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On the derivatives of B-Tribonacci polynomials

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Abstract: In this paper, B-Tribonacci polynomials which are extensions of Fibonacci polynomials are defined. Some identities relating B-Tribonacci polynomials and their derivatives are established.

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1 Introduction

The Fibonacci sequence and polynomials play an important role in the enrichment of Fibonacci theory. Various extensions of this sequence and polynomials are found in the literature, for example [8]. In [6], Fibonacci and Lucas polynomials are generalized to introduce $h(x)$-Fibonacci polynomials and $h(x)$-Lucas polynomials and properties of these polynomials are studied. It is to be noted that the Catalan and Bryd Fibonacci polynomials are generalisations of $h(x)$-Fibonacci polynomials. In [4], the authors have discussed derivative sequences of Fibonacci and Lucas polynomials. The k-th derivative of sequences of polynomials of Fibonacci and Lucas polynomials are discussed in [10]. Properties involving the second order partial derivative sequences of Fibonacci and Lucas polynomials are established in [11]. Generalized Fibonacci and Lucas polynomials, with their associated diagonal polynomials are found in [9]. In the same paper, the author discusses associated polynomials like Lucas, Chebyshev, Fermat, Pell, and Jacobsthal polynomials. The general results of Lucas polynomials with special values of the coefficients are found in [2, 5, 7]. In [3], properties of k-Fibonacci polynomials, which are extensions of k-Fibonacci numbers, are studied.

In this paper, we establish some properties relating B-Tribonacci polynomials and their derivatives. B-Tribonacci polynomials are extensions of k-Fibonacci polynomials discussed in [3]. We first define B-Tribonacci polynomials.

Definition 1.1. *Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. We denote by $t_n(x)$, in short, t_n, the n-th B-Tribonacci polynomial and let $t_0 = 0$, $t_1 = 0$, $t_2 = 1$. Then B-Tribonacci polynomials, t_n, are defined by*

$$t_{n+2} = x^2 t_{n+1} + 2x t_n + t_{n-1}, \tag{1}$$

where the coefficients x^2, $2x$, 1 of $t_{n+1},\ t_n,\ t_{n-1}$ respectively, are the terms of the expansion of $(x+1)^2$.

We list below few terms of (1):

$$t_{-1}(x) = 1, \quad t_0(x) = 0, \quad t_1(x) = 0, \quad t_2(x) = 1, \quad t_3(x) = x^2, \quad t_4(x) = x^4 + 2x.$$

Note that B-Tribonacci polynomials can also be obtained from $h(x)$-B-Tribonacci polynomials discussed in [1] by taking $h(x) = x$.

We list below some of the identities of B-Tribonacci polynomials.

(1) The generating function of (1), for $n \geq 0$ is given by

$$G(z) = \frac{z^2}{1 - z(x+z)^2}. \tag{2}$$

That is, $\displaystyle\sum_{n=0}^{\infty} t_n z^n = \frac{z^2}{1 - z(x+z)^2}$.

(2) The n-th term of (1) in combinatorial form is written as

$$t_n = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \binom{2n-4-2r}{r} x^{2n-4-3r}, n \geq 2. \tag{3}$$

Differentiating (3) w.r.t. x, we get

$$\frac{dt_n}{dx} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} (2n-4-3r) \binom{2n-4-2r}{r} x^{2n-5-3r}, n \geq 2. \tag{4}$$

2 Identities relating B-Tribonacci polynomials and their derivatives

In this section, we prove some relations between B-Tribonacci polynomials and their derivatives.

Theorem 2.1. *For $n \geq 0$,*

$$\left[\frac{dt_{n+1}}{dx} + 2x\,\frac{dt_{n-1}}{dx} + 2\frac{dt_{n-2}}{dx}\right] = 2(n-1)x\ t_n + 2(n-2)\ t_{n-1}. \tag{5}$$

Proof. Clearly, the result holds for $n = 0, 1, 2$. Now, take $n = 3m$, then

$$\begin{aligned}
&t_{3m+1} + 2x\, t_{3m-1} + 2t_{3m-2} \\
&= \sum_{r=0}^{2m-1} \binom{6m-2-2r}{r} x^{6m-2-3r} + 2 \sum_{r=0}^{2m-2} \binom{6m-6-2r}{r} x^{6m-5-3r} \\
&\quad + 2 \sum_{r=0}^{2m-3} \binom{6m-8-2r}{r} x^{6m-8-3r} \\
&= x^{6m-2} + (6m-2)x^{6m-5} \\
&\quad + \sum_{r=2}^{2m-1} \left(\binom{6m-2-2r}{r} + 2 \binom{6m-4-2r}{r-1} + 2 \binom{6m-4-2r}{r-2} \right) x^{6m-2-3r} \\
&= x^{6m-2} + (6m-2)x^{6m-5} + \sum_{r=2}^{2m-1} \left(\binom{6m-2-2r}{r} + 2 \binom{6m-3-2r}{r-1} \right) x^{6m-2-3r} \\
&= x^{6m-2} + \sum_{r=1}^{2m-1} \left(\binom{6m-2-2r}{r} + 2 \binom{6m-3-2r}{r-1} \right) x^{6m-2-3r} \\
&= x^{6m-2} + \sum_{r=1}^{2m-1} \frac{6m-2}{r} \binom{6m-3-2r}{r-1} x^{6m-2-3r}.
\end{aligned}$$

Now, differentiating both sides w.r.t. x, we get

$$\begin{aligned}
&\frac{dt_{3m+1}}{dx} + 2x \frac{dt_{3m-1}}{dx} + 2 \frac{dt_{3m-2}}{dx} + 2\, t_{3m-1} \\
&= (6m-2)x^{6m-3} + \sum_{r=1}^{2m-1} \frac{6m-2}{r} (6m-2-3r) \binom{6m-3-2r}{r-1} x^{6m-3-3r} \\
&= (6m-2) \sum_{r=0}^{2m-1} \binom{6m-3-2r}{r} x^{6m-3-3r} \\
&= (6m-2) \left(x \sum_{r=0}^{2m-1} \binom{6m-4-2r}{r} x^{6m-4-3r} + \sum_{r=1}^{2m-1} \binom{6m-4-2r}{r-1} x^{6m-3-3r} \right) \\
&= (6m-2) \left(x \sum_{r=0}^{2m-1} \binom{6m-4-2r}{r} x^{6m-4-3r} + \sum_{r=0}^{2m-2} \binom{6m-6-2r}{r} x^{6m-6-3r} \right) \\
&= (6m-2) \Big(x\, t_{3m} + t_{3m-1} \Big).
\end{aligned}$$

Therefore, we have

$$\frac{dt_{3m+1}}{dx} + 2x \frac{dt_{3m-1}}{dx} + 2 \frac{dt_{3m-2}}{dx} = 2(3m-1)x\, t_{3m} + 2(3m-2)t_{3m-1}.$$

Similarly, the statement can be proved for $n = 3m + 1$ and $n = 3m + 2$.
Hence, the result follows by induction. □

Theorem 2.2. *For* $n \geq 1$,

$$\frac{dt_{n+2}}{dx} = \sum_{r=0}^{\lfloor \frac{n-1}{3} \rfloor} 4^r \Big[(2n-6r)\, xt_{n+1-3r} - (2n-6-6r)\, t_{n-3r} \Big]. \tag{6}$$

Proof. We prove the statement (6) by induction on n. For $n = 1, 2, 3$, (6) can be easily verified. Assume that it is true for $n \le m+2$ and $m \le 3k-1$. We now prove that it holds for $m = 3k$. Equation (1) implies,

$$\begin{aligned}
\frac{dt_{3k+2}}{dx} &= 2xt_{3k+1} + 2t_{3k} + x^2\frac{dt_{3k+1}}{dx} + 2x\frac{dt_{3k}}{dx} + \frac{dt_{3k-1}}{dx} \\
&= 2xt_{3k+1} + 2t_{3k} + \sum_{r=0}^{k-1} 4^r \Big[(6k-2-6r)\, x^3 t_{3k-3r} - (6k-8-6r)\, x^2 t_{3k-1-3r}\Big] \\
&\quad + \sum_{r=0}^{k-1} 4^r \Big[(6k-4-6r)\, 2x^2 t_{3k-1-3r} - (6k-10-6r)\, 2xt_{3k-2-3r}\Big] \\
&\quad + \sum_{r=0}^{k-1} 4^r \Big[(6k-6-6r)\, xt_{3k-2-3r} - (6k-12-6r)\, t_{3k-3-3r}\Big] \\
&= 2x^3 t_{3k} + 6x^2 t_{3k-1} + 6xt_{3k-2} + 2t_{3k-3} \\
&\quad + \sum_{r=0}^{k} 4^r \Big[(6k-6r)xt_{3k+1-3r} - (6k-6-6r)t_{3k-3r}\Big] \\
&\quad + \sum_{r=0}^{k-1} 4^r \Big[-2x^3 t_{3k-3r} - 6x^2 t_{3k-1-3r} + 2xt_{3k-2-3r} + 6t_{3k-3-3r}\Big] \\
&= \sum_{r=0}^{k} 4^r \Big[(6k-6r)xt_{3k+1-3r} - (6k-6-6r)t_{3k-3r}\Big] + 8xt_{3k-2} + 8t_{3k-3} \\
&\quad + \sum_{r=1}^{k-1} 4^r \Big[-2x^3 t_{3k-3r} - 6x^2 t_{3k-1-3r} + 2xt_{3k-2-3r} + 6t_{3k-3-3r}\Big] \\
&= \sum_{r=0}^{k} 4^r \Big[(6k-6r)xt_{3k+1-3r} - (6k-6-6r)t_{3k-3r}\Big] \\
&\quad + 4(2x^3 t_{3k-3} + 6x^2 t_{3k-4} + 6xt_{3k-5} + 2t_{3k-6}) \\
&\quad + \sum_{r=1}^{k-1} 4^r \Big[-2x^3 t_{3k-3r} - 6x^2 t_{3k-1-3r} + 2xt_{3k-2-3r} + 6t_{3k-3-3r}\Big].
\end{aligned}$$

Hence, further simplification leads to the following conclusion.

$$\begin{aligned}
\frac{dt_{3k+2}}{dx} &= \sum_{r=0}^{k} 4^r \Big[(6k-6r)xt_{3k+1-3r} - (6k-6-6r)t_{3k-3r}\Big] \\
&\quad + 4^{(k-1)}\Big[2x^3 t_3 + 6x^2 t_2 + 6xt_1 + 2t_0\Big] \\
&\quad + 4^{(k-1)}\Big[-2x^3 t_3 - 6x^2 t_2 + 2xt_1 + 6t_0\Big] \\
&= \sum_{r=0}^{k} 4^r \Big[(6k-6r)xt_{3k+1-3r} - (6k-6-6r)t_{3k-3r}\Big].
\end{aligned}$$

Therefore, the result holds for $m = 3k$. Similarly, the theorem can be proved for $m = 3k+1$ and $m = 3k+2$.

Hence, the statement is true for $n = m+3$. Thus, the result is proved. $\square$

Theorem 2.3. *For all $n \geq 0$,*

$$\frac{dt_n}{dx} = 2x \sum_{r=0}^{n+1} t_r t_{n+1-r} + 2 \sum_{r=0}^{n} t_r t_{n-r}. \tag{7}$$

Proof. The generating function of (1) is

$$G(z) = \frac{z^2}{1 - z(x+z)^2}.$$

Therefore,

$$\sum_{n=0}^{\infty} t_n z^n = \frac{z^2}{1 - z(x+z)^2}.$$

Differentiating both sides with respect to x, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{dt_n}{dx} z^n &= \frac{2z^3(x+z)}{[1 - z(x+z)^2]^2} \\
&= 2z(x+z) \sum_{n=0}^{\infty} \Big(\sum_{r=0}^{n} t_r\ t_{n-r} \Big) z^{n-2} \\
&= \sum_{n=0}^{\infty} \Big(2x \sum_{r=0}^{n} t_r\ t_{n-r} \Big) z^{n-1} + \sum_{n=0}^{\infty} \Big(2 \sum_{r=0}^{n} t_r\ t_{n-r} \Big) z^{n}.
\end{aligned}$$

Comparing the coefficient of z^n, we have

$$\frac{dt_n}{dx} = 2x \sum_{r=0}^{n+1} t_r t_{n+1-r} + 2 \sum_{r=0}^{n} t_r t_{n-r}.$$

This completes the proof. ☐

Theorem 2.4. *For all $n, r \geq 0$,*

$$\frac{d^r t_{n+2}}{dx^r} = \begin{cases} 0, & r > 2n, \\ (2n)!, & r = 2n, \\ \frac{1}{(n-2r)} \Big[n(x^2) \frac{d^r t_{n+1}}{dx^r} + n(2x) \frac{d^r t_n}{dx^r} + (n+r) \frac{d^r t_{n-1}}{dx^r} & \\ \qquad - nr(r-1) \frac{d^{r-2} t_{n+1}}{dx^{r-2}} - r\ \frac{d^r (x^2 t_{n+1})}{dx^r} \Big], & r < 2n. \end{cases} \tag{8}$$

Proof. We prove (8) using induction on r.
Note that

$$\begin{aligned}
(n-2)\frac{dt_{n+2}}{dx} &= (n-1)\frac{dt_{n+2}}{dx} - \frac{dt_{n+2}}{dx} \\
&= (n-1)\Big[(2x\ t_{n+1} + 2\ t_n) + x^2 \frac{dt_{n+1}}{dx} + 2x \frac{dt_n}{dx} + \frac{dt_{n-1}}{dx}\Big] - \frac{dt_{n+2}}{dx} \\
&= \frac{dt_{n+2}}{dx} + 2x \frac{dt_n}{dx} + 2\frac{dt_{n-1}}{dx} - 2x\ t_{n+1} \\
&\quad + (n-1)x^2 \frac{dt_{n+1}}{dx} + (n-1)2x \frac{dt_n}{dx} + (n-1)\frac{dt_{n-1}}{dx} - \frac{dt_{n+2}}{dx} \\
&= (n-1)x^2 \frac{dt_{n+1}}{dx} + n(2x)\frac{dt_n}{dx} + (n+1)\frac{dt_{n-1}}{dx} - 2x\ t_{n-1} \\
&= nx^2 \frac{dt_{n+1}}{dx} + n(2x)\frac{dt_n}{dx} + (n+1)\frac{dt_{n-1}}{dx} - \frac{d}{dx}\big(x^2 t_{n+1}\big).
\end{aligned}$$

Hence, the statement is true for $r = 1$. We shall assume that the statement holds for $r = k$. Therefore, we have

$$\frac{d^k t_{n+2}}{dx^k} = \frac{1}{(n-2k)}\Big[n(x^2)\frac{d^k t_{n+1}}{dx^k} + n(2x)\frac{d^k t_n}{dx^k} + (n+k)\frac{d^k t_{n-1}}{dx^k} \\ -nk(k-1)\frac{d^{k-2}t_{n+1}}{dx^{k-2}} - k\,\frac{d^k(x^2 t_{n+1})}{dx^k}\Big].$$

Now,

$$(n-2k)\frac{d^{k+1}t_{n+2}}{dx^{k+1}} = \Big[n(x^2)\frac{d^{k+1}t_{n+1}}{dx^{k+1}} + n(2x)\frac{d^{k+1}t_n}{dx^{k+1}} + (n+k)\frac{d^{k+!}t_{n-1}}{dx^{k+!}} \\ -nk(k-1)\frac{d^{k-2}t_{n+1}}{dx^{k-2}} - k\,\frac{d^{k+1}(x^2 t_{n+1})}{dx^{k+1}} + n(2x)\frac{d^k t_{n+1}}{dx^k} + 2n\frac{d^k t_n}{dx^k}\Big].$$

Also, differentiating (1), $(k+1)$ times w.r.t. x and using Leibniz theorem, we get

$$\frac{d^{k+1}t_{n+2}}{dx^{k+1}} - x^2\frac{d^{k+1}t_{n+1}}{dx^{k+1}} - 2x\,\frac{d^{k+1}t_n}{dx^{k+1}} - \frac{d^{k+1}t_{n-1}}{dx^{k+1}} - (k+1)k\,\frac{d^{k-2}t_{n+1}}{dx^{k-2}} \\ = (k+1)\Big[(2x)\,\frac{d^k t_{n+1}}{dx^k} + 2\,\frac{d^k t_n}{dx^k}\Big].$$

Therefore,

$$(k+1)(n-2k)\frac{d^{k+1}t_{n+2}}{dx^{k+1}} \\ = \Big[(k+1)n(x^2)\frac{d^{k+1}t_{n+1}}{dx^{k+1}} + (k+1)n(2x)\frac{d^{k+1}t_n}{dx^{k+1}} + (k+1)(n+k)\frac{d^{k+1}t_{n-1}}{dx^{k+1}} \\ - nk(k-1)(k+1)\frac{d^{k-2}t_{n+1}}{dx^{k-2}} - k(k+1)\frac{d^{k+1}(x^2 t_{n+1})}{dx^{k+1}} \\ + n\Big(\frac{d^{k+1}t_{n+2}}{dx^{k+1}} - x^2\frac{d^{k+1}t_{n+1}}{dx^{k+1}} - 2x\,\frac{d^{k+1}t_n}{dx^{k+1}} - \frac{d^{k+1}t_{n-1}}{dx^{k+1}} - (k+1)k\,\frac{d^{k-2}t_{n+1}}{dx^{k-2}}\Big)\Big].$$

Hence

$$\Big((k+1)(n-2k) - n\Big)\frac{d^{k+1}t_{n+2}}{dx^{k+1}} = \Big[kn\,(x^2)\frac{d^{k+1}t_{n+1}}{dx^{k+1}} + kn\,(2x)\frac{d^{k+1}t_n}{dx^{k+1}} \\ + k(n+k+1)\,\frac{d^{k+1}t_{n-1}}{dx^{k+1}} - n(k+1)k^2\,\frac{d^{k-2}t_{n+1}}{dx^{k-2}} \\ - (k+1)k\frac{d^{k+1}(x^2 t_{n+1})}{dx^{k+1}}\Big]$$

Thus

$$\Big(n - 2(k+1)\Big)\frac{d^{k+1}t_{n+2}}{dx^{k+1}} = \Big[n\,(x^2)\frac{d^{k+1}t_{n+1}}{dx^{k+1}} + n\,(2x)\frac{d^{k+1}t_n}{dx^{k+1}} + (n+k+1)\,\frac{d^{k+1}t_{n-1}}{dx^{k+1}} \\ - n(k+1)k\,\frac{d^{k-2}t_{n+1}}{dx^{k-2}} - (k+1)\frac{d^{k+1}(x^2 t_{n+1})}{dx^{k+1}}\Big].$$

Hence, the result. □

Theorem 2.5. *For $n \geq 0$,*

$$(x^4 + 8x)\frac{dt_n}{dx} + 4\frac{dt_{n-1}}{dx} - \frac{dt_{n-4}}{dx} = 2nxt_{n+1} + 2(n-3-2x^3)t_n \\ + 4x^2(n-2)t_{n-1} + 4x(n-1)t_{n-2} + 4t_{n-3}. \quad (9)$$

Proof. Consider,

$$t_{n+2} + 2xt_n + 2t_{n-1} + 2x\Big(t_n + 2x\ t_{n-2} + 2t_{n-3}\Big) = (x^4 + 8x)t_n + 4t_{n-1} - t_{n-4}.$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}
&(x^4 + 8x)\frac{dt_n}{dx} + (4x^3 + 8)t_n + 4\frac{dt_{n-1}}{dx} - \frac{dt_{n-4}}{dx} \\
&\quad = \Big[\frac{dt_{n+2}}{dx} + 2x\ \frac{dt_n}{dx} + 2\frac{dt_{n-1}}{dx}\Big] + 2x\Big[\frac{dt_n}{dx} + 2x\ \frac{dt_{n-2}}{dx} + 2\frac{dt_{n-3}}{dx}\Big] \\
&\qquad + 2(2t_n + 2xt_{n-2} + 2t_{n-3}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(x^4 + 8x)\frac{dt_n}{dx} + (4x^3 + 8)t_n + 4\frac{dt_{n-1}}{dx} - \frac{dt_{n-4}}{dx} \\
&\quad = 2(n)x\ t_{n+1} + 2(n-1)\ t_{n-1} + 2x\Big(2(n-2)x\ t_{n-1} + 2(n-3)\ t_{n-2}\Big) \\
&\qquad + 2(2t_n + 2xt_{n-2} + 2t_{n-3}).
\end{aligned}$$

Hence, we conclude

$$\begin{aligned}
&(x^4 + 8x)\frac{dt_n}{dx} + 4\frac{dt_{n-1}}{dx} - \frac{dt_{n-4}}{dx} \\
&\quad = 2nxt_{n+1} + 2(n-3-2x^3)t_n + 4x^2(n-2)t_{n-1} + 4x(n-1)t_{n-2} + 4t_{n-3}.
\end{aligned}$$

□

Theorem 2.6. *For all* $n \geq 0$*,*

$$x\ \frac{dt_{n+1}}{dx} + 3\ \frac{dt_n}{dx} = 2(n-1)\ t_{n+1}. \tag{10}$$

Proof. Note that for $n = 0, 1, 2$, the result holds. Now, take $n = 3m$, then LHS of (10), implies

$$\begin{aligned}
x\ \frac{dt_{3m+1}}{dx} + 3\ \frac{dt_{3m}}{dx} &= \sum_{r=0}^{2m-1} (6m-2-3r)\binom{6m-2-2r}{r}\ x^{6m-2-3r} \\
&\quad + 3\ \sum_{r=0}^{2m-2} (6m-4-3r)\binom{6m-4-2r}{r}\ x^{6m-5-3r}. \\
&= (6m-2)\ x^{6m-2} + \sum_{r=1}^{2m-1} (6m-2-3r)\binom{6m-2-2r}{r}\ x^{6m-2-3r} \\
&\quad + 3\ \sum_{r=1}^{2m-1} (6m-1-3r)\binom{6m-2-2r}{r-1}\ x^{6m-2-3r}.
\end{aligned}$$

After further simplifications, we get

$$\begin{aligned}
x\ \frac{dt_{3m+1}}{dx} + 3\ \frac{dt_{3m}}{dx} &= (6m-2)\sum_{r=0}^{2m-1} \binom{6m-2-2r}{r}\ x^{6m-2-3r} \\
&= 2(3m-1)\ t_{3m+1}.
\end{aligned}$$

Similarly, the statement can be proved for $n = 3m+1$ and $n = 3m+2$. Hence, by induction the theorem is proved. □

Theorem 2.7. *For all $n \geq 0$,*

$$\frac{dt_n}{dx} = nt_{n+1} - \sum_{r=0}^{n+3} t_r t_{n+3-r}. \tag{11}$$

Proof. We prove the statement by induction on n. For $n = 0, 1, 2$, clearly, the statement holds. Assume that statement (11) is true for $n \leq k$, so we have

$$\frac{dt_k}{dx} = kt_{k+1} - \sum_{r=0}^{k+3} t_r t_{k+3-r}.$$

Next, for $n = k + 1$, we have

$$\begin{aligned}
\frac{dt_{k+1}}{dx} &= x^2 \frac{dt_k}{dx} + 2x \frac{dt_{k-1}}{dx} + \frac{dt_{k-2}}{dx} + 2x\, t_k + 2\, t_{k-1} \\
&= x^2 \Big(kt_{k+1} - \sum_{r=0}^{k+3} t_r t_{k+3-r} \Big) + 2x \Big((k-1)t_k - \sum_{r=0}^{k+2} t_r t_{k+2-r} \Big) \\
&\quad + (k-2)t_{k-1} - \sum_{r=0}^{k+1} t_r t_{k+1-r} + 2x\, t_k + 2\, t_{k-1}. \\
&= (k+1)t_{k+2} - \sum_{r=0}^{k+4} t_r t_{k+4-r}.
\end{aligned}$$

Hence, by induction the result follows. □

3 Conclusion

In this paper B-Tribonacci polynomials are defined. Generating function and combinatorial representation are obtained. Identities relating B-Tribonacci polynomials with their derivatives are established.

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