/(2+2\theta)$. The proof follows closely that of Theorem 1.1 in the case $k \ge 2$ above. By the Cauchy inequality and the hypothesis (5), we have

$$\sum_{d\le x} |g(d)| \ll x^{(1+\theta)/2}.$$

By using the partial summation, we have

$$\sum_{d>x} \frac{|g(d)|}{d^2} \ll x^{(\theta-3)/2}.$$

Thus,

$$N \sum_{d\le N^{c/2}} \frac{g(d)}{d^2} = N \sum_{d=1}^{\infty} \frac{g(d)}{d^2} + O\Big(N^{c\theta/4-3c/4}\Big). \tag{14}$$

Using again the partial summation with the hypothesis (5), we have

$$\sum_{d\le N^{c/2-1/4}} \frac{N^{(c+1)/3}|g(d)|}{d^{2/3}} \ll N^{c\theta/4+c/4+3/8-\theta/8} \tag{15}$$

and

$$\sum_{N^{c/2-1/4}<d\le N^{c/2}} \frac{N^{c}|g(d)|}{d^{2}} \ll N^{c\theta/4+c/4+3/8-\theta/8}. \tag{16}$$

Then, Theorem 1.4 follows from (14)–(16). □

References

[1] Akbal, Y. (2017). Friable values of Piatetski-Shapiro sequences. *Proceedings of the American Mathematical Society*, 145, 4255–4268.

[2] Baker, R. C., & Banks, W. D. (2015). Character sums with Piatetski-Shapiro sequences. *The Quarterly Journal of Mathematics*, 66, 393–416.

[3] Baker, R. C., Banks, W. D., Brüdern, J., Shparlinski, I. E., & Weingartner, A. (2013). Piatetski-Shapiro sequences. *Acta Arithmetica*, 157, 37–68.

[4] Banks, W. D., Guo, V. Z., & Shparlinski, I. E. (2016). Almost primes of the form $\lfloor p^c \rfloor$. *Indagationes Mathematicae*, 27, 423–436.

[5] Bordellés, O., Dai, L., Heyman, R., Pan, H., & Shparlinski, I. E. (2019). On a sum involving the Euler function. *Journal of Number Theory*, 202, 278–297.

[6] Cao, X., & Zhai, W. (1998). The distribution of square-free numbers of the form $[n^c]$. *Journal de Théorie des Nombres de Bordeaux*, 10(2), 287–299.

[7] Deshouillers, J. M. (2019). A remark on cube-free numbers in Segal–Piatestki-Shapiro sequences. *Hardy–Ramanujan Journal*, 41, 127–132.

[8] Ivić, A. (1985) *The Riemann Zeta-function: The Theory of the Riemann Zeta-function with Applications*. John Wiley & Sons, Inc., New York.

[9] Liu, K., Shparlinski, I. E., & Zhang, T. (2017). Squares in Piatetski-Shapiro sequences. *Acta Arithmetica*, 181, 239–252.

[10] Ma, J., Wu, J., & Zhao, F. (2021). On a generalisation of Bordellés–Dai–Heyman–Pan–Shparlinski's conjecture. *Journal of Number Theory*, 236, 334–348.

[11] Piatetski-Shapiro, I. I. (1953). On the distribution of prime numbers in sequences of the form $f(n)$. *Math. Sbornik*, 33, 559–566.

[12] Rieger, G. (1978). Remark on a paper of Stux concerning squarefree numbers in non-linear sequences. *Pacific Journal of Mathematics*, 78(1), 241–242.

[13] Srichan, T. (2021). On the Distribution of (k, r)-Integers in Piatetski-Shapiro Sequences. *Czechoslovak Mathematical Journal*, 71(4), 1063–1070.

[14] Srichan, T. Multiplicative functions of special type on Piatetski-Shapiro sequences. *Mathematica Slovaca* (to appear).

[15] Srichan, T., & Tangsupphathawat, P. (2020). Square-full numbers in Piatetski-Shapiro sequences. *Annales mathématiques du Québec*, 44(2), 385–391.

[16] Stux, I. (1975). Distribution of squarefree integers in non-linear sequences. *Pacific Journal of Mathematics*, 59(2), 577–584.

[17] Wu, J. (2020). Note on a paper by Bordellés, Dai, Heyman, Pan and Shparlinski. *Periodica Mathematica Hungarica*, 80, 95–102.

[18] Zhai, W. (2020). On a sum involving the Euler function. *Journal of Number Theory*, 211, 199–219.

[19] Zhang, M., & Jinjiang, L. (2017). Distribution of cube-free numbers with form $[n^c]$. *Frontiers of Mathematics in China*, 12(6), 1515–1525.

[20] Zhao, F., & Wu, J. (2022). Note on a paper by Bordellés, Dai, Heyman, Pan and Shparlinski, II. *Acta Arithmetica*, 202, 185–194.