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Bi-unitary multiperfect numbers, IV(c)

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Dedicated to the memory of Prof. Varanasi Sitaramaiah

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Abstract: A divisor d of a positive integer n is called a unitary divisor if $\gcd(d, n/d) = 1$; and d is called a bi-unitary divisor of n if the greatest common unitary divisor of d and n/d is unity. The concept of a bi-unitary divisor is due to D. Surynarayana (1972). Let $\sigma^{**}(n)$ denote the sum of the bi-unitary divisors of n. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis (1987) proved that there are no odd bi-unitary multiperfect numbers. The present paper is part IV(c) in a series of papers on even bi-unitary multiperfect numbers. In parts I, II and III we determined all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 6$ and u is odd. In part V we fixed the case $a = 8$. The case $a = 7$ is more difficult. In Parts IV(a-b) we solved partly this case, and in the present paper (Part IV(c)) we continue the study of the same case ($a = 7$).

Keywords: Perfect numbers, Triperfect numbers, Multiperfect numbers, Bi-unitary analogues.
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1 Introduction

Throughout this paper, all lower case letters denote positive integers; p and q denote primes. The letters u, v and w are reserved for odd numbers.

A divisor d of n is called a unitary divisor if $\gcd(d, n/d) = 1$. If d is a unitary divisor of n, we write $d \| n$. A divisor d of n is called a *bi-unitary* divisor if $(d, n/d)^{**} = 1$, where the symbol

$(a,b)^{**}$ denotes the greatest common unitary divisor of a and b. The concept of a bi-unitary divisor is due to D. Suryanarayana (cf. [11]). Let $\sigma^{**}(n)$ denote the sum of bi-unitary divisors of n. The function $\sigma^{**}(n)$ is multiplicative, that is, $\sigma^{**}(1) = 1$ and $\sigma^{**}(mn) = \sigma^{**}(m)\sigma^{**}(n)$ whenever $(m,n) = 1$. If p^α is a prime power and α is odd, then every divisor of p^α is a bi-unitary divisor; if α is even, each divisor of p^α is a bi-unitary divisor except for $p^{\alpha/2}$. Hence

$$\sigma^{**}(p^\alpha) = \begin{cases} \sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1}, & \text{if } \alpha \text{ is odd,} \\ \sigma(p^\alpha) - p^{\alpha/2}, & \text{if } \alpha \text{ is even.} \end{cases} \tag{1.3}$$

If α is even, say $\alpha = 2k$, then $\sigma^{**}(p^\alpha)$ can be simplified to

$$\sigma^{**}(p^\alpha) = \left(\frac{p^k - 1}{p-1}\right).(p^{k+1} + 1). \tag{1.4}$$

From (1.3), it is not difficult to observe that $\sigma^{**}(n)$ is odd only when $n = 1$ or $n = 2^\alpha$.

The concept of a bi-unitary perfect number was introduced by C. R. Wall [12]; a positive integer n is called a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. C. R. Wall [12] proved that there are only three bi-unitary perfect numbers, namely $6, 60$ and 90. A positive integer n is called a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$ for some $k \geq 3$. For $k = 3$ we obtain the bi-unitary triperfect numbers.

Peter Hagis [1] proved that there are no odd bi-unitary multiperfect numbers. Our present paper is part IV(c) in a series of papers on even bi-unitary multiperfect numbers. In Parts I, II and III (see [2–4]) we found all bi-unitary triperfect numbers of the form $n = 2^a u$, where $1 \leq a \leq 6$ and u is odd. In part V we fixed the case $a = 8$. The case $a = 7$ seems to be more difficult. In parts IV(a-b) we solved partly the case $a = 7$. In this paper we continue this study and obtain some further results in this case.

For general accounts on various perfect-type numbers, we refer to [8, 9].

Note. Investigation of the case $c = 2$ below bases on notes by Professor Varanasi Sitaramaiah [10]. He sent them to me before he passed away in Oct 2020.

2 Preliminaries

We assume that the reader has Parts I, II, III, IV(a-b), V (see [2–7]) available. We, however, recall Lemmas 2.1–2.3 from these parts because they are so important also here.

Lemma 2.1. (I) *If α is odd, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} > \frac{\sigma^{**}(p^{\alpha+1})}{p^{\alpha+1}}$$

for any prime p.

(II) *For any* $\alpha \geq 2\ell - 1$ *and any prime* p,

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} \geq \left(\frac{1}{p-1}\right)\left(p - \frac{1}{p^{2\ell}}\right) - \frac{1}{p^\ell} = \frac{1}{p^{2\ell}}\left(\frac{p^{2\ell+1}-1}{p-1} - p^\ell\right).$$

(III) *If* p *is any prime and* α *is a positive integer, then*

$$\frac{\sigma^{**}(p^\alpha)}{p^\alpha} < \frac{p}{p-1}.$$

Remark 2.1. (I) and (III) of Lemma 2.1 are mentioned in C. R. Wall [12]; (II) of Lemma 2.1 has been used by him [12] without explicitly stating it.

Lemma 2.2. *Let* $a > 1$ *be an integer not divisible by an odd prime* p, *and let* α *be a positive integer. Let* r *denote the least positive integer such that* $a^r \equiv 1 \pmod{p^\alpha}$; *then* r *is usually denoted by* $ord_{p^\alpha} a$. *We have the following properties.*

(i) *If* r *is even, then* $s = r/2$ *is the least positive integer such that* $a^s \equiv -1 \pmod{p^\alpha}$. *Also,* $a^t \equiv -1 \pmod{p^\alpha}$ *for a positive integer* t *if and only if* $t = su$, *where* u *is odd.*

(ii) *If* r *is odd, then* $p^\alpha \nmid a^t + 1$ *for any positive integer* t.

Remark 2.2. Let a, p, r and $s = r/2$ be as in Lemma 2.2 ($\alpha = 1$). Then $p|a^t - 1$ if and only if $r|t$. If t is odd and r is even, then $r \nmid t$. Hence $p \nmid a^t - 1$. Also, $p|a^t + 1$ if and only if $t = su$, where u is odd. In particular, if t is even and s is odd, then $p \nmid a^t + 1$. In order to check the divisibility of $a^t - 1$ (when t is odd) by an odd prime p, we can confine to those p for which $ord_p\, a$ is odd. Similarly, for examining the divisibility of $a^t + 1$ by p when t is even we need to consider primes p with $s = (ord_p\, a)/2$ even.

Lemma 2.3. *Let* k *be odd and* $k \geq 3$. *Let* $p \neq 5$.

(a) *If* $p \in [3, 2520] \setminus \{11, 19, 31, 71, 181, 829, 1741\}$, $ord_p 5$ *is odd and* $p|5^k - 1$, *then we can find a prime* p' *(depending on* p*) such that* $p' | \frac{5^k - 1}{4}$ *and* $p' \geq 2521$.

(b) *If* $q \in [3, 2520] \setminus \{13, 313, 601\}$, $s = \frac{1}{2} ord_q 5$ *is even and* $q|5^{k+1} + 1$, *then we can find a prime* q' *(depending on* q*) such that* $q' | \frac{5^{k+1}+1}{2}$ *and* $q' \geq 2521$.

Lemma 2.4. *Let* m *be odd and* $m \geq 3$. *Let* $p \neq 29$.

(a) *If* $p \in [3, 519] \setminus \{7, 13, 67\}$, $ord_p 29$ *is odd and* $p|29^m - 1$, *then there exists an odd prime* p' *such that* $p' | \frac{29^m - 1}{28}$ *and* $p' > 519$.

(b) *If* $q \in [3, 519] \setminus \{37, 61, 313, 421\}$, $s = \frac{1}{2} \operatorname{ord}_q 29$ *is even and* $q|29^{m+1} + 1$, *then there exists an odd prime* q' *such that* $q' | \frac{29^{m+1}+1}{2}$ *and* $q' > 519$.

Proof. (a) Let $p|29^m - 1$. If $r = ord_p 29$, that is, r is the least positive integer such that $29^r \equiv 1 \pmod{p}$, then $r|m$. Since m is odd, r must be odd. Also, $29^r - 1|29^m - 1$. Let

$$S_{29} = \{(p, r) : p \in [3, 519],\ p \neq 29 \text{ and } r = ord_p 29 \text{ odd}\}.$$

From Appendix A, we have

$$S_{29} = \{(7,1),(13,3),(23,11),(59,29),(67,3),(71,35),(83,41),(103,51),(107,53),\\ (139,69),(149,37),(151,25),(167,83),(173,43),(178,89),(181,15),(197,49),\\ (199,99),(223,111),(227,113),(239,119),(283,47),(347,173),(373,93),(383,191),\\ (397,99),(419,209),(431,215),(439,219),(463,231),(487,81),(499,249)\}.$$

Let $p|29^m - 1$ and $p \in [3,519] \setminus \{7,13,67\}$. Then $(p,r) \in S_{29} \setminus \{(7,1),(13,3),(67,3)\}$, where $r = ord_p 29$. Also, $29^r - 1|29^m - 1$. To prove (a), it is enough to show that $\frac{29^r-1}{28}$ is divisible by a prime $p' > 519$. From Appendix B, we know the factors of $29^r - 1$. By examining the factors of $29^r - 1$ for $r \notin \{1,3,3\} = \{1,3\}$, which correspond to the primes $7, 13$ and 67 respectively, we infer that we can a find a prime $p'|\frac{29^r-1}{28}|\frac{29^m-1}{28}$ satisfying $p' > 519$. This proves (a).

For example, if $p = 23$, then $r = 11$. From Appendix B,

$$29^{11} - 1 = \{\{2,2\},\{7,1\},\{23,1\},\{18944890940537,1\}\}.$$

Thus if $23|29^m - 1$, then $p' = 18944890940537|\frac{29^m-1}{4}$ and trivially $p' > 519$.

(b) Let $q|29^{m+1} + 1$ and $q \in [3,519] \setminus \{37,61,313,421\}$. Let $r = ord_q 29$. If r is odd, then $q \nmid 29^{m+1} + 1$ (see Remark 2.2 $(a = 29)$). We may assume that r is even. Let $s = r/2$. Then s is the least positive integer such that $q|29^s + 1$. Again from Remark 2.2 $(a = 29)$, $q \nmid 29^{m+1} + 1$ if s is odd. Since $q|29^{m+1} + 1$, we have that s is even. Also, $m + 1 = su$, where u is odd. This implies that $29^s + 1|29^{m+1} + 1$.

Let

$$T_{29} = \{(q,s) : q \neq 29, q \in [3,519] \text{ and } s = \frac{1}{2} ord_q 29 \text{ even}\}.$$

From Appendix A, we have

$$T_{29} = \{(17,8),(37,6),(41,20),(61,6),(73,36),(89,44),(97,48),(101,50),\\ (113,56),(137,68),(157,26),(193,32),(229,114),(241,60),(257,64),\\ (269,134),(293,146),(313,6),(317,158),(337,168),(353,44),\\ (389,194),(409,204),(421,2),(433,216),(449,224),(461,230)\}.$$

Let $q|29^{m+1} + 1$ and $q \in [3,519] \setminus \{37,61,313,421\}$. Then

$$(q,s) \in T_{29} \setminus \{(37,6),(61,6),(313,6),(421,2)\},$$

where $s = \frac{1}{2} ord_q 29$. To prove (b), it is enough to show that $\frac{29^s+1}{2}$ is divisible by a prime $q' > 519$ for all $s \in T' = \{s : (q,s) \in T_{29} \setminus \{(37,6),(61,6),(313,6),(421,2)\}$. This follows by examining the factors of $29^t + 1$ given in Appendix C.

For example, if $q = 41$, then $s = 20$. Also,

$$29^{20} + 1 = \{\{2,1\},\{41,1\},\{353641,1\},\{6103563899172302171321,1\}\}.$$

We can take $q' = 353641$.

The proof of Lemma 2.4 is complete. □

3 Partial results on bi-unitary triperfect numbers of the form $n = 2^7u$

In part IV(a) we solved partly the case $n = 2^7u$. We proved that if n is a bi-unitary triperfect number of the form $n = 2^7.5^b.17^c.v$, where $(v, 2.5.17) = 1$, then $b \geq 2$ and $c \geq 1$. We then solved completely the case $b = 2$. We proved that in this case $c = 1$ and further showed that $n = 2^7.3^2.5^2.7.13.17 = 44553600$ is the only bi-unitary triperfect number of this form. In part IV(b), we presented some partial results concerning the case $b \geq 3$ under the assumption $3 \nmid n$ and $7|n$. The object of the present paper (part IV(c)) is to provide some further results under the assumption $3 \nmid n$ (which implies that $b \geq 3$).

Let n be a bi-unitary triperfect number divisible unitarily by 2^7 so that $\sigma^{**}(n) = 3n$ and $n = 2^7u$, where u is odd. In addition, assume that $3 \nmid n$. Since $\sigma^{**}(2^7) = 2^8 - 1 = 255 = 3.5.17$, we get the following equations:

$$n = 2^7.5^b.17^c.v$$

and

$$2^7.5^{b-1}.17^{c-1}.v = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(v), \text{ where } (v, 2.3.5.17) = 1.$$

Here $b \geq 3$ and $c \geq 1$. In fact, the case $b = 2$ is not possible since it implies that $3 \mid n$.

Theorem 3.1. *Assume that n is a bi-unitary triperfect number such that $2^7\|n$ and $3 \nmid n$. Let p $(\neq 2,3)$ be a prime divisor of n. Denote by α the largest number such that p^α divides n, that is, $p^\alpha\|n$. If $3 \nmid (p-1)$, then $\alpha = 2k$, where k is odd and ≥ 1.*

Proof. If α is odd, say $\alpha = 2m - 1$, then

$$\sigma^{**}(p^\alpha) = \frac{p^{\alpha+1} - 1}{p-1} = \frac{(p^2)^m - 1}{p-1}.$$

Since $p^2 \equiv 1 \pmod 3$ and (by assumption) $3 \nmid (p-1)$, we have $3|\sigma^{**}(p^\alpha)$. Further, since $\sigma^{**}(p^\alpha) \mid n$, we have $3|n$. This is not possible, since by our assumption $3 \nmid n$.

If $4 \mid \alpha$, say $b = 4m$, then

$$\sigma^{**}(p^\alpha) = \frac{p^{2m} - 1}{p-1}(p^{2m+1} + 1).$$

This leads to a contradiction, too. Therefore, $\alpha = 2k$, where k is odd and ≥ 1. □

Theorem 3.2. *Assume that n is a bi-unitary triperfect number such that $2^7 \| n$ and $3 \nmid n$. Denote $n = 2^7.5^b.17^c.v$, where $(v, 2.3.5.17) = 1$. Then $b = 2k$ and $c = 2\ell$, where k is odd (≥ 3) and ℓ is odd (≥ 1).*

Proof. We may apply Theorem 3.1 for $p = 5$ and $p = 17$, since $3 \nmid (5-1)$ and $3 \nmid (17-1)$. This shows that $b = 2k$ and $c = 2\ell$, where k and ℓ are odd (≥ 1). The case $k = 1$ (that is, $b = 2$) is not possible as noted above. □

It appears that the case $\ell = 1$ (that is, $c = 2$) is not possible if we make an additonal assumption $7 \nmid n$. The proof seems to be lengthy and is carried out below.

Let n be a bi-unitary triperfect number divisible unitarily by 2^7 so that $\sigma^{**}(n) = 3n$ and $n = 2^7 u$, where u is odd. In addition, assume that $3 \nmid n$ and $7 \nmid n$. Then we get the following equations:

$$n = 2^7.5^b.17^c.v \tag{3.1a}$$

and

$$2^7.5^{b-1}.17^{c-1}.v = \sigma^{**}(5^b).\sigma^{**}(17^c).\sigma^{**}(v), \tag{3.1b}$$

where

$$b \geq 3,\ c \geq 1,\ (v, 2.3.5.7.17) = 1 \text{ and } v \text{ has not more than five odd prime factors.} \tag{3.1c}$$

The number of prime factors of v is restricted on the basis of parity of the appropriate values of the function σ^{**}.

Theorem 3.3. *Assume that n is a bi-unitary triperfect number such that $2^7 \| n$, $3 \nmid n$ and $7 \nmid n$. Denote $n = 2^7.5^b.17^c.v$, where $(v, 2.3.5.7.17) = 1$. Then $c \neq 2$.*

Corollary 3.1. *Assume that n is a bi-unitary triperfect number such that $2^7 \| n$, $3 \nmid n$ and $7 \nmid n$. Denote $n = 2^7.5^b.17^c.v$, where $(v, 2.3.5.7.17) = 1$. Then $b = 2k$ and $c = 2\ell$, where k and ℓ are odd and ≥ 3 (and thus $b, c \geq 6$).*

Corollary 3.2. *Let $n = 2^7.5^b.17^c.v$, where $(v, 2.3.5.7.17) = 1$. Then n is not a bi-unitary triperfect number if $4 \mid b$ or $4 \mid c$ or b is odd or c is odd or $b = 2$ or $c = 2$.*

Corollary 3.1 follows from Theorems 3.2 and 3.3, and Corollary 3.2 follows from Corollary 3.1.

For the proof of Theorem 3.3, we consider the case $c = 2$ (that is, $\ell = 1$). We show that this case is impossible. The rest of this paper is devoted to this case. Let $c = 2$. We have $\sigma^{**}(17^2) = 290 = 2.5.29$. From (3.1b), we see that $29 | v$. Let $v = 29^d.w$. We obtain the following equations from (3.1a)–(3.1c):

$$n = 2^7.5^b.17^2.29^d.w, \tag{3.2a}$$

and

$$2^6.5^{b-2}.17.29^{d-1}.w = \sigma^{**}(5^b).\sigma^{**}(29^d).\sigma^{**}(w), \tag{3.2b}$$

where

$$(w, 2.3.5.7.17.29) = 1 \text{ and } w \text{ cannot have more than four odd prime factors.} \qquad (3.2c)$$

Remark 3.1. It follows from Theorem 3.1 (with $p = 29$) that $d = 2m$, where m is odd, and so $d \geq 2$.

Lemma 3.1. *Let $c = 2$ in (3.1a)–(3.1c) so that the equations (3.2a)–(3.2c) can be used. Then $29^3|n$, that is, $d \geq 3$.*

Proof. We need to prove that $d \geq 3$. On the contrary let $d = 2$.

We have $\sigma^{**}(29^2) = 842 = 2.421$. Taking $d = 2$ in (3.2b), we see that $421|w$. Let $w = 421^e.w'$. From (3.2a) and (3.2b) we get

$$n = 2^7.5^b.17^2.29^2.(421)^e.w', \qquad (3.3a)$$

and

$$2^5.5^{b-2}.17.29.(421)^{e-1}.w' = \sigma^{**}(5^b).\sigma^{**}((421)^e).\sigma^{**}(w'), \qquad (3.3b)$$

where

$$(w', 2.3.5.7.17.29.421) = 1 \text{ and } w' \text{ cannot have more than three odd prime factors.} \qquad (3.3c)$$

We prove that $11 \nmid w'$. On the contrary, let $w' = 11^f.w''$. From (3.3a) and (3.3b), we obtain

$$n = 2^7.5^b.17^2.29^2.(421)^e.11^f.w'', \qquad (3.4a)$$

and

$$2^5.5^{b-2}.17.29.(421)^{e-1}.11^f.w'' = \sigma^{**}(5^b).\sigma^{**}((421)^e).\sigma^{**}(11^f)\sigma^{**}(w''), \qquad (3.4b)$$

where

$$(w'', 2.3.5.7.11.17.29.421) = 1 \text{ and } w'' \text{ cannot have more than two odd prime factors.} \qquad (3.4c)$$

Since $3|11^t - 1$ if and only if t is even, it follows that $3|\sigma^{**}(11^f)$ if and only if f is odd or $4|f$. From (3.4b), $3 \nmid \sigma^{**}(11^f)$. Hence we may assume that $f = 2u$, where u is odd.

We prove that $u \geq 3$. Assume that $u = 1$ so that $f = 2$. We have $\sigma^{**}(11^2) = 122 = 2.61$. From (3.4b), $61|w''$. Let $w'' = 61^g.w'''$. From (3.4c), w''' is unity or a power of an odd prime. If $w''' = p^\alpha$, then from (3.4b), we can assume that $p \geq 13$. Hence form (3.4a), we have $n = 2^7.5^b.17^2.29^2.(421)^e.11^2.61^g.p^\alpha$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{842}{841}.\frac{421}{420}.\frac{122}{121}.\frac{61}{60}.\frac{13}{12} = 2.784866823 < 3,$$

a contradiction.

We may now assume that $f = 2u$, where u is odd and $u \geq 3$. We have

$$\sigma^{**}(11^f) = \left(\frac{11^u - 1}{10}\right).(11^{u+1} + 1).$$

We prove that:

(A) $\frac{11^u-1}{10}$ is divisible by an odd prime $p'|w''$ and $p' \geq 23$; and

(B) $\frac{11^{u+1}+1}{2}$ is divisible by an odd prime $q'|w''$ and $q' \geq 31$.

Proof of (A). We observe the following:

(i) $2\|11^u - 1$.

(ii) $7|11^u - 1 \iff 3|u \iff 19|11^u - 1$. Since $7 \nmid n$ by our assumption, $7 \nmid 11^u - 1$. Hence $19 \nmid 11^u - 1$.

(iii) $421|11^u - 1 \iff 105|u$. In particular, $421|11^u - 1$ implies that $3|u$. By (ii) above it follows that $7|11^u - 1$. This is not possible. Hence $421 \nmid 11^u - 1$.

(iv) Since u is odd, $11^u - 1$ is not divisible by 3, 13, 17 and 29; trivially not divisible by 11.

(v) From (i)-(iv) above, it follows that $\frac{11^u-1}{10}$ is odd, > 1 and not divisible by 3, 7, 11, 13, 17, 19, 29 and 421.

(vi) If $5|\frac{11^u-1}{10}$, then $5^2|11^u - 1$, which is equivalent to $5|u$. In such a case, $11^5 - 1|11^u - 1$. Also, $11^5 - 1 = 2.5^2.3221$. Hence $3221|\frac{11^u-1}{10}|\sigma^{**}(11^f)$. From (3.4b), $3221|w''$. So, we may take $p' = 3221$, and thus (A) holds in this case.

(vii) Assume that $5 \nmid \frac{11^u-1}{10}$. Then from (v), $\frac{11^u-1}{10}$ is odd, > 1 and not divisible by 3, 5, 7, 11, 13, 17, 19, 29 and 421. From (3.4b), if $p'|\frac{11^u-1}{10}$, then $p'|w''$ and $p' \geq 23$.

From (vi) and (vii), it is clear that $\frac{11^u-1}{10}$ is divisible by an odd prime $p'|w''$ and $p' \geq 23$. The proof of (A) is complete.

Proof of (B). We have the following:

(viii) $\frac{11^{u+1}+1}{2}$ is odd and > 1.

(ix) $11^t + 1$ is not divisible by $3, 5, 7, 11$ and 19 for any even positive integer t. In particular, $11^{u+1} + 1$ is so.

(x) $13|11^{u+1} + 1 \iff u + 1 = 6u'$. Hence $13|11^{u+1} + 1$ implies that

$$13.61.1117 = \frac{11^6 + 1}{2}|\frac{11^{u+1} + 1}{2}|\sigma^{**}(11^f).$$

From (3.4b), it follows that w'' is divisible by $13, 61$ and 1117. This contradicts (3.4c). Hence $13 \nmid 11^{u+1} + 1$.

(xi) $17|11^{u+1}+1 \iff u+1 = 8u'$. Hence $17|11^{u+1}+1$ implies that

$$17.6304673|\frac{11^8+1}{2}|\frac{11^{u+1}+1}{2}|\sigma^{**}(11^f).$$

From (3.4b), it follows that $6304673|w''$. Already by (A), $p'|\frac{11^u-1}{10}$, $p'|w''$ and $p' \geq 23$. Also, $p' \neq 6304673$ since $\frac{11^u-1}{10}$ and $11^{u+1}+1$ are relatively prime. By (3.4c), it follows that $w'' = (p')^g.(6304673)^h$. From (3.4a),

$$n = 2^7.5^b.17^2.29^2.(421)^e.11^f.(p')^g.(6304673)^h,$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{842}{841}.\frac{421}{420}.\frac{11}{10}.\frac{23}{22}.\frac{6304673}{6304672} = 2.88394643 < 3,$$

a contradiction. Thus $17 \nmid 11^{u+1}+1$.

(xii) $29|11^{u+1}+1$ if and only if $u+1 = 14u'$. Also, $11^{14}+1 = 2.29.61.1933.55527473$. It follows from (3.4b) that if $29|11^{u+1}+1$, then w'' will be divisible by three odd primes, namely $61, 1933$ and 55527473. This violates (3.4c). Hence $29 \nmid 11^{u+1}+1$.

(xiii) $421 \nmid 11^t+1$ for any positive integer t. In particular $421 \nmid 11^{u+1}+1$.

From (viii)–(xiii), we can conclude that $\frac{11^{u+1}+1}{2}$ is odd, > 1 and not divisible by 3, 5, 7, 11, 13, 17, 19, 29 and 421. Since $\frac{11^{u+1}+1}{2}|\sigma^{**}(11^f)$ it follows from (3.4b) that if $q'|\frac{11^{u+1}+1}{2}$, then $q'|w''$ and from (3.4b), $q' \geq 23$. Since $p' \neq q'$, we can assume that $p' \geq 23$ and $q' \geq 31$ as $q' \neq 29$. The proof of (B) is complete.

From (3.4c), it follows that $w'' = (p')^g.(q')^h$. From (3.4a), we have

$$n = 2^7.5^b.17^2.29^2.(421)^e.11^f.(p')^g.(q')^h,$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{842}{841}.\frac{421}{420}.\frac{11}{10}.\frac{23}{22}.\frac{31}{30} = 2.98008 < 3,$$

a contradiction.

Thus $11 \nmid w'$ in (3.3a) and (3.3b).

By (3.3c), we can assume that (in the most unfavourable situation) w' is divisible by three distinct odd primes say p_1, p_2 and p_3, where $p_1 \geq 13$, $p_2 \geq 19$ and $p_3 \geq 23$. Hence from (3.3c), $w' = (p_1)^f.(p_2)^g.(p_3)^h$ and so from (3.3a), $n = 2^7.5^b.17^2.29^2.(421)^e.(p_1)^f.(p_2)^g.(p_3)^h$. Hence we obtain

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{842}{841}.\frac{421}{420}.\frac{13}{12}.\frac{19}{18}.\frac{23}{22} = 2.99804148 < 3,$$

a contradiction. This proves that $d \geq 3$. The proof of Lemma 3.1 is complete. □

Remark 3.2. By Lemma 3.1 and Remark 3.1, we can assume that $d = 2m$, where m is odd and $m \geq 3$ (in the case $c = 2$).

Lemma 3.2. *Let $c = 2$ in (3.1a)–(3.1c) so that the equations (3.2a)–(3.2c) can be used. Then*
(a) n is not divisible by 11 and 13 simultaneously,
(b) n is not divisible by 11 and 19 simultaneously,
(c) if $11 \nmid n$, then n is not divisible by 13 and 19 simultaneously.

Proof. (a) We assume that n given in (3.2a)–(3.2c) is divisible by 11 and 13. Hence $w = 11^e.13^f.w'$. From (3.2a) and (3.2b), we obtain the following:

$$n = 2^7.5^b.17^2.29^d.11^e.13^f.w' \qquad (3.5a)$$

and

$$2^6.5^{b-2}.17.29^{d-1}.11^e.13^f.w' = \sigma^{**}(5^b).\sigma^{**}(29^d).\sigma^{**}(11^e).\sigma^{**}(13^f).\sigma^{**}(w'), \qquad (3.5b)$$

where

$$(w', 2.3.5.7.17.29.11.13) = 1 \text{ and } w' \text{ has not more than two odd prime factors.} \qquad (3.5c)$$

By Lemma 3.1, we have $d \geq 3$. By Remark 3.2, we may assume that $d = 2m$, where m is odd and $m \geq 3$. We have

$$\sigma^{**}(29^d) = \left(\frac{29^m - 1}{28}\right).(29^{m+1} + 1).$$

We now prove the following by making use of (b) of Lemma 2.4:
(C) $\frac{29^{m+1}+1}{2}$ is divisible by an odd prime $q' > 519$ and $q'|w'$.

Proof of (C). Let

$$T'_{29} = \{q|29^{m+1} + 1 : \ q \in [3, 519] \setminus \{37, 61, 313, 421\} \text{ and } s = \frac{1}{2}ord_q 29 \text{ is even.}\}$$

If T'_{29} is non-empty, by (b) of Lemma 2.4, we can find an odd prime $q'|\frac{29^{m+1}+1}{2}$ and $q' > 519$. From (3.5b), clearly $q'|w'$.

Suppose that T'_{29} is empty. Since $q \nmid 29^{m+1} + 1$, if $s = \frac{1}{2}ord_q 29$ is odd, it follows that $q \nmid 29^{m+1} + 1$ for any $q \in [3, 519]$ except possibly for $q \in \{37, 61, 313, 421\}$.

We note that $37|29^{m+1}+1 \iff 61|29^{m+1}+1 \iff 313|29^{m+1}+1 \iff m+1 = 6u$. Assume that $37|29^{m+1} + 1$. Then $29^6 + 1|29^{m+1} + 1$. Also, $29^6 + 1 = 2.37.61.313.421$. Hence from (3.5b), w' is divisible by $37, 61, 313$ and 421. This violates (3.5c). Thus $37 \nmid 29^{m+1} + 1$ and consequently $29^{m+1} + 1$ is not divisible by 61 and 313.

If $421 \nmid 29^{m+1}+1$, then $29^{m+1}+1$ is not divisible by any prime in $[3,519]$. This is true with respect to $\frac{29^{m+1}+1}{2}$ also. If $q'|\frac{29^{m+1}+1}{2}$, then $q' > 519$.

We may assume that $421|\frac{29^{m+1}+1}{2}$. We claim that $\frac{29^{m+1}+1}{2}$ is divisible by an odd prime $q' \neq 421$. If this is not so, then we must have $\frac{29^{m+1}+1}{2} = (421)^{\alpha}$, for some positive integer α. If $\alpha \geq 2$, then $421^2|29^{m+1}+1$. But this is equivalent to $m+1 = 842.u$. But $6737|\frac{29^{842}+1}{2}|\frac{29^{m+1}+1}{2} = (421)^{\alpha}$. This is impossible. Hence $\alpha = 1$ and so $\frac{29^{m+1}+1}{2} = 421$ or $m+1 = 2$ or $m = 1$. But $m \geq 3$. This contradiction proves that $\frac{29^{m+1}+1}{2}$ is divisible by an odd prime $q' \neq 421$. Hence $q' \notin [3,519]$ so that $q' > 519$.

This proves (C).

By Lemma 2.1, we have $\frac{\sigma^{**}(5^b)}{5^b} \geq \frac{19406}{15625}$ $(b \geq 5)$; $\frac{\sigma^{**}(29^d)}{29^d} \geq \frac{731700}{707281}$ $(d \geq 3)$; $\frac{\sigma^{**}(11^e)}{11^e} \geq \frac{15984}{14641}$ $(e \geq 5)$; and $\frac{\sigma^{**}(13^f)}{13^f} \geq \frac{30772}{28561}$ $(f \geq 3)$.

Hence if $e \geq 3$ and $f \geq 3$, using the above results and from (3.5a), we obtain

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{19406}{15625}.\frac{290}{289}.\frac{731700}{707281}.\frac{15984}{14641}.\frac{30772}{28561} = 3.021234777 > 3,$$

a contradiction.

Hence $e \leq 2$ or $f \leq 2$. Since e is even, $e = 2$. Also, if f is odd or $4|f$, then $7|\sigma^{**}(13^f)$. This is not possible from (3.5b). Hence $f = 2u'$, where u' is odd. Hence $f \leq 2$ implies $f = 2$. Thus $e = 2$ or $f = 2$.

Let $e = 2$. Since $\sigma^{**}(11^2) = 122 = 2.61$, by taking $e = 2$ in (3.5b), we see that $61|w'$. Already from (C), $q'|w'$. Hence form (3.5c), $w' = 61^g.(q')^h$. Hence from (3.5a) $(e = 2)$, we have $n = 2^7.5^b.17^2.29^d.11^2.13^f.61^g.(q')^h$, so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{122}{121}.\frac{13}{12}.\frac{61}{60}.\frac{521}{520} = 2.879584826 < 3,$$

a contradiction. Thus $e = 2$ is not admissible.

We now show that $f = 2$ is also not admissible. Let $f = 2$ in (3.5a) and (3.5b). We have $\sigma^{**}(13^2) = 170 = 2.5.17$. Taking $f = 2$ in (3.5a) and (3.5b), we obtain

$$n = 2^7.5^b.17^2.29^d.11^e.13^2.w' \qquad (3.6a)$$

and

$$2^5.5^{b-3}.29^{d-1}.11^e.13^2.w' = \sigma^{**}(5^b).\sigma^{**}(29^d).\sigma^{**}(11^e).\sigma^{**}(w'), \qquad (3.6b)$$

where

$$(w', 2.3.5.7.17.29.11.13) = 1 \text{ and } w' \text{ has not more than two odd prime factors.} \qquad (3.6c)$$

We now prove that (by making use of (a) of Lemma 2.4):
(D) $\frac{29^m-1}{28}$ is divisible by an odd prime $p' > 519$ and $p'|w'$.

Proof of (D). Let

$$S'_{29} = \{p|29^m - 1 : \ p \in [3,519] \smallsetminus \{7,13,67\} \text{ and } ord_p 29 \text{ is odd}\}.$$

If S'_{29} is non-empty, by (a) of Lemma 2.4, there exists $p'|\frac{29^m-1}{28}$ and $p' > 519$. From (3.5b), it is clear that $p'|w'$. Thus (D) holds in this case.

Let S'_{29} be empty. Since $p \nmid 29^m - 1$ when $ord_p 29$ is even, it follows that $p \nmid 29^m - 1$ for any $p \in [3,519]$ except for possibly $p \in \{7,13,67\}$. Since by our assumption $7 \nmid n$ in (3.5a) or (3.6a), it follows that $7 \nmid \frac{29^m-1}{28}$.

We may note that $13|29^m - 1 \iff 3|m \iff 67|29^m - 1$. Assume that $13|29^m - 1$. Then $67|\frac{29^m-1}{28}|\sigma^{**}(29^d)$. From (3.6b), $67|w'$. From (C), $q'|w'$. From (3.6c), $w' = 67^g.(q')^h$, and $q' > 519$. From (3.6a), we have $n = 2^7.5^b.17^2.29^d.11^e.13^2.67^g.(q')^h$ and so we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{11}{10}.\frac{170}{169}.\frac{67}{66}.\frac{521}{520} = 2.91273178 < 3,$$

a contradiction. Hence $13 \nmid 29^m - 1$ and consequently $67 \nmid 29^m - 1$.

Thus $\frac{29^m-1}{28}$ which is odd, > 1 and is not divisible by any prime in $[3,519]$. If $p'|\frac{29^m-1}{28}$, then $p' > 519$ and from (3.6b), $p'|w'$.

The proof of (D) is complete.

Already from (C), $q'|w'$. Hence from (3.6c), $w' = (p')^g.(q')^h$. Since $p' > 519$, $q' > 519$ and $p' \neq q'$, we may assume that $p' \geq 521$ and $q' \geq 523$. From (3.6a), we have

$$n = 2^7.5^b.17^2.29^d.11^e.13^2.(p')^g.(q')^h$$

and so we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{11}{10}.\frac{170}{169}.\frac{521}{520}.\frac{523}{522} = 2.874754834 < 3,$$

a contradiction. This proves that $f = 2$ is not admissible.

This completes the proof of (a) of Lemma 3.2.

(b) *Proof of Lemma 3.2 (b).* Suppose 11 and 19 divide n in (3.2a) and (3.2b) so that $w = 11^e.19^f.w'$. From (3.2a) and (3.2b), we have

$$n = 2^7.5^b.17^2.29^d.11^e.19^f.w', \tag{3.7a}$$

and

$$2^6.5^{b-2}.17.29^{d-1}.11^e.19^f.w' = \sigma^{**}(5^b).\sigma^{**}(29^d).\sigma^{**}(11^e).\sigma^{**}(19^f).\sigma^{**}(w'), \quad (3.7b)$$

where

$$(w', 2.3.5.7.17.29.11.19) = 1 \text{ and } w' \text{ has at most two odd prime factors.} \quad (3.7c)$$

Since $3|\sigma^{**}(11^e)$ if e is odd or $4|e$, we may assume that $e = 2u'$, where u' is odd; also, in (3.7a) and (3.7b), we can assume that $e \neq 2$. For, let $e = 2$ in (3.7b). Since $\sigma^{**}(11^2) = 122 = 2.61$, from (3.7b), $61|w'$. Let $w' = 61^f.w''$, where w'' is 1 or a prime power p^α with $p \geq 23$. Hence $\frac{\sigma^{**}(w'')}{w''} < \frac{23}{22}$. Since $n = 2^7.5^b.17^2.29^d.11^2.19^f.61^g.w''$, we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{122}{121}.\frac{19}{18}.\frac{61}{60}.\frac{23}{22} = 2.927653275 < 3,$$

a contradiction. Thus we may assume that $e \neq 2$, so that $e \geq 6$ as $e = 2u'$ and u' is odd.

Remark 3.3. As in (C) after (3.5c), it can be shown exactly in the same manner (see the proof of (a) of Lemma 3.3 below) that $\frac{29^{m+1}+1}{2}$ is divisible by an odd prime $q' > 519$ and from (3.7b), $q'|w'$.

We now prove that in (3.7a) and (3.7b), the exponents b, e and f cannot exceed 7 simultaneously. On the contrary, let $b \geq 7, e \geq 7$ and $f \geq 7$. By Lemma 2.1, we have $\frac{\sigma^{**}(29^d)}{29^d} \geq \frac{616042622}{594823321}$ $(d \geq 5)$; $\frac{\sigma^{**}(11^e)}{11^e} \geq \frac{235780128}{214358881}$ $(e \geq 7)$; $\frac{\sigma^{**}(19^f)}{19^f} \geq \frac{17926964000}{16983563041}$ $(f \geq 7)$ and $\frac{\sigma^{**}(5^b)}{5^b} \geq \frac{487656}{390625}$ $(b \geq 7)$.

Using the above results, from (3.7a), for $b \geq 7, e \geq 7$ and $f \geq 7$, we have

$$3 = \frac{\sigma^{**}(n)}{n} \geq \frac{255}{128}.\frac{487656}{390625}.\frac{290}{289}.\frac{616042622}{594823321}.\frac{235780128}{214358881}.\frac{17926964000}{16983563041}$$
$$= 3.000891774524 > 3,$$

a contradiction. Thus $b \geq 7, e \geq 7$ and $f \geq 7$ cannot hold simultaneously.

Recalling that b and e are even and ≥ 6, the following cases arise.

(i) $b = 6,\ e = 6,\ f \geq 7$; (ii) $b = 6,\ e \geq 7,\ f \leq 6$; (iii) $b \geq 7,\ e = 6,\ f \leq 6$;

(iv) $b = 6,\ e \geq 7,\ f \geq 7$; (v) $b \geq 7,\ e = 6,\ f \geq 7$;

(vi) $b \geq 7,\ e \geq 7,\ f \leq 6$; (vii) $b = 6,\ e = 6,\ f \leq 6$.

Since $\sigma^{**}(11^6) = \{\{2,1\},\{7,1\},\{19,1\},\{7321,1\}\}$, we have $7|\sigma^{**}(11^6)$. Taking $e = 6$ in (3.7b), we see that $7|w'$. But w' is prime to 7. Thus $e = 6$ is not admissible. It follows that we need to only examine the cases (ii), (iv) and (vi).

We now prove that $f \leq 6$ is not possible so that the cases (ii) and (vi) would be wiped away.

Let $f = 1$. We have $\sigma^{**}(19) = 20 = 2^2.5$. Taking $f = 1$ in (3.7b), it follows that its right hand side is divisible by 2^5 and its left hand side is unitarily divisible by 2^6. Hence w' is 1 or an odd prime power. Thus by taking $f = 1$ in (3.7a) and (3.7b), we obtain

$$n = 2^7.5^b.17^2.29^d.11^e.19.w', \tag{3.8a}$$

and

$$2^4.5^{b-3}.17.29^{d-1}.11^e.19.w' = \sigma^{**}(5^b).\sigma^{**}(29^d).\sigma^{**}(11^e).\sigma^{**}(w'), \tag{3.8b}$$

where

$$(w', 2.3.5.7.11.17.19.29) = 1 \text{ and } w' \text{ has at most one odd prime factor.} \tag{3.8c}$$

By Remark 3.3, $\frac{29^{m+1}+1}{2}$ is divisible by an odd prime $q' > 519$ and from (3.8b), $q'|w'$. Consider the factor $\frac{29^m-1}{28}$; we now use Lemma 2.4 to show that this factor is divisible by an odd prime $p' > 519$ and $p'|w'$. That w' will be divisible by two distinct primes p' and q' leads to a contradiction in virtue of (3.8c). Let

$$S'_{29} = \{p|29^m - 1 : \ p \in [3, 519] \setminus \{7, 13, 67\} \text{ and } ord_p 29 \text{ is odd}\}.$$

If S'_{29} is non-empty, by (a) of Lemma 2.4, there exists an odd prime $p'|\frac{29^m-1}{2}$, $p' > 519$ and from (3.8b), $p'|w'$.

Suppose that S'_{29} is empty. Since $p \nmid 29^m - 1$ if $ord_p 29$ is even, it follows that $p \nmid 29^m - 1$ for any prime p in $[3, 519]$ except for possible $p \in \{7, 13, 67\}$. We have $7|29^m - 1$ but $7 \nmid \frac{29^m-1}{28}$ as $7 \nmid w'$ in (3.8b) (by our assumption that $7 \nmid n$). We note that $13|29^m - 1 \Longleftrightarrow 67|29^m - 1$. By (a) of Lemma 3.2, $13 \nmid 29^m - 1$. Hence $67 \nmid 29^m - 1$.

Thus $\frac{29^m-1}{28} > 1$, is odd and not divisible by any prime in $[3, 519]$. Hence if $p'|\frac{29^m-1}{28}$, then $p' > 519$ and $p'|w'$. This proves that $f = 1$ is not admissible.

We now prove that $f = 2$ is not admissible. Let $f = 2$. Since $\sigma^{**}(19^2) = 362 = 2.181$, taking $f = 2$ in (3.7b), we see that $181|w'$. By Remark 3.3, $q'|w'$ and $q' > 519$. Hence w' is divisible by 181 and q'. From (3.7c), we have $w' = (181)^g.(q')^h$. Hence $n = 2^7.5^b.17^2.29^d.11^e.19^2.(181)^g.(q')^h$ and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{11}{10}.\frac{362}{361}.\frac{181}{180}.\frac{521}{520} = 2.876171965 < 3,$$

a contradiction.

Next we prove that $f = 3$ is not admissible. Let $f = 3$. We have $\sigma^{**}(19^3) = \frac{19^4-1}{18} = 2^3.5.181$. Taking $f = 3$ in (3.7b), we see that 2^6 is a factor of its right-hand side and 2^6 is a unitary divisor of its left hand side. Hence $w' = 1$. But by Remark 3.3, $q'|w'$ and $q' > 519$. This is a contradiction.

We note that $f = 4$ is not admissible since $7|\sigma^{**}(19^4) = 2^4.5^2.7^3$ and from (3.7b) $(f = 4)$, it follows that $7|w'$ which is false.

Further, $f = 5$ is not admissible since $7|\sigma^{**}(19^5) = 2^2.5.7^3.381$.

Also, $f = 6$ is not admissible since $3|\sigma^{**}(19^6) = 2.3.17.127.3833$ and from (3.7b) $(f = 6)$, it follows that $3|w'$ which is false.

The only case remaining is (vi): $b = 6$, $e \geq 7$ and $f \geq 7$. Then, we have $\sigma^{**}(5^6) = 2.31.313$, and taking $b = 6$ in (3.7b), we see that w' is divisible by 31 and 313. Also by Remark 3.3, w' is divisible by $q' > 519$. Thus w' is divisible by three primes $31, 313$ and q'. This is a contradiction to (3.7c).

This proves (b).

(c) *Proof of Lemma 3.2 (c).* Suppose 13 and 19 divide n in (3.2a) and (3.2b) so that $w = 13^e.19^f.w'$. From (3.2a) and (3.2b), we have

$$n = 2^7.5^b.17^2.29^d.13^e.19^f.w', \tag{3.9a}$$

and

$$2^6.5^{b-2}.17.29^{d-1}.13^e.19^f.w' = \sigma^{**}(5^b).\sigma^{**}(29^d).\sigma^{**}(13^e).\sigma^{**}(19^f).\sigma^{**}(w'), \tag{3.9b}$$

where

$$(w', 2.3.5.7.17.29.13.19) = 1 \text{ and } w' \text{ has at most two odd prime factors.} \tag{3.9c}$$

We recall that $b = 2k$, where k is odd and $k \geq 3$. We have

$$\sigma^{**}(5^b) = \left(\frac{5^k - 1}{4}\right).(5^{k+1} + 1).$$

We now show that:

(E) $\frac{5^k-1}{4}$ is divisible by an odd prime $P > 2520$ and $P|w'$,

(F) $\frac{5^{k+1}+1}{2}$ is divisible by an odd prime $Q > 2520$ and $Q|w'$.

Proof of (E). Let

$$S'_5 = \{p|5^k - 1 : p \in [3, 2520] \setminus \{11, 19, 31, 71, 181, 829, 1741\} \text{ and } ord_p 5 \text{ is odd}\}.$$

If S'_5 is non-empty, then by (a) of Lemma 2.3, (E) holds. We may assume that S'_5 is empty. Since $p \nmid 5^k - 1$ if $ord_p 5$ is even, it follows that $5^k - 1$ is not divisible by any prime $p \in [3, 2520]$ except for possibly $p \in \{11, 19, 31, 71, 181, 829, 1741\}$. We observe the following:

(i) $11|5^k-1 \iff k=5u \iff 71|5^k-1$. Since by hypothesis, $11 \nmid 5^k-1$, it follows that $71 \nmid 5^k-1$.

(ii) $19|5^k-1 \iff k=9u \iff 829|5^k-1$. Assume that $19|5^k-1$ so that $829|5^k-1$. From (3.9a), we have $n=2^7.5^b.17^2.29^d.13^e.19^f.(829)^g.(q')^h$, where $q'>519$, $q'|\frac{29^{m+1}+1}{2}$ and $q'|w'$ (see Remark 3.3). Hence we have

$$3=\frac{\sigma^{**}(n)}{n}<\frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{13}{12}.\frac{19}{18}.\frac{829}{828}.\frac{521}{520}=2.968808064<3,$$

a contradiction. Hence $19 \nmid 5^k-1$ and consequently $829 \nmid 5^k-1$.

(iii) $181|5^k-1 \iff k=15u \iff 1741|5^k-1$. Suppose $181|5^k-1$. Then $1741|5^k-1$. Hence from (3.9a), $n=2^7.5^b.17^2.29^d.13^e.19^f.(181)^g.(1741)^h$, so that we have

$$3=\frac{\sigma^{**}(n)}{n}<\frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{13}{12}.\frac{19}{18}.\frac{181}{180}.\frac{1741}{1740}=2.977087654<3,$$

a contradiction. Hence 5^k-1 is neither divisible by 181 nor 1741.

(iv) If $31 \nmid 5^k-1$, then from (i)–(iii) above, it follows that $\frac{5^k-1}{4}$ is not divisible by any prime in $[3,2520]$. If $P|\frac{5^k-1}{4}$, then $P>2520$ and by (3.9b), $P|w'$. This proves (E) in this case.

(v) Suppose that $31|5^k-1$. We show that $\frac{5^k-1}{4}$ is divisible by an odd prime $P\neq 31$. If this is not the case, let $\frac{5^k-1}{4}=31^\alpha$, for some positive integer α. If $\alpha\geq 2$, then $31^2|5^k-1$, which is equivalent to $k=93u=31u'$. Hence $1861|\frac{5^{31}-1}{4}\frac{5^k-1}{4}=31^\alpha$ and this is impossible. Hence $\alpha=1$ so that $\frac{5^k-1}{4}=31$ or $k=3$ and $b=6$. We now show that $b=6$ is not admissible in (3.9b). We have $\sigma^{**}(5^6)=2.31.313$. Taking $b=6$ in (3.9b), we obtain

$$\begin{aligned}&2^5.5^4.17.29^{d-1}.13^e.19^f.31^{g-1}.(313)^{h-1}\\ &\quad=\sigma^{**}(29^d).\sigma^{**}(13^e).\sigma^{**}(19^f).\sigma^{**}(31^g).\sigma^{**}(313^h).\end{aligned}\tag{3.9d}$$

By Remark 3.3, $q'|\frac{29^{m+1}+1}{2}|\sigma^{**}(29^d)$ and $q'>519$. From (3.9d), it follows that its left hand side is not divisible by q'. This proves that $b=6$ is not admissible.

Thus $\frac{5^k-1}{4}$ is divisible by an odd prime $P\neq 31$. Clearly, $P\notin[3,2520]$ so that $P>2520$. From (3.9b), $P|w'$.

The proof of (E) is complete.

Proof of (F). Let

$$T_5'=\{q|5^{k+1}+1 : q\in[3,2520]\smallsetminus\{13,313,601\} \text{ and } s=\frac{1}{2}ord_q 5 \text{ is even}\}.$$

If T_5' is non-empty, then by (b) of Lemma 2.3, (F) holds. We may assume that T_5' is empty. Since $q \nmid 5^{k+1}+1$ if $s=\frac{1}{2}ord_q 5$ is odd, it follows that $5^{k+1}+1$ is not divisible by any prime $q \in [3,2520]$ except for possibly $q \in \{13,313,601\}$.

We have the following:

(vi) Assume that $313|5^{k+1}+1$. From (3.9b), it follows that $313|w'$. From (E), w' is divisible by $P > 2520$. Hence from (3.9c) and (3.9a), we have

$$n = 2^7.5^b.17^2.29^d.13^e.19^f.(313)^g.P^h.$$

Therefore we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{13}{12}.\frac{19}{18}.\frac{313}{312}.\frac{2521}{2520} = 2.970199332 < 3,$$

a contradiction. Thus $313 \nmid 5^{k+1}+1$.

(vii) Suppose that $601|5^{k+1}+1$. As in (vi) above, we have

$$n = 2^7.5^b.17^2.29^d.13^e.19^f.(601)^g.P^h$$

and so

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{13}{12}.\frac{19}{18}.\frac{601}{600}.\frac{2521}{2520} = 2.965644393 < 3,$$

a contradiction. Thus $601 \nmid 5^{k+1}+1$.

(viii) If $13 \nmid 5^{k+1}+1$, it follows from (vi) and (viii) that $\frac{5^{k+1}+1}{2}$ is not divisible by any prime in $[3,2520]$. Consequently, if $Q|\frac{5^{k+1}+1}{2}$, then $Q > 2520$ and by (b), $Q|w'$. This proves (F) in this case.

(ix) Assume that $13|5^{k+1}+1$. We claim that $\frac{5^{k+1}+1}{2}$ is divisible by an odd prime $Q \neq 13$. On the other hand, let $\frac{5^{k+1}+1}{2} = 13^\alpha$, for some positive integer α.
If $\alpha \geq 2$, then $13^2|5^{k+1}+1$. This is equivalent to $k+1 = 26u$. Hence we have

$$53|\frac{5^{26}+1}{2}|\frac{5^{k+1}+1}{2} = 13^\alpha,$$

which is not possible.
Therefore, we have $\alpha = 1$ so that $\frac{5^{k+1}+1}{2} = 13$, i.e, $k = 1$. But $k \geq 3$. It now follows that $\frac{5^{k+1}+1}{2}$ is divisible by an odd prime $Q \neq 13$. Hence $Q \notin [3,2520]$ so that $Q > 2520$ and from (3.9b), $Q|w'$.

The proof of (F) is complete.

We are now in a position to complete the proof of (c). By (3.9c), (3.9a), (E) and (F), we have

$$n = 2^7.5^b.17^2.29^d.13^e.19^f.(P)^g.(Q)^h,$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{13}{12}.\frac{19}{18}.\frac{2521}{2520}.\frac{2531}{2530} = 2.805991691 < 3,$$

a contradiction.

This proves (c).

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Consider the equations (3.2a)–(3.2c) corresponding to the case $c = 2$. In (3.2b), we can assume that $d = 2m$, where m is odd and ≥ 3 (see Remark 3.2). Then*

(a) $\frac{29^{m+1}+1}{2}$, a factor of $\sigma^{**}(29^d)$ is divisible by an odd prime $q' > 519$ and $q'|w$,

(b) n is divisible by exactly one of the primes 11, 13 and 19,

(c) $\frac{29^m-1}{28}$, a factor of $\sigma^{**}(29^d)$ is divisible by an odd prime $p' > 519$ and $p'|w$.

Proof. (a) Let

$$T'_{29} = \{q|29^{m+1}+1 : q \in [3,519] \setminus \{37,61,313,421\} \text{ and } s = \frac{1}{2}ord_q 29 \text{ is even}\}.$$

If T'_{29} is non-empty, then (a) holds by Lemma 2.4(b). We may assume that T'_{29} is empty. Since $s = \frac{1}{2}ord_q 29$ is odd implies that $q \nmid 29^{m+1}+1$, it follows that $29^{m+1}+1$ is not divisible by any prime q in $[3,519]$ except for possibly $q \in \{37,61,313,421\}$.

We note that $37|29^{m+1}+1 \iff m+1 = 6u \iff 61|29^{m+1}+1 \iff 313|29^{m+1}+1$.

Assume that $37|29^{m+1}+1$ so that $m+1 = 6u$. Hence $29^6+1|29^{m+1}+1$. But $29^6+1 = 2.37.61.313.421$. It follows that $\frac{29^{m+1}+1}{2}$ (a factor of $\sigma^{**}(29^d)$ in (3.2b)) is divisible by the four primes $37,61,313$ and 421. By (3.2c), it follows that $w = 37^e.61^f.313^g.421^h$ and so

$$n = 2^7.5^b.17^2.29^d.37^e.61^f.313^g.421^h.$$

Hence we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{37}{36}.\frac{61}{60}.\frac{313}{312}.\frac{421}{420} = 2.71945 < 3,$$

a contradiction.

Hence $37 \nmid 29^{m+1}+1$. As a consequence $29^{m+1}+1$ is not divisible by 61 and 313.

If $421 \nmid 29^{m+1}+1$, it follows that $29^{m+1}+1$ is not divisible by any prime in $[3,519]$; the same holds with respect to $\frac{29^{m+1}+1}{2}$. Hence if $q'|\frac{29^{m+1}+1}{2}$, then $q' > 519$ and $q'|w$ from (3.2b). Suppose $421|29^{m+1}+1$. We claim that $\frac{29^{m+1}+1}{2}$ is divisible by an odd prime $\neq 421$. On the other hand, let $\frac{29^{m+1}+1}{2} = (421)^\alpha$, for some positive integer α. If $\alpha \geq 2$, then $421^2|29^{m+1}+1$. But this is equivalent to $m+1 = 842.u$. Hence

$$6737|\left(\frac{29^{842}+1}{2}\right)|\left(\frac{29^{m+1}+1}{2}\right) = (421)^\alpha,$$

and this is impossible. Hence $\alpha = 1$ so that $\frac{29^{m+1}+1}{2} = 421$ or $m = 1$. But $m \geq 3$. Thus $\frac{29^{m+1}+1}{2}$ is divisible by an odd prime $q' \neq 421$. Hence $q' \notin [3, 519]$ and so $q' > 519$. From (3.2b), $q'|w$.

This completes the proof of (a).

(b) *Proof of (b).* We first prove that n is divisible by at least one of 11, 13 and 19. On the contrary assume that n is divisible by none of 11, 13 and 19. From (3.2b), it follows that every prime factor of w in (3.2b) is at least 23. By (a), $q'|w$ and $q' > 519$. From (3.2a) and (3.2c), we have

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{23}{22}.\frac{31}{30}.\frac{37}{36}.\frac{521}{520} = 2.87912 < 3,$$

a contradiction. Thus, n is divisible by at least one of 11, 13 and 19. Now, part (b) follows from Lemma 3.2.

(c) *Proof of (c).* Let

$$S'_{29} = \{p|29^m - 1 : p \in [3, 519] \setminus \{7, 13, 67\} \text{ and } ord_p 29 \text{ is odd}\}.$$

We distinguish three cases on the basis of part (b):

Case 1. Suppose that $11|n$. Then w in (3.2a) and (3.2b) cannot have more than three other odd prime factors. If S'_{29} is non-empty, by (a) of Lemma 2.4 and (3.2b), the statement in (c) holds. We may assume that S'_{29} is empty. Since $p \nmid 29^m - 1$ if $ord_p 29$ is even, it follows that $29^m - 1$ is not divisible by any prime $p \in [3, 519]$ except for possibly $p \in \{7, 13, 67\}$. The same is true with respect to $\frac{29^m-1}{28}$; this is not divisible by 7 as $7|\frac{29^m-1}{28}$ would imply that $7|w|n$, from (3.2b). But by our assumption $7 \nmid n$.

By part (b), $13 \nmid 29^m - 1$ since $11|n$. Also, $13|29^m - 1 \iff 67|29^m - 1$. Hence $67 \nmid 29^m - 1$. Thus $\frac{29^m-1}{28}$ is not divisible by $7, 13$ and 67. It follows that $\frac{29^m-1}{28}$ is not divisible by any prime in $[3, 519]$. Hence if $p'|\frac{29^m-1}{28}$, then $p' > 519$ and from (3.2b), $p'|w$. Thus in this case, (c) holds.

Case 2. Assume that $13|n$. Then w in (3.2a) and (3.2b) cannot have more than three other odd prime factors. If S'_{29} is non-empty, by (a) of Lemma 2.4 and (3.2b), the statement in (c) holds. We may assume that S'_{29} is empty. Since $p \nmid 29^m - 1$ if $ord_p 29$ is even, it follows that $29^m - 1$ is not divisible by any prime $p \in [3, 519]$ except for possibly $p \in \{7, 13, 67\}$. The same is true with respect to $\frac{29^m-1}{28}$; this is not divisible by 7 as in Case 1.

Assume that $13|\frac{29^m-1}{28}$. Then $67|\frac{29^m-1}{28}$, since $13|29^m - 1 \iff 67|29^m - 1$. Hence w is divisible by 67. By (a), w is divisible by $q' > 519$. Since $19 \nmid n$, a possible third prime factor of w, say r is at least 23. Hence from (3.2a), we have

$$n = 2^7.5^b.17^2.29^d.13^e.67^f.(q')^f.r^g,$$

and so we obtain

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{13}{12}.\frac{67}{66}.\frac{521}{520}.\frac{23}{22} = 2.981349246 < 3,$$

a contradiction. Hence $\frac{29^m-1}{28}$ is neither divisible by 13 nor 67.
It follows that $\frac{29^m-1}{28}$ is not divisible by any prime in $[3,519]$. Hence if $p'|\frac{29^m-1}{28}$, then $p' > 519$ and from (3.2b), $p'|w$. This proves (c), in this case.

Case 3. Assume that $19 \mid n$. Then w in (3.2a) and (3.2b) cannot have more than three other odd prime factors. If S'_{29} is non-empty, by (a) of Lemma 2.4 and (3.2b), the statement in (c) holds. We may assume that S'_{29} is empty. Since $p \nmid 29^m - 1$ if $ord_p 29$ is even, it follows that $29^m - 1$ is not divisible by any prime $p \in [3,519]$ except for possibly $p \in \{7,13,67\}$. The same is true with respect to $\frac{29^m-1}{28}$; this is not divisible by 7 as in Case 1. Since by our assumption, $13 \nmid 29^m - 1$, we have $67 \nmid 29^m - 1$. It follows that $\frac{29^m-1}{28}$ is not divisible by any prime in $[3,519]$. Hence if $p'|\frac{29^m-1}{28}$, then $p' > 519$ and from (3.2b), $p'|w$. This proves (c), in this case also.

The proof of (c) is complete.

The proof of Lemma 3.3 is complete. □

Proof of Theorem 3.3. Assume that $c = 2$. Then we have the equations (3.2a)–(3.2c). By Lemma 3.3, $t|n$ for exactly one $t \in \{11,13,19\}$. By (3.2a)–(3.2c) and Lemma 3.3, we have $w = t^e.(p')^f.(q')^g.(t')^h$, where $p' \geq 521$, $q' \geq 523$ and t' is the possible fourth prime factor of w with $t' \geq 23$. From (3.2a), we have

$$n = 2^7.5^b.17^2.29^d.t^e.(p')^f.(q')^g.(t')^h,$$

so that

$$3 = \frac{\sigma^{**}(n)}{n} < \frac{255}{128}.\frac{5}{4}.\frac{290}{289}.\frac{29}{28}.\frac{11}{10}.\frac{521}{520}.\frac{523}{522}.\frac{23}{22} = 2.987746535 < 3,$$

a contradiction. Hence $c \neq 2$. The proof of Theorem 3.3 is complete.

Professor Sitaramaiah [10] proposes that the above results lead to the following inequality. The present author has not verified the proof. Therefore the inequality is presented as a conjecture.

Conjecture. Let n be a bi-unitary perfect number of the form $n = 2^7.5^b.17^c.t^d.w$, where $b \geq 3$, $t \in \{11,13,19\}$ and w is prime to $2.3.5.7.17.t$. (Here w cannot have more than four odd prime factors.) Then

$$n > \begin{cases} 2.605 \times 10^{134} & \text{if } 5^6 \| n \text{ and } 11|n, \\ 9.25 \times 10^{167} & \text{if } 5^7|n \text{ and } 11|n, \\ 7.65 \times 10^{171} & \text{if } 13|n, \\ 6.079 \times 10^{180} & \text{if } 19|n. \end{cases}$$

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Appendix A Table of $ord_p 29$

Let p denote an odd prime $\neq 29$. In the following table, r denotes the smallest positive integer such that $29^r \equiv 1 \pmod{p}$; that is, $r = ord_p 29$, and s denotes the smallest positive integer such that $29^s \equiv -1 \pmod{p}$ if s exists. If s does not exist, that is, if $29^t + 1$ is not divisible by p for any positive integer t, the entry in column s will be filled up by dash sign. If r is even, then $s = r/2$, and if r is odd, s does not exist.

SL.No	p	r	s
1	3	2	1
2	5	2	1
3	7	1	–
4	11	10	5
5	13	3	–
6	17	16	8
7	19	18	9
8	23	11	–
9	29	–	–
10	31	10	5
11	37	12	6
12	41	40	20
13	43	42	21
14	47	46	23
15	53	26	13
16	59	29	–
17	61	12	6
18	67	3	–
19	71	35	–
20	73	72	36
21	79	78	39
22	83	41	–
23	89	88	44
24	97	96	48
25	101	100	50
26	103	51	–
27	107	53	–
28	109	54	27
29	113	112	56
30	127	126	63
31	131	130	65
32	137	136	68
33	139	69	–
34	149	37	–
35	151	25	–
36	157	52	26
37	163	162	81
38	167	83	–
39	173	43	–
40	179	89	–
41	181	15	–
42	191	190	95
43	193	64	32
44	197	49	–
45	199	99	–
46	211	210	105
47	223	111	–
48	227	113	–
49	229	228	114
50	233	58	29
51	239	119	–
52	241	120	60
53	251	250	125
54	257	128	64
55	263	262	181
56	269	268	134
57	271	6	3
58	277	138	69
59	281	70	35
60	283	47	–
61	293	292	146
62	307	306	153
63	311	310	155
64	313	12	6
65	317	316	158
66	331	330	165
67	337	336	168
68	347	173	–
69	349	174	87
70	353	88	44
71	359	358	179
72	367	122	61
73	373	93	–
74	379	126	63
75	383	191	–
76	389	388	194
77	397	99	–
78	401	10	5
79	409	408	204
80	419	209	–
81	421	4	2
82	431	215	–
83	433	432	216
84	439	219	–
85	443	442	221
86	449	448	224
87	457	114	57
88	461	460	230
89	463	231	–
90	467	466	233
91	479	478	239
92	487	81	–
93	491	490	245
94	499	249	–
95	503	502	251
96	509	254	127

Appendix B Factors of $29^t - 1$

$$29^{11} - 1 = \{\{2,2\},\{7,1\},\{23,1\},\{18944890940537,1\}\}$$

$$29^{15} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{181,1\},\{22111,1\},\ldots\}$$

$$29^{25} - 1 = \{\{2,2\},\{7,1\},\{151,1\},\{732541,1\},\ldots\}$$

$$29^{29} - 1 = \{\{2,2\},\{7,1\},\{59,1\},\{16763,1\},\ldots\}$$

$$29^{35} - 1 = \{\{2,2\},\{7,2\},\{71,1\},\{732541,1\},\ldots\}$$

$$29^{37} - 1 = \{\{2,2\},\{7,1\},\{149,1\},\{13913,1\},\ldots\}$$

$$29^{41} - 1 = \{\{2,2\},\{7,1\},\{83,1\},\{2789,1\},\ldots\}$$

$$29^{43} - 1 = \{\{2,2\},\{7,1\},\{173,1\},\{13933,1\},\ldots\}$$

$$29^{47} - 1 = \{\{2,2\},\{7,1\},\{283,1\},\{659693,1\},\ldots\}$$

$$29^{49} - 1 = \{\{2,2\},\{7,3\},\{197,1\},\{88009573,1\},\ldots\}$$

$$29^{51} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{103,1\},\{3911,1\},\ldots\}$$

$$29^{53} - 1 = \{\{2,2\},\{7,1\},\{107,1\},\{10601,1\},\ldots\}$$

$$29^{69} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{139,1\},\{131327761273,1\},\ldots\}$$

$$29^{81} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{487,1\},\{14437,1\},\ldots\}$$

$$29^{83} - 1 = \{\{2,2\},\{7,1\},\{167,1\},\{5118695830412449740993707190291471468836668205749640789916257545333338736071795560037899154822689107089157792756778513,1\}\}$$

$$29^{89} - 1 = \{\{2,2\},\{7,1\},\{179,1\},\{1069,1\},\ldots\}$$

$$29^{93} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{373,1\},\{36767,1\},\ldots\}$$

$$29^{99} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{23,1\},\{67,1\},\{199,1\},\{397,1\},\ldots\}$$

$$29^{111} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{149,1\},\{223,1\},\{13913,1\},\ldots\}$$

$$29^{113} - 1 = \{\{2,2\},\{7,1\},\{227,1\},\{280407660520833927530540107069533152661694069630684961289427326872431929489749753183807662058116475449521522961049909031188217178659584119469734915010632321126523,1\}\}$$

$$29^{119} - 1 = \{\{2,2\},\{7,2\},\{239,1\},\{3911,1\},\ldots\}$$

$$29^{173} - 1 = \{\{2,2\},\{7,1\},\{347,1\},\{58129,1\},\ldots\}$$

$$29^{191} - 1 = \{\{2,2\},\{7,1\},\{383,1\},\{40111,1\},\ldots\}$$

$$29^{209} - 1 = \{\{2,2\},\{7,1\},\{23,1\},\{419,1\},\{6271,1\},\ldots\}$$

$$29^{215} - 1 = \{\{2,2\},\{7,1\},\{173,1\},\{431,1\},\{13933,1\},\ldots\}\}$$

$$29^{219} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{439,1\},\{6053603111,1\},\ldots\}$$

$$29^{231} - 1 = \{\{2,2\},\{7,2\},\{13,1\},\{23,1\},\{67,1\},\{463,1\},\{6637,1\},\ldots\}$$

$$29^{249} - 1 = \{\{2,2\},\{7,1\},\{13,1\},\{67,1\},\{167,1\},\{499,1\},\{29712205413756639028349675614801497135602368219563778259864144834894553169889665110560085610522473351082608322557539966718721333911128435649453801835929153090374481352770504963666022731188458841724568980018901135915638961809133780621611276327757376892520873789540573231789116235762632534094201009245510500265082025644139231,1\}\}.$$

Appendix C Factors of $29^t + 1$

$$
\begin{aligned}
29^6+1 =&\{\{2,1\},\{37,1\},\{61,1\},\{313,1\},\{421,1\}\}\\
29^8+1 =&\{\{2,1\},\{17,1\},\{26209,1\},\ldots\}\\
29^{20}+1 =&\{\{2,1\},\{41,1\},\{353641,1\},\{6103563899172302171321,1\}\}\\
29^{26}+1 =&\{\{2,1\},\{157,1\},\{421,1\},\{6917,1\},\ldots\}\\
29^{32}+1 =&\{\{2,1\},\{193,1\},\{63354497,1\},\ldots\}\\
29^{36}+1 =&\{\{2,1\},\{73,1\},\{9001,1\},\ldots\}\\
29^{44}+1 =&\{\{2,1\},\{89,1\},\{353,1\},\{617,1\},\{353641,1\},\ldots\}\\
29^{48}+1 =&\{\{2,1\},\{97,1\},\{807796875876007901354096217940924731897896044763983 39\\
&268267830745473,1\}\}\\
29^{50}+1 =&\{\{2,1\},\{101,1\},\{421,1\},\{1061,1\},\ldots\}\\
29^{56}+1 =&\{\{2,1\},\{17,1\},\{113,1\},\{26209,1\},\ldots\}\\
29^{60}+1 =&\{\{2,1\},\{41,1\},\{241,1\},\{9001,1\},\ldots\}\\
29^{64}+1 =&\{\{2,1\},\{257,1\},\{641,1\},\{7937,1\},\ldots\}\\
29^{68}+1 =&\{\{2,1\},\{137,1\},\{132329,1\},\{353641,1\},\ldots\}\}\\
29^{114}+1 =&\{\{2,1\},\{37,1\},\{61,1\},\{229,1\},\{313,1\},\{421,1\},\{131101,1\},\ldots\}\\
29^{134}+1 =&\{\{2,1\},\{269,1\},\{421,1\},\\
&\{4038939491763675949776013984702126604972242754516010094863907796 53\\
&2557197391099291642318406711490210523652835313142583804949561622260\\
&4242907529468849630911488133406834061825422209051664115309,1\}\}\\
29^{146}+1 =&\{\{2,1\},\{293,1\},\{421,1\},\{139999693,1\},\ldots\}\\
29^{158}+1 =&\{\{2,1\},\{317,1\},\{421,1\},\\
&\{4290543548015008446012807492232313781960923969999169308715227691 14\\
&1237324030732627347577996363252523654348272312548929299582663172416\\
&2150335813957152962966396123618197571059583125451593548425657091827\\
&403065061596849037756827 33,1\}\}\\
29^{168}+1 =&\{\{2,1\},\{17,1\},\{113,1\},\{337,1\},\{673,1\},\ldots\}\\
29^{194}+1 =&\{\{2,1\},\{389,1\},\{421,1\},\{1553,1\},\ldots\}\\
29^{204}+1 =&\{\{2,1\},\{137,1\},\{409,1\},\{9001,1\},\ldots\}\\
29^{216}+1 =&\{\{2,1\},\{17,1\},\{433,1\},\{673,1\},\ldots\}\\
29^{224}+1 =&\{\{2,1\},\{193,1\},\{449,1\},\{63354497,1\},\ldots\}\\
29^{230}+1 =&\{\{2,1\},\{421,1\},\{461,1\},\{829,1\},\ldots\}.
\end{aligned}
$$