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Note on the natural density of r-free numbers

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Abstract: Let P be a finite set of prime numbers. By using an elementary method, the proportion of all r-free numbers which are divisible by at least one element in P is studied.
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1 Introduction and result

Let $r > 1$ be a fixed integer. A positive integer n is r-free if each of its prime factors appears to the power of at most $r - 1$. As usual, 2-free and 3-free numbers are called square-free and cube-free, respectively. The density of a finite set A of distinct positive integers, $\delta[A]$, is the ratio of the number of its elements to its largest element. The natural density of an infinite increasing sequence of positive integers, a_n, is

$$\delta[A] = \lim_{n\to\infty} \frac{n}{a_n}.$$

The natural density of the set of square-free integers was studied first by Gegenbauer. He proved that the natural density of the set of square-free integers is $6/\pi^2$, [2]. Later, in [6] Scott gave a conjecture that the natural density of the set of odd square-free integers is $4/\pi^2$ and it was proven by Jameson in [3]. Very recently, the authors [8] generalized this problem to the case of r-free integers by using an elementary method, and showed that the asymptotical ratio of odd to even r-free numbers is asymptotically $2^r : 2^r-2$. In the same year, Brown [1] reproved Jameson's result and generalized it. Brown proved that the proportion of all numbers which are square-free and divisible by all of the primes in T and by none of the primes in P is

$$\frac{6}{\pi^2}\prod_{p\in T}\frac{1}{1+p}\prod_{p\in P}\frac{p}{1+p}, \tag{1}$$

where P and T are disjoint sets of prime numbers with T finite. In this paper, we use the elementary method in [7] to prove a similar result as in [1] and generalize it to r-free integers.

Let A be a given set, for $x > 1$, we denote $A(x)$ to be the number of elements in $A \cap [1, x]$. $f(x) \sim g(x)$ means $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = 1$ and we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$. Let $P = \{p_1, p_2, \ldots, p_k\}$ be a finite set of prime numbers. For $r \geq 2$ integer, let Q be the set of all r-free numbers. Let C_P be the set of r-free numbers not divisible by any $p \in P$, and C'_P be the set of r-free numbers divisible by some $p \in P$. Here (n, P) denotes the greatest common divisor of $p_1p_2\cdots p_k$ and n. For $0 \leq \alpha_i \leq r-1$, $1 \leq i \leq k$, we denote

$$\mathcal{A}_{\alpha_1,\ldots,\alpha_k} := \left\{n \in C'_P \;\middle|\; n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}m,\ m \in C_P\right\}$$

and $\mathcal{A}_{0,\cdots,0} = C_P$. We prove the following Theorem.

Theorem 1.1. *Let $P = \{p_1, p_2, \ldots, p_k\}$ be a finite set of prime numbers. As $x \to \infty$, we have*

$$\frac{C'_P(x)}{C_P(x)} \to \prod_{p\in P}\frac{p^r-1}{p^r-p^{r-1}} - 1.$$

Remark 1.2. *From*

$$\frac{C'_P(x)}{C_P(x)} = \frac{Q(x)-C_P(x)}{C_P(x)} = \frac{Q(x)}{C_P(x)} - 1,$$

we have, as $x \to \infty$,

$$\frac{C_P(x)}{Q(x)} \to \prod_{p\in P}\frac{p^r-p^{r-1}}{p^r-1}. \tag{2}$$

In the case $T = \varnothing$, we can see that the equation (2) covers Brown's result in (1).

2 Lemma

We will use the following Lemma, which deals with the case where P has a single element.

Lemma 2.1. *Let p be a given prime number and $P = \{p\}$. As $x \to \infty$, we have*

$$\frac{C_P(x)}{C'_P(x)} \to \frac{p^r-p^{r-1}}{p^{r-1}-1}.$$

Proof. The proof is very similar of the proof of Theorem 1 in [8].
We assume that

$$C_P(x) \sim ax \quad \text{and} \quad C'_P(x) \sim bx, \quad \text{for some } a, b \in \mathbb{R}^+. \tag{3}$$

First, we wish to show that

$$\frac{a}{b} = \frac{p^r - p^{r-1}}{p^{r-1} - 1}. \tag{4}$$

For $1 \leq i \leq r-1$, we denote

$$A_i = \{n \in C'_P \mid n = p^i m,\ m \in C_P\}.$$

We note that, for $1 \leq i \leq r-1$, A_i are disjoint sets. Thus,

$$C'_P = \bigcup_{i=1}^{r-1} A_i. \tag{5}$$

The definition of A_i implies that, for $1 \leq i \leq r-1$,

$$A_i(x) = C_P\Big(\frac{x}{p^i}\Big). \tag{6}$$

In view of (5) and (6), we have

$$C'_P(x) = \sum_{i=1}^{r-1} C_P\Big(\frac{x}{p^i}\Big). \tag{7}$$

From (3) and (7), we get, as $x \to \infty$,

$$bx \to \sum_{i=1}^{r-1} a\frac{x}{p^i} = ax\frac{p^{r-1}-1}{p^r - p^{r-1}}.$$

This proves (4). Now it remains to prove the existence of a and b. In view of (7), we write

$$Q(x) = C_P(x) + C'_P(x) = \sum_{i=0}^{r-1} C_P\Big(\frac{x}{p^i}\Big). \tag{8}$$

We replace x in (8) by x/p and subtract this with (8). We have

$$Q(x) - Q\Big(\frac{x}{p}\Big) = C_P(x) - C_P\Big(\frac{x}{p^r}\Big). \tag{9}$$

Replace x in (9) by x/p^r, we have

$$Q\Big(\frac{x}{p^r}\Big) - Q\Big(\frac{x}{p^{r+1}}\Big) = C_P\Big(\frac{x}{p^r}\Big) - C_P\Big(\frac{x}{p^{2r}}\Big). \tag{10}$$

In view of (9) and (10), we have

$$Q(x) - Q\Big(\frac{x}{p}\Big) + Q\Big(\frac{x}{p^r}\Big) - Q\Big(\frac{x}{p^{r+1}}\Big) = C_P(x) - C_P\Big(\frac{x}{p^{2r}}\Big).$$

Repeating this, we have

$$C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) = \sum_{i=0}^{k} Q\Big(\frac{x}{p^{ri}}\Big) - \sum_{i=0}^{k} Q\Big(\frac{x}{p^{ri+1}}\Big), \text{ for all } k \in \mathbb{N}. \tag{11}$$

It is well known that $Q(x) \sim cx$, where $c = 1/\zeta(r)$. Then, for $\epsilon > 0$, we take x_0 such that

$$(c-\epsilon)x \leq Q(x) \leq (c+\epsilon)x, \text{ for } x \geq x_0. \tag{12}$$

To apply inequality (12) with (11), we take k such that $\frac{x}{p^{rk+r+1}} < x_0 \leq \frac{x}{p^{rk+1}}$ and from (11) and (12), we have

$$\begin{aligned} C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) &\leq \sum_{i=0}^{k}(c+\epsilon)\frac{x}{p^{ri}} - \sum_{i=0}^{k}(c-\epsilon)\frac{x}{p^{ri+1}} \\ &= x\Big(\frac{p(c+\epsilon)-(c-\epsilon)}{p}\Big)\sum_{i=0}^{k}\frac{1}{p^{ri}} \\ &\leq x\Big(\frac{p(c+\epsilon)-(c-\epsilon)}{p}\Big)\frac{p^r}{p^r-1}. \end{aligned}$$

From choosing k so that $\frac{x}{p^{rk+r+1}} < x_0 \leq \frac{x}{p^{rk+1}}$, we have $C_P\Big(\frac{x}{p^{r(k+1)}}\Big) \leq Q(px_0) < px_0$. Then, we have

$$\begin{aligned} C_P(x) &\leq x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1} + px_0 \\ &\leq x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1} + \frac{p^{r+1}}{p^r-1}x_0. \end{aligned}$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$\begin{aligned} C_P(x) &\leq x\Big(p(c+\epsilon)-(c-\epsilon)\Big)\frac{p^{r-1}}{p^r-1} - (p^2\epsilon x)\frac{p^{r-1}}{p^r-1} \\ &= x\Big(p(c+\epsilon)-p^2\epsilon - c + \epsilon\Big)\frac{p^{r-1}}{p^r-1}. \end{aligned}$$

By a similar proof we deal with the lower bound. In view of (11) and (12), for the integer k such that $\frac{x}{p^{rk+r+1}} < x_0 \leq \frac{x}{p^{rk+1}}$, we have

$$\begin{aligned} C_P(x) - C_P\Big(\frac{x}{p^{r(k+1)}}\Big) &\geq \sum_{i=0}^{k}(c-\epsilon)\frac{x}{p^{ri}} - \sum_{i=0}^{k}(c+\epsilon)\frac{x}{p^{ri+1}} \\ &= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\sum_{i=0}^{k}\frac{1}{p^{ri}} \\ &= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\Big(\frac{p^r - p^{r-rk}}{p^r-1}\Big) \\ &= x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\Big(\frac{p^r}{p^r-1}\Big) - x\Big(\frac{p(c-\epsilon)-(c+\epsilon)}{p}\Big)\Big(\frac{p^{r-rk}}{p^r-1}\Big) \\ &\geq \frac{x\Big(p(c-\epsilon)-(c+\epsilon)\Big)(p^{r-1})}{p^r-1} - cx\Big(\frac{p^{r-rk}}{p^r-1}\Big). \end{aligned}$$

We note that $p^{r+1}x_0 > \frac{x}{p^{rk}}$. Then, we have

$$\begin{aligned} C_P(x) &\geq C_P(x) - C_P\Big(\frac{x}{2^{r(k+1)}}\Big) \\ &\geq \frac{x\big(p(c-\epsilon)-(c+\epsilon)\big)(p^{r-1})}{p^r-1} - \frac{cx_0p^{2r+1}}{p^r-1}. \end{aligned}$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$C_P(x) \geq \frac{x\big(p(c-\epsilon)-(c+\epsilon)\big)\big(p^{r-1}\big)}{p^r-1} - \frac{cx\epsilon p^{2r+1}}{p^r-1} = \frac{x\big(p(c-\epsilon)-(c+\epsilon)-c\epsilon p^{r+2}\big)\big(p^{r-1}\big)}{p^r-1}.$$

This proves the existence of a. The existence of b follows from the existence of a, since $Q(x) = C_P(x) + C'_P(x)$. □

3 Proof of Theorem 1.1

The proof is similar to the proof of Lemma 2.1 but much more complex. First, we assume that

$$C_P(x) \sim \delta x \quad \text{and} \quad C'_P(x) \sim \beta x, \quad \text{for some } \delta, \beta \in \mathbb{R}^+. \tag{13}$$

Now, we note that each element $n \in \mathcal{A}_{\alpha_1,\ldots,\alpha_k}$ is the form $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} m,\ m \in C_P$, $0 \leq \alpha_i \leq r-1$ and for some $\alpha_i \neq 0,\ 1 \leq i \leq k$. Thus $\frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$ is an element in C_P. This implies that, for $0 \leq \alpha_i \leq r-1,\ 1 \leq i \leq k$,

$$\mathcal{A}_{\alpha_1,\ldots,\alpha_k}(x) = C_P\Big(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}\Big). \tag{14}$$

From (14), we sum $\mathcal{A}_{\alpha_1,\ldots,\alpha_k}(x)$ for all $0 \leq \alpha_i \leq r-1$ but not all zero, $1 \leq i \leq k$, and get

$$\begin{aligned} C'_P(x) &= \sum_{\substack{0 \leq \alpha_i \leq r-1 \\ \alpha_i \text{ are not all zero}}} C_P\Big(\frac{x}{\prod_{i=1}^k p_i^{\alpha_i}}\Big) \\ &= \sum_{1 \leq i \leq k} \sum_{1 \leq \alpha_i \leq r-1} C_P\left(\frac{x}{p_i^{\alpha_i}}\right) + \sum_{1 \leq i_1 < i_2 \leq k} \sum_{1 \leq \alpha_{i_1}, \alpha_{i_2} \leq r-1} C_P\left(\frac{x}{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}}}\right) \\ &\quad + \cdots + \sum_{1 \leq \alpha_1,\ldots,\alpha_k \leq r-1} C_P\left(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}\right). \end{aligned} \tag{15}$$

In view of (13), we have

$$\begin{aligned} \beta x &= \delta x \sum_{1 \leq i \leq k} \sum_{1 \leq \alpha_i \leq r-1} \left(\frac{1}{p_i^{\alpha_i}}\right) + \delta x \sum_{1 \leq i_1 < i_2 \leq k} \sum_{1 \leq \alpha_{i_1}, \alpha_{i_2} \leq r-1} \left(\frac{1}{p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}}}\right) \\ &\quad + \cdots + \delta x \sum_{1 \leq \alpha_1,\ldots,\alpha_k \leq r-1} \left(\frac{1}{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}}\right) \\ &= \delta x \Big(\prod_{p \in P} \frac{p^r - 1}{p^r - p^{r-1}} - 1\Big). \end{aligned} \tag{16}$$

Now, it remains to show the existence of δ and β. To do this, we use mathematical induction on k, where k is the number of primes in P. Let P_j be a set of prime numbers with j elements.

We assume that, for $1 \leq j < k$, there exists δ_j such that $C_{P_j}(x) \sim \delta_j x$. From Lemma 2.1, δ_1 exists. Let q be a prime number with $q \notin P_{k-1}$. Note that

$$C_{P_{k-1}}(x) = C_{P_{k-1} \cup \{q\}}(x) + C^*_{P_{k-1} \cup \{q\}}(x), \tag{17}$$

where $C^*_{P_{k-1} \cup \{q\}} = \{n \in Q \mid (n, P_{k-1}) = 1 \text{ and } q \mid n\}$. With the same reason from (14), we have

$$C^*_{P_{k-1}\cup\{q\}}(x) = \sum_{1\leq\alpha\leq r-1} C_{P_{k-1}}\left(\frac{x}{q^\alpha}\right). \tag{18}$$

In view of (17) and (18), we have

$$C_{P_{k-1}}(x) = C_{P_{k-1}\cup\{q\}}(x) + \sum_{1\leq\alpha\leq r-1} C_{P_{k-1}}\left(\frac{x}{q^\alpha}\right). \tag{19}$$

Since $C_{P_{k-1}}(x) \sim \delta_{k-1}x$, we have

$$C_{P_{k-1}\cup\{q\}}(x) \sim \Big(1 - \sum_{1\leq\alpha\leq r-1}\left(\frac{1}{q^\alpha}\right)\Big)\delta_{k-1}x.$$

This shows that δ_k exists. By mathematical induction the proof is completed. □

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