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Generalized Pisano numbers

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Abstract: In this paper, we define and investigate the generalized Pisano sequences and we deal with, in detail, two special cases, namely, Pisano and Pisano–Lucas sequences. We present Binet's formulas, generating functions and Simson's formulas for these sequences. Moreover, we give some identities and matrices associated with these sequences. Furthermore, we show that there are close relations between Pisano and Pisano–Lucas numbers and modified Oresme, Oresme–Lucas and Oresme numbers.

Keywords: Pisano numbers, Pisano–Lucas numbers, Tribonacci numbers, Modified Oresme numbers, Oresme–Lucas numbers, Oresme numbers.
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1 Introduction

Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme–Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$
\begin{aligned}
G_{n+2} &= G_{n+1} - \frac{1}{4}G_n, \quad G_0 = 0, G_1 = 1, & (1)\\
H_{n+2} &= H_{n+1} - \frac{1}{4}H_n, \quad H_0 = 2, H_1 = 1, & (2)\\
O_{n+2} &= O_{n+1} - \frac{1}{4}O_n, \quad O_0 = 0, O_1 = \frac{1}{2}. & (3)
\end{aligned}
$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$
\begin{aligned}
G_{-n} &= 4G_{-(n-1)} - 4G_{-(n-2)},\\
H_{-n} &= 4H_{-(n-1)} - 4H_{-(n-2)},\\
O_{-n} &= 4O_{-(n-1)} - 4O_{-(n-2)},
\end{aligned}
$$

for $n = 1, 2, 3, \ldots$, respectively. Therefore, recurrences (1)–(3) hold for all integers n.

Oresme sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to this sequence (see [1–5, 8, 9]).

Now, we describe two sequences as Pisano and Pisano–Lucas numbers related to modified Oresme, Oresme–Lucas, Oresme numbers. These sequences are defined as follows

$$
\begin{aligned}
P_n &= P_{n-1} - \frac{1}{4}P_{n-2} + 1, \quad \text{with } P_0 = 0, P_1 = 1, \ n \geq 2,\\
R_n &= R_{n-1} - \frac{1}{4}R_{n-2} + \frac{1}{4}, \quad \text{with } R_0 = 3, R_1 = 2, \ n \geq 2,
\end{aligned}
$$

respectively. The first few values of Pisano and Pisano–Lucas numbers are

$$0, 1, 2, \frac{11}{4}, \frac{13}{4}, \frac{57}{16}, \frac{15}{4}, \frac{247}{64}, \ldots$$

and

$$3, 2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \frac{33}{32}, \frac{65}{64}, \ldots,$$

respectively. The sequences $\{P_n\}$ and $\{R_n\}$ satisfy the following third order linear recurences:

$$
\begin{aligned}
P_n &= 2P_{n-1} - \frac{5}{4}P_{n-2} + \frac{1}{4}P_{n-3}, \quad P_0 = 0, P_1 = 1, P_2 = 2,\\
R_n &= 2R_{n-1} - \frac{5}{4}R_{n-2} + \frac{1}{4}R_{n-3}, \quad R_0 = 3, R_1 = 2, R_2 = \frac{3}{2}.
\end{aligned}
$$

There are close relations between Pisano and Pisano–Lucas and modified Oresme, Oresme–Lucas, Oresme numbers. For example, they satisfy the following interrelations:

$$
\begin{aligned}
nP_n &= -(n+2)\,G_n + 4n,\\
P_n &= -(n+2)\,H_n + 4,\\
nP_n &= -2\,(n+2)\,O_n + 4n,
\end{aligned}
$$

and

$$
\begin{aligned}
nR_n &= G_n + n,\\
R_n &= H_n + 1,\\
nR_n &= 2O_n + n.
\end{aligned}
$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Pisano, Pisano–Lucas numbers). First, we recall some properties of generalized Tribonacci numbers.

The generalized (r,s,t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{4}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r,s,t are real numbers.

This sequence has been studied by many authors (see [7]). The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1,2,3,...$ when $t \neq 0$. Therefore, recurrence (4) holds for all integer n. As $\{W_n\}$ is a third-order recurrence sequence (difference equation), its characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{5}$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{r}{3} + A + B, \\
\beta &= \frac{r}{3} + \omega A + \omega^2 B, \\
\gamma &= \frac{r}{3} + \omega^2 A + \omega B,
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}, \\
\Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \ \ \omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).
\end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.1. *(Two Distinct Roots Case: $\alpha = \beta \neq \gamma$) Binet's formula of generalized Tribonacci numbers is*

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n \tag{6}$$

where

$$\begin{aligned}
A_1 &= \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2}, \\
A_2 &= \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)}, \\
A_3 &= \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}.
\end{aligned}$$

Next, we present the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.1. *(Lemma 1.1 in [7]) Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized* (r,s,t) *sequence (the generalized Tribonacci sequence)* $\{W_n\}_{n\geq 0}$. *Then,* $f_{W_n}(x)$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{7}$$

2 Generalized Pisano sequence

In this paper, we consider the case $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$. A generalized Pisano sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} - \frac{5}{4}W_{n-2} + \frac{1}{4}W_{n-3} \tag{8}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 5W_{-(n-1)} - 8W_{-(n-2)} + 4W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (8) holds for all integers n.

Characteristic equation of the third-order recurrence sequence W_n is the cubic equation

$$x^3 - 2x^2 + \frac{5}{4}x - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 (x-1) = 0$$

whose roots are

$$\alpha = \tfrac{1}{2}, \ \ \beta = \tfrac{1}{2}, \ \ \gamma = 1.$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 2, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= \tfrac{5}{4}, \\ \alpha\beta\gamma &= \tfrac{1}{4}, \end{aligned}$$

or

$$\alpha + \beta = 1, \ \ \alpha\beta = \tfrac{1}{4}.$$

Equation (6) can be used to obtain Binet formula of generalized Pisano numbers (two distinct roots case: $\alpha = \beta \neq \gamma$). Binet formula of generalized Pisano numbers can be given as

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2} = -4W_2 + 4W_1, \\ A_2 &= \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)} = -4W_2 + 6W_1 - 2W_0, \\ A_3 &= \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2} = 4W_2 - 4W_1 + W_0, \end{aligned}$$

i.e.,

$$W_n = ((-4W_2 + 4W_1) + (-4W_2 + 6W_1 - 2W_0)n) \times \left(\frac{1}{2}\right)^n + (4W_2 - 4W_1 + W_0).$$

The first few generalized Pisano numbers with positive subscript and negative subscript are given in the following Table 1.

n	W_n	W_{-n}
0	W_0	$W_0 = x_0 = W_0$
1	W_1	$W_{-1} = x_1 = 5W_0 - 8W_1 + 4W_2$
2	W_2	$W_{-2} = x_2 = 17W_0 - 36W_1 + 20W_2$
3	$W_3 = \frac{1}{4}(W_0 - 5W_1 + 8W_2)$	$W_{-3} = x_3 = 49W_0 - 116W_1 + 68W_2$
4	$W_4 = \frac{1}{4}(2W_0 - 9W_1 + 11W_2)$	$W_{-4} = x_4 = 129W_0 - 324W_1 + 196W_2$
5	$W_5 = \frac{1}{16}(11W_0 - 47W_1 + 52W_2)$	$W_{-5} = x_5 = 321W_0 - 836W_1 + 516W_2$
6	$W_6 = \frac{1}{16}(13W_0 - 54W_1 + 57W_2)$	$W_{-6} = x_6 = 769W_0 - 2052W_1 + 1284W_2$
7	$W_7 = \frac{1}{64}(57W_0 - 233W_1 + 240W_2)$	$W_{-7} = x_7 = 1793W_0 - 4868W_1 + 3076W_2$
8	$W_8 = \frac{1}{64}(60W_0 - 243W_1 + 247W_2)$	$W_{-8} = x_8 = 4097W_0 - 11268W_1 + 7172W_2$

Table 1. A few generalized Pisano numbers.

Now, we define two special cases of the sequence $\{W_n\}$. Pisano sequence $\{P_n\}_{n\geq 0}$ and Pisano–Lucas sequence $\{R_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$P_n = 2P_{n-1} - \frac{5}{4}P_{n-2} + \frac{1}{4}P_{n-3}, \quad P_0 = 0, P_1 = 1, P_2 = 2, \tag{9}$$

$$R_n = 2R_{n-1} - \frac{5}{4}R_{n-2} + \frac{1}{4}R_{n-3}, \quad R_0 = 3, R_1 = 2, R_2 = \frac{3}{2}. \tag{10}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = 5P_{-(n-1)} - 8P_{-(n-2)} + 4P_{-(n-3)},$$
$$R_{-n} = 5R_{-(n-1)} - 8R_{-(n-2)} + 4R_{-(n-3)},$$

for $n = 1, 2, 3, \ldots$, respectively. Therefore, recurrences (9)–(10) hold for all integer n.

Next, we present the first few values of the Pisano and Pisano–Lucas numbers with positive and negative subscripts (Table 2).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
P_n	0	1	2	$\frac{11}{4}$	$\frac{13}{4}$	$\frac{57}{16}$	$\frac{15}{4}$	$\frac{247}{64}$	$\frac{251}{64}$	$\frac{1013}{256}$	$\frac{509}{128}$	$\frac{4083}{1024}$	$\frac{4089}{1024}$
P_{-n}		0	4	20	68	196	516	1284	3076	7172	16388	36868	81924
R_n	3	2	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{17}{16}$	$\frac{33}{32}$	$\frac{65}{64}$	$\frac{129}{128}$	$\frac{257}{256}$	$\frac{513}{512}$	$\frac{1025}{1024}$	$\frac{2049}{2048}$
R_{-n}		5	9	17	33	65	129	257	513	1025	2049	4097	8193

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

For all integers $n,$ Pisano and Pisano–Lucas numbers can be expressed by Binet's formulas as

$$\begin{aligned} P_n &= -(n+2)\times 2^{-n+1}+4 \\ R_n &= 2^{-n+1}+1 \end{aligned}$$

respectively. Note that Binet's formulas of modified Oresme, Oresme–Lucas and Oresme numbers, respectively, are

$$\begin{aligned} G_n &= n\alpha^{n-1}=\frac{n}{2^{n-1}}, \\ H_n &= 2\alpha^n=\frac{1}{2^{n-1}}, \\ O_n &= n\alpha^n=\frac{n}{2^n}, \end{aligned}$$

and so

$$\begin{aligned} nP_n &= -(n+2)\,G_n+4n, \\ P_n &= -(n+2)\,H_n+4, \\ nP_n &= -2\,(n+2)\,O_n+4n, \end{aligned}$$

$$\begin{aligned} nR_n &= G_n+n, \\ R_n &= H_n+1, \\ nR_n &= 2O_n+n. \end{aligned}$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 2.1. *Suppose that $f_{W_n}(x)=\sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Pisano sequence $\{W_n\}_{n\geq 0}$. Then, $f_{W_n}(x)$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-2W_0)x+(W_2-2W_1+\frac{5}{4}W_0)x^2}{1-2x+\frac{5}{4}x^2-\frac{1}{4}x^3}.$$

Proof. Take $r=2, s=-\frac{5}{4}, t=\frac{1}{4}$ in Lemma 1.1. □

The previous lemma gives the following results as particular examples.

Corollary 2.1. *Generating functions of Pisano and Pisano–Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} P_n x^n &= \frac{x}{1-2x+\frac{5}{4}x^2-\frac{1}{4}x^3}, \\ \sum_{n=0}^{\infty} R_n x^n &= \frac{6-5x}{2-3x+x^2}, \end{aligned}$$

respectively.

3 Simson formulas

There is a well-known Simson Identity (formula) for modified Oresme sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$

which was derived first by R. Simson in 1753 and it is now called Cassini Identity (formula), as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Pisano sequence $\{W_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized Pisano Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -4^{-n}(4W_2 - 4W_1 + W_0)(2W_2 - 3W_1 + W_0)^2.$$

Proof. Take $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$ in Theorem 2.2 in [6]. □

The previous theorem has the following results as particular examples.

Corollary 3.1. *For all integers n, Simson formula of Pisano and Pisano–Lucas numbers are given as*

$$\begin{vmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \end{vmatrix} = -4^{-n+1},$$

$$\begin{vmatrix} R_{n+2} & R_{n+1} & R_n \\ R_{n+1} & R_n & R_{n-1} \\ R_n & R_{n-1} & R_{n-2} \end{vmatrix} = 0,$$

respectively.

4 Some identities

In this section, we derive some identities of Pisano and Pisano–Lucas numbers. Initially, we can introduce a few basic relations between $\{W_n\}$ and $\{P_n\}$.

Lemma 4.1. *The following identities hold for all integers n.*

(a) $4W_n = 4(49W_0 - 116W_1 + 68W_2)P_{n+4} + 4(196W_1 - 81W_0 - 116W_2)P_{n+3}$
$+ (129W_0 - 324W_1 + 196W_2)P_{n+2}.$

(b) $4W_n = 4(17W_0 - 36W_1 + 20W_2)P_{n+3} - 4(29W_0 - 64W_1 + 36W_2)P_{n+2}$
$+ (49W_0 - 116W_1 + 68W_2)P_{n+1}.$

(c) $4W_n = 4(5W_0-8W_1+4W_2)P_{n+2}-4(9W_0-16W_1+8W_2)P_{n+1}+(17W_0-36W_1+20W_2)P_n.$

(d) $4W_n = 4W_0P_{n+1} - 4(2W_0 - W_1)P_n + (5W_0 - 8W_1 + 4W_2)P_{n-1}.$

(e) $4W_n = 4W_1P_n - 4(2W_1 - W_2)P_{n-1} + W_0P_{n-2}.$

(f) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)^2P_n = -16(-5W_1^2 - 4W_2^2 + W_0W_1 + 8W_1W_2)W_{n+4}$
$+ 16(-9W_1^2 - 8W_2^2 + 2W_0W_1 - W_0W_2 + 16W_1W_2)W_{n+3}$
$+ 4(W_0^2 + 25W_1^2 + 20W_2^2 - 10W_0W_1 + 8W_0W_2 - 44W_1W_2)W_{n+2}.$

(g) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)^2P_n = -16(-W_1^2 + W_0W_2)W_{n+3}$
$+4(W_0^2-5W_0W_1+8W_0W_2-4W_1W_2)W_{n+2}-4(-5W_1^2-4W_2^2+W_0W_1+8W_1W_2)W_{n+1}.$

(h) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)^2P_n = 4(W_0^2 + 8W_1^2 - 5W_0W_1 - 4W_1W_2)W_{n+2} -$
$4(-4W_2^2 + W_0W_1 - 5W_0W_2 + 8W_1W_2)W_{n+1} - 4(-W_1^2 + W_0W_2)W_n.$

(i) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)^2P_n$
$= 4(2W_0^2 + 16W_1^2 + 4W_2^2 - 11W_0W_1 + 5W_0W_2 - 16W_1W_2)W_{n+1}$
$- (5W_0^2 + 36W_1^2 - 25W_0W_1 + 4W_0W_2 - 20W_1W_2)W_n$
$+ (W_0^2 + 8W_1^2 - 5W_0W_1 - 4W_1W_2)W_{n-1}.$

(j) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)^2P_n$
$= (11W_0^2 + 92W_1^2 + 32W_2^2 - 63W_0W_1 + 36W_0W_2 - 108W_1W_2)W_n$
$- (9W_0^2 + 72W_1^2 + 20W_2^2 - 50W_0W_1 + 25W_0W_2 - 76W_1W_2)W_{n-1}$
$+ (2W_0^2 + 16W_1^2 + 4W_2^2 - 11W_0W_1 + 5W_0W_2 - 16W_1W_2)W_{n-2}.$

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times P_{n+4} + b \times P_{n+3} + c \times P_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times P_4 + b \times P_3 + c \times P_2, \\ W_1 &= a \times P_5 + b \times P_4 + c \times P_3, \\ W_2 &= a \times P_6 + b \times P_5 + c \times P_4, \end{aligned}$$

we find that

$$a = 49W_0 - 116W_1 + 68W_2,$$
$$b = 196W_1 - 81W_0 - 116W_2, c = \frac{1}{4}(129W_0 - 324W_1 + 196W_2).$$

The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{R_n\}$.

Lemma 4.2. *The following equalities are true:*

(a) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)R_n = 4(5W_0 - 19W_1 + 18W_2)W_{n+4}$
$- 4(7W_0 - 27W_1 + 26W_2)W_{n+3} + (9W_0 - 35W_1 + 34W_2)W_{n+2}$.

(b) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)R_n = 4(3W_0 - 11W_1 + 10W_2)W_{n+3}$
$- 4(4W_0 - 15W_1 + 14W_2)W_{n+2} + (5W_0 - 19W_1 + 18W_2)W_{n+1}$.

(c) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)R_n = 4(2W_0 - 7W_1 + 6W_2)W_{n+2}$
$- 2(5W_0 - 18W_1 + 16W_2)W_{n+1} + (3W_0 - 11W_1 + 10W_2)W_n$.

(d) $(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)R_n = 2(3W_0 - 10W_1 + 8W_2)W_{n+1}$
$- (7W_0 - 24W_1 + 20W_2)W_n + (2W_0 - 7W_1 + 6W_2)W_{n-1}$.

(e) $2(W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)R_n = 2(5W_0 - 16W_1 + 12W_2)W_n$
$- (11W_0 - 36W_1 + 28W_2)W_{n-1} + (3W_0 - 10W_1 + 8W_2)W_{n-2}$.

Now, we give a few basic relations between $\{P_n\}$ and $\{R_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{aligned} 4R_n &= 68P_{n+4} - 100P_{n+3} + 33P_{n+2}, \\ 4R_n &= 36P_{n+3} - 52P_{n+2} + 17P_{n+1}, \\ 4R_n &= 20P_{n+2} - 28P_{n+1} + 9P_n, \\ 4R_n &= 12P_{n+1} - 16P_n + 5P_{n-1}, \\ 4R_n &= 8P_n - 10P_{n-1} + 3P_{n-2}. \end{aligned}$$

5 On the recurrence properties of generalized Pisano sequence

Taking $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$ as in Theorem 2 in [10], we obtain the following proposition.

Proposition 5.1. *For $n \in \mathbb{Z}$, generalized Pisano numbers (the case $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$) have the following identity:*

$$W_{-n} = 2^{2n}(W_{2n} - R_nW_n + \frac{1}{2}(R_n^2 - R_{2n})W_0).$$

From the above Proposition 5.1 and Corollary 6 in [10], we have the following corollary which gives the connection between the special cases of generalized Pisano sequence at the positive index and the negative index: for modified Pisano, Pisano–Lucas and Pisano numbers: take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2$ and take $W_n = R_n$ with $R_0 = 3, R_1 = 2, R_2 = \frac{3}{2}$, respectively. Note that in this case $R_n = H_n$ where the sequence H_n is the Lucas (r, s, t) sequence given in [10].

Corollary 5.1. *For $n \in \mathbb{Z}$, we have the following extension to negative subscripts:*

(a) *Pisano sequence:*

$$P_{-n} = 2^{2n}(P_{2n} - P_nR_n).$$

(b) *Pisano–Lucas sequence:*

$$R_{-n} = 2^{2n-1}\left(R_n^2 - R_{2n}\right).$$

By using the identity $4R_n = 20P_{n+2} - 28P_{n+1} + 9P_n$ (and Proposition 5.1 or Corollary 5.1), we get

$$P_{-n} = 2^{2n-2}(28P_nP_{n+1} - 20P_nP_{n+2} - 9P_n^2 + 4P_{2n}).$$

Note also that since

$$\begin{aligned} nP_n &= -(n+2)\,G_n + 4n, \\ P_n &= -(n+2)\,H_n + 4, \\ nP_n &= -2\,(n+2)\,O_n + 4n, \end{aligned}$$

and

$$\begin{aligned} G_{-n} &= -4^nG_n = -n \times 2^{n+1}, \\ H_{-n} &= 4^nH_n = 2^{n+1}, \\ O_{-n} &= -4^nO_n = -n \times 2^n, \end{aligned}$$

we get

$$\begin{aligned} nP_{-n} &= 2^{2n}\,(n-2)\,G_n + 4n, \\ P_{-n} &= 4^n\,(n-2)\,H_n + 4, \\ nP_{-n} &= 2^{2n+1}\,(n-2)\,O_n + 4n, \end{aligned}$$

and since

$$\begin{aligned} nR_n &= G_n + n, \\ R_n &= H_n + 1, \\ nR_n &= 2O_n + n, \end{aligned}$$

we obtain

$$\begin{aligned} nR_{-n} &= 2^{2n}G_n + n, \\ R_{-n} &= 4^nH_n + 1, \\ nR_{-n} &= 2^{2n+1}O_n + n. \end{aligned}$$

6 Sum formulas

Next, we present some sum formulas of generalized Pisano numbers.

Theorem 6.1. *For $n \geq 0$ and $m, j \in \mathbb{Z}$, generalized Pisano numbers have the following properties:*

(a) $\sum_{k=0}^{n} W_k = -\dfrac{(A_1 + (n+2)A_2)}{2^n} + (n+1)A_3 + 2(A_1 + A_2).$

(b) $\sum_{k=0}^{n} W_{2k} = -\dfrac{(3A_1 + 2(3n+4)A_2)}{9 \times 2^{2n}} + (n+1)A_3 + \frac{4}{9}(3A_1 + 2A_2).$

(c) $\sum_{k=0}^{n} W_{2k+1} = -\dfrac{(3A_1 + (6n+11)\,A_2)}{9 \times 2^{2n+1}} + (n+1)A_3 + \frac{2}{9}(3A_1 + 5A_2).$

(d) $\sum_{k=0}^{n} W_{mk+j} = \dfrac{1}{2^{mn+j}(2^m-1)^2}\Big(2^{mn+j}(n+1)(2^m-1)^2A_3 + (mn+j)A_2$
$+ (2^m-1)(2^{mn+m}-1)A_1 + 2^m(-mn-m-j+(2^mj+m-j)2^{mn})A_2\Big).$

(e) *If x is a real (or complex number) with $x \neq 1$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(1-x^{n+1})A_3}{1-x} + \frac{2^{m-j}}{(x-2^m)^2}((-x+2^m)A_1 + (2^m j - jx + mx)A_2) - \frac{1}{2^{mn+j}} \frac{x^{n+1}}{(x-2^m)^2}((-x+2^m)A_1 + ((mn+m+j)2^m - (mn+j)x)A_2).$$

Proof. To prove (e) use Binet's formula of W_n, i.e.,

$$W_n = \frac{(A_1 + A_2 n)}{2^n} + A_3,$$

where

$$\begin{aligned} A_1 &= -4W_2 + 4W_1, \\ A_2 &= -4W_2 + 6W_1 - 2W_0, \\ A_3 &= 4W_2 - 4W_1 + W_0. \end{aligned}$$

There is no need to prove (d) separately. Identity (d) is just the $x = 1$ limit of identity (e). The singularity at $x = 1$ in (e) is removable since

$$\lim_{x \to 1} \frac{1 - x^{n+1}}{1 - x} = \lim_{x \to 1} \frac{-(n+1)x^n}{-1} = n + 1,$$

by L'Hospital's rule. Identities (a), (b) and (c) are special cases of (e). □

From Theorem 6.1, we have the following corollary which gives sum formulas of Pisano numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2$).

Corollary 6.1. *For $n \geq 0$ and $m, j \in \mathbb{Z}$, Pisano numbers have the following properties:*

(a) $\sum_{k=0}^{n} P_k = 4(n-2) + \frac{n+4}{2^{n-1}}$.

(b) $\sum_{k=0}^{n} P_{2k} = \frac{4}{9}(9n-7) + \frac{3n+7}{9 \times 2^{2n-2}}$.

(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{4}{9}(9n-2) + \frac{6n+17}{9 \times 2^{2n}}$.

(d) $$\sum_{k=0}^{n} P_{mk+j} = \frac{1}{2^{mn+j}(2^m-1)^2}\Big(2^{mn+j+2}(n+1)(2^m-1)^2 - 2(mn+j) - 4(2^m-1)(2^{mn+m}-1) - 2^{m+1}(-mn-m-j+(2^m j+m-j)2^{mn})\Big).$$

(e) *If x is a real (or complex number) with $x \neq 1$, then*

$$\sum_{k=0}^{n} x^k P_{mk+j} = \frac{4(1-x^{n+1})}{1-x} + \frac{2^{m-j}}{(x-2^m)^2}\Big(-4(-x+2^m) - 2(2^m j - jx + mx)\Big) + \frac{1}{2^{mn+j-1}} \frac{x^{n+1}}{(x-2^m)^2}\Big(2(-x+2^m) + ((mn+m+j)2^m - (mn+j)x)\Big).$$

Taking $W_n = R_n$ with $R_0 = 3, R_1 = 2, R_2 = \frac{3}{2}$ in Theorem 6.1, we have the following corollary which presents sum formulas of Pisano–Lucas numbers.

Corollary 6.2. *For $n \geq 0$ and $m, j \in \mathbb{Z}$, Pisano–Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} R_k = n + 5 - \dfrac{1}{2^{n-1}}$.

(b) $\sum_{k=0}^{n} R_{2k} = \frac{1}{3}(3n+11) - \dfrac{1}{3 \times 2^{2n-1}}$.

(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{3}(3n+7) - \dfrac{1}{3 \times 2^{2n}}$.

(d) $\sum_{k=0}^{n} R_{mk+j} = \dfrac{1}{2^{mn+j}(2^m - 1)}(2^{mn+j}(n+1)(2^m-1) + 2(2^{mn+m} - 1))$.

(e) *If x is a real (or complex number) with $x \neq 1$, then*

$$\sum_{k=0}^{n} x^k R_{mk+j} = \frac{(1 - x^{n+1})}{1 - x} + \frac{x^{n+1} - 2^{mn+m}}{2^{mn+j-1}(x - 2^m)}.$$

7 Matrices associated with generalized Pisano numbers

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 2 & -\frac{5}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, $\det A = \frac{1}{4}$. From (8), we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 2 & -\frac{5}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix}. \tag{11}$$

Matrix formulation of W_n can be given by

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 2 & -\frac{5}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{12}$$

If we take $W = P$ in (11), we have

$$\begin{pmatrix} P_{n+2} \\ P_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} 2 & -\frac{5}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n+1} \\ P_n \\ P_{n-1} \end{pmatrix}.$$

We also define

$$B_n = \begin{pmatrix} P_{n+1} & -\frac{5}{4}P_n + \frac{1}{4}P_{n-1} & \frac{1}{4}P_n \\ P_n & -\frac{5}{4}P_{n-1} + \frac{1}{4}P_{n-2} & \frac{1}{4}P_{n-1} \\ P_{n-1} & -\frac{5}{4}P_{n-2} + \frac{1}{4}P_{n-3} & \frac{1}{4}P_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -\frac{5}{4}W_n + \frac{1}{4}W_{n-1} & \frac{1}{4}W_n \\ W_n & -\frac{5}{4}W_{n-1} + \frac{1}{4}W_{n-2} & \frac{1}{4}W_{n-1} \\ W_{n-1} & -\frac{5}{4}W_{n-2} + \frac{1}{4}W_{n-3} & \frac{1}{4}W_{n-2} \end{pmatrix}$$

Theorem 7.1. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$, *i.e., Pisano numbers have the following representation:.*

$$A^n = \begin{pmatrix} 2 & -\frac{5}{4} & \frac{1}{4} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{n+1} & -\frac{5}{4}P_n + \frac{1}{4}P_{n-1} & \frac{1}{4}P_n \\ P_n & -\frac{5}{4}P_{n-1} + \frac{1}{4}P_{n-2} & \frac{1}{4}P_{n-1} \\ P_{n-1} & -\frac{5}{4}P_{n-2} + \frac{1}{4}P_{n-3} & \frac{1}{4}P_{n-2} \end{pmatrix}. \tag{13}$$

(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$ in Theorem 5.1 in [7]. □

A property of matrix A^n can be given as

$$A^n = 2A^{n-1} - \frac{5}{4}A^{n-2} + \frac{1}{4}A^{n-3}$$

for all integers n.

Theorem 7.2. *For all integers m, n, we have*

$$W_{n+m} = W_n P_{m+1} + \left(-\frac{5}{4}W_{n-1} + \frac{1}{4}W_{n-2}\right) P_m + \frac{1}{4}W_{n-1}P_{m-1}. \tag{14}$$

Proof. Take $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$ in Theorem 5.2. in [7]. □

By Lemma 4.1, we know that

$$\begin{aligned} & (W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)^2 P_m \\ = \;& 4(W_0^2 + 8W_1^2 - 5W_0W_1 - 4W_1W_2)W_{m+2} \\ & -4(-4W_2^2 + W_0W_1 - 5W_0W_2 + 8W_1W_2)W_{m+1} - 4(-W_1^2 + W_0W_2)W_m, \end{aligned}$$

so (14) can be written in the following form

$$\begin{aligned} & (W_0 - 4W_1 + 4W_2)(W_0 - 3W_1 + 2W_2)^2 W_{n+m} \\ = \;& W_n(4(W_0^2 + 8W_1^2 - 5W_0W_1 - 4W_1W_2)W_{m+3} - 4(-4W_2^2 + W_0W_1 \\ & -5W_0W_2 + 8W_1W_2)W_{m+2} - 4(-W_1^2 + W_0W_2)W_{m+1}) \\ & + \left(-\frac{5}{4}W_{n-1} + \frac{1}{4}W_{n-2}\right)(4(W_0^2 + 8W_1^2 - 5W_0W_1 - 4W_1W_2)W_{m+2} \\ & -4(-4W_2^2 + W_0W_1 - 5W_0W_2 + 8W_1W_2)W_{m+1} - 4(-W_1^2 + W_0W_2)W_m) \\ & +\frac{1}{4}W_{n-1}(4(W_0^2 + 8W_1^2 - 5W_0W_1 - 4W_1W_2)W_{m+1} \\ & -4(-4W_2^2 + W_0W_1 - 5W_0W_2 + 8W_1W_2)W_m - 4(-W_1^2 + W_0W_2)W_{m-1}). \end{aligned}$$

Corollary 7.1. *For all integers m, n, we have*

$$\begin{aligned} P_{n+m} &= P_n P_{m+1} + \left(-\frac{5}{4}P_{n-1} + \frac{1}{4}P_{n-2}\right) P_m + \frac{1}{4}P_{n-1}P_{m-1}, \\ R_{n+m} &= R_n P_{m+1} + \left(-\frac{5}{4}R_{n-1} + \frac{1}{4}R_{n-2}\right) P_m + \frac{1}{4}R_{n-1}P_{m-1}. \end{aligned}$$

Taking $m = n$ in the last corollary we obtain the following identities:

$$P_{2n} = P_nP_{n+1} + \left(-\frac{5}{4}P_{n-1} + \frac{1}{4}P_{n-2}\right)P_n + \frac{1}{4}P_{n-1}^2,$$
$$R_{2n} = R_nP_{n+1} + \left(-\frac{5}{4}R_{n-1} + \frac{1}{4}R_{n-2}\right)P_n + \frac{1}{4}R_{n-1}P_{n-1}.$$

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