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Bounds on some energy-like invariants of corona and edge corona of graphs

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Abstract: The Laplacian-energy-like invariant of a finite simple graph is the sum of square roots of all its Laplacian eigenvalues and the incidence energy is the sum of square roots of all its signless Laplacian eigenvalues. In this paper, we give the bounds on the Laplacian-energy-like invariant and incidence energy of the corona and edge corona of two graphs. We also observe that the bounds on the Laplacian-energy-like invariant and incidence energy of the corona and edge corona are sharp when the graph is the corona or edge corona of two complete graphs.
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1 Introduction

We consider only finite simple graphs (i.e. no loops and no multiple edges) throughout this paper. For a graph G of order n with m edges and vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the *adjacency matrix* $A(G)$ of G is an $n \times n$ real symmetric matrix whose (i, j)-entry is 1 if v_i is adjacent to v_j and 0 otherwise. Let d_i denotes the degree of a vertex v_i and if $d_i = r$ for $i = 1, 2, \ldots, n$, then G is called *r-regular*. Let $D(G)$ be the degree diagonal matrix of G with diagonal entries

$d_1, d_2, \ldots, d_n$ and 0 elsewhere. Then $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the *Laplacian* and *signless Laplacian matrices* of G, respectively. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $\theta_1 \leq \theta_2 \cdots \leq \theta_n$ denote the eigenvalues of $L(G)$ and $Q(G)$ known as the *Laplacian* and *signless Laplacian eigenvalues* of G, respectively. It is well known [6] that $\mu_1 = 0$. Moreover, if the graph is r-regular, then $\mu_n \leq 2r$ and $\theta_n = 2r$.

The *energy* of G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i is the eigenvalue of $A(G)$. The roots of this graph invariant traced back to the 1940s and was motivated by the so-called *Hückel molecular orbital theory* of conjugated hydrocarbons in chemistry [19]. In 1978, Gutman proposed the energy of a graph, independent of its chemical motivations, in mathematics [11]. It did not attract any noteworthy attention among mathematicians in the beginning but from the early 21st century onwards remarkable mathematical research started [19]. Important variants of the graph energy have also been proposed in the literature. The Laplacian-spectral analogue of the graph energy, known as the *Laplacian energy*, is defined in [14] as $LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$. Beyond the applications in molecular orbital theory of conjugated molecules, the Laplacian energy is found to have remarkable chemical applications [4]. Similar to the graph energy and Laplacian energy, Liu and Liu [20] proposed another graph invariant, known as the *Laplacian-energy-like*, which is defined as $LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$. The chemical applications of the Laplacian-energy-like invariant are much studied and it is also described as a newly designed molecular descriptor [24]. The concept of graph energy was extended to any matrix by Nikiforov as the sum of its singular values [21]. Motivated by Nikiforov's idea, Jooyandeh et al. [18] defined the concept of *incidence energy* of a graph G, denoted by $IE(G)$, as the sum of the singular values of its incidence matrix. Moreover, the incidence energy can also be expressed [12] as $IE(G) = \sum_{i=1}^{n} \sqrt{\theta_i}$.

The Laplacian-energy-like invariant and the incidence energy of a graph share many interesting relations. In particular, if the graph G is bipartite, then $IE(G) = LEL(G)$. For more relations on the Laplacian-energy-like invariant and incidence energy of a graph, we refer the readers to [12]. Recently, the Laplacian-energy-like invariants are explored for some graph operations on regular and semi-regular graphs in [23] and [26]. New bounds on the Laplacian- energy-like invariant and incidence energy of the line graph, subdivision graph and total graph of regular graphs have been obtained in [3] and [5]. Motivated by such results, in this paper, we give the bounds on the Laplacian-energy-like invariant and incidence energy for some important graph operations, namely the *corona* and *edge corona* of two graphs. The corona of two graphs was introduced in [10] with the goal of constructing a graph whose automorphism group is the wreath product of the two component automorphism groups. A variant called edge corona of two graphs was introduced in [15]. It is worth mentioning that the corona and edge corona can be used to construct pairs of nonisomorphic graphs with the same spectrum for the adjacency, Laplacian and signless Laplacian matrices. So, if G_1, G_1' (respectively, G_2, G_2') are two r_1- (respectively, r_2-) regular nonisomorphic graphs with the same spectrum for the adjacency, Laplacian and signless Laplacian matrices, then the (edge) corona of G_1 and G_2; and the (edge) corona of G_1' and G_2'

(not necessarily regular) have the same spectrum for all of the adjacency, Laplacian and signless Laplacian matrices [2].

The paper is arranged as follows: In Section 2, we recall the definitions of the corona and edge corona of two graphs. We also present some well known results which are needed in our work. In Section 3, we state and prove our main results for the Laplacian-energy-like invariant of the corona and edge corona of two graphs. In Section 4, we prove our main results for the incidence energy of the corona and edge corona of two graphs. We observe that the bounds on the Laplacian-energy-like invariant and incidence energy of the corona and edge corona are sharp when all graphs considered are the complete graph K_n. We end the paper with some concluding remarks on our main results in Section 5.

2 Preliminaries

In this section, we recall the definitions of the corona and edge corona of two graphs and present some well-known results that are required to prove our results in the next two sections.

Definition 2.1. [10] *Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices, respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2.*

Note that the corona $G_1 \circ G_2$ has $n_1(n_2+1)$ vertices and $n_1n_2 + n_1m_2 + m_1$ edges, where m_1 and m_2 denote the number of edges of G_1 and G_2, respectively.

Definition 2.2. [15] *Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices with m_1 and m_2 edges, respectively. The edge corona $G_1 \diamond G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2 and then joining the two end-vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2.*

Note that the edge corona $G_1 \diamond G_2$ has $n_1 + m_1n_2$ vertices and $m_1 + 2m_1n_2 + m_1m_2$ edges. In [1], Barik et. al. have given a complete description of the spectra of the corona of graphs. In particular, they have computed the Laplacian spectrum of the corona of any two graphs.

Theorem 2.1. [1] *Let G_1 and G_2 be any graphs of order n_1 and n_2 with m_1 and m_2 edges, respectively. Let $G = G_1 \circ G_2$. Let the Laplacian eigenvalues of G_1 and G_2 be respectively $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n_1}$ and $0 = \eta_1 \leq \eta_2 \leq \cdots \leq \eta_{n_2}$. Let*

$$\alpha_i, \overline{\alpha_i} = \frac{\mu_i + n_2 + 1 \pm \sqrt{(n_2+1)^2 - 4\mu_i}}{2}$$

for every μ_i. Then the Laplacian spectrum of G is

$$\begin{pmatrix} \eta_2 + 1 & \ldots & \eta_{n_2} + 1 & \alpha_1 & \overline{\alpha_1} & \ldots & \alpha_{n_1} & \overline{\alpha_{n_1}} \\ n_1 & \ldots & n_1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

where entries in the first row are the eigenvalues with the corresponding multiplicities written below.

In [15], Hou and Shiu computed the Laplacian spectrum of the edge corona of a regular graph and an arbitrary graph.

Theorem 2.2. [15] *Let* G_1 *be an* r_1*-regular* $(r_1 \geq 2)$ *graph of order* n_1 *with* m_1 *edges and let* G_2 *be any graph of order* n_2 *with* m_2 *edges. Let* $G = G_1 \diamond G_2$*. Let the Laplacian eigenvalues of* G_1 *and* G_2 *be respectively* $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n_1}$ *and* $0 = \eta_1 \leq \eta_2 \leq \cdots \leq \eta_{n_2}$*. Let*

$$\beta_i, \overline{\beta_i} = \frac{r_1 n_2 + \mu_i + 2 \pm \sqrt{(r_1 n_2 + \mu_i + 2)^2 - 4(n_2 + 2)\mu_i}}{2}$$

for every μ_i*. Then the Laplacian spectrum of* G *is*

$$\begin{pmatrix} \eta_1 + 2 & \eta_2 + 2 & \ldots & \eta_{n_2} + 2 & \beta_1 & \overline{\beta_1} & \ldots & \beta_{n_1} & \overline{\beta_{n_1}} \\ m_1 - n_1 & m_1 & \ldots & m_1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

where entries in the first row are the eigenvalues with the corresponding multiplicities written below.

In [25], Wang and Zhou computed the signless Laplacian spectrum of the corona of an arbitrary graph and a regular graph and they also computed the signless Laplacian spectrum of the edge corona of a connected regular graph and a regular graph.

Theorem 2.3. [25] *Let* G_1 *be any graph of order* n_1 *with* m_1 *edges and let* G_2 *be an* r_2*-regular graph of order* n_2 *with* m_2 *edges. Let* $G = G_1 \circ G_2$*. Let the signless Laplacian eigenvalues of* G_1 *and* G_2 *be respectively* $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{n_1}$ *and* $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n_2}$*. Let*

$$\gamma_i, \overline{\gamma_i} = \frac{\theta_i + n_2 + 1 + 2r_2 \pm \sqrt{(\theta_i + n_2 + 1 + 2r_2)^2 - 4[2r_2 n_2 + (2r_2 + 1)\theta_i]}}{2}$$

for every θ_i*. Then the signless Laplacian spectrum of* G *is*

$$\begin{pmatrix} \tau_1 + 1 & \ldots & \tau_{n_2 - 1} + 1 & \gamma_1 & \overline{\gamma_1} & \ldots & \gamma_{n_1} & \overline{\gamma_{n_1}} \\ n_1 & \ldots & n_1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

where entries in the first row are the eigenvalues with the corresponding multiplicities written below.

Theorem 2.4. [25] *Let* G_1 *be a connected* r_1*-regular* $(r_1 \geq 1)$ *of order* n_1 *with* m_1 *edges and let* G_2 *be an* r_2*-regular graph of order* n_2 *with* m_2 *edges. Let* $G = G_1 \diamond G_2$*. Let the signless Laplacian eigenvalues of* G_1 *and* G_2 *be respectively* $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{n_1}$ *and* $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n_2}$*. Let*

$$\delta_i, \overline{\delta_i} = \frac{r_1 n_2 + 2 + 2r_2 + \theta_i \pm \sqrt{(r_1 n_2 + 2 + 2r_2 + \theta_i)^2 - 4[(r_1 n_2 + \theta_i)(2 + 2r_2) - n_2 \theta_i]}}{2}$$

for every θ_i*. Then the signless Laplacian spectrum of* G *when* $r_1 \geq 2$ *is*

$$\begin{pmatrix} \tau_1 + 2 & \tau_2 + 2 & \ldots & \tau_{n_2 - 1} + 2 & \tau_{n_2} + 2 & \delta_1 & \overline{\delta_1} & \ldots & \delta_{n_1} & \overline{\delta_{n_1}} \\ m_1 & m_1 & \ldots & m_1 & m_1 - n_1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

where entries in the first row are the eigenvalues with the corresponding multiplicities written below. Also if $r_1 = 1$*, then the signless Laplacian eigenvalues of* G *are* $\tau_1 + 2, \tau_2 + 2, \ldots, \tau_{n_2 - 1} + 2, n_2, \delta_2$ *and* $\bar{\delta_2}$*.*

Definition 2.3. [16] *The first Zagreb index of G is defined as*

$$M_1(G) = \sum_{v_i \in V(G)} d_i^{\,2}.$$

Remark 2.5. *We note that* $\sum_{i=1}^{n} \mu_i^{\,2} = tr(D-A)^2 = \sum_{i=1}^{n} d_i^{\,2} + \sum_{i=1}^{n} d_i = M_1(G) + 2m$. *Thus* $\sum_{i=1}^{n} \mu_i^{\,2} = M_1(G) + rn$ *if G is r-regular.*

The first Zagreb index satisfies the following well-known upper bound. The inequality is obtained in [9] and the characterisation of the graphs is proved in [7].

Lemma 2.1. [7,9] *Let G be a graph of order n with $m \geq 1$ edges. Then the first Zagreb index of G satisfies the following.*

$$M_1(G) \leq m\left(\frac{2m}{n-1} + n - 2\right)$$

with the equality if and only if $G \cong S_n$ or K_n or K_n with one isolated vertex, where S_n denotes the star graph of n vertices.

Lemma 2.2. [16] *Let G be a graph of order n with m edges. Then the first Zagreb index of G satisfies the following.*

$$M_1(G) \geq \frac{4m^2}{n}$$

The equality holds if and only if G is isomorphic to a regular graph.

Lemma 2.3. [8] *Let G be a graph of order n with $m \geq 1$ edges. Let $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ denote the Laplacian eigenvalues of G. Then $\mu_2 = \mu_3 = \cdots = \mu_n$ if and only if $G \cong K_n$.*

Lemma 2.4. [5] *Let G be a graph of order n with $m \geq 1$ edges. Let $q_1 \leq q_2 \cdots \leq q_n$ denote the signless Laplacian eigenvalues of G. Then $q_1 = q_2 = \cdots = q_{n-1}$ if and only if $G \cong K_n$.*

The following inequality by Ozeki is a very useful one for obtaining lower bounds.

Lemma 2.5 (Ozeki's inequality)**.** [22] *Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ be two positive n-tuples with $0 < p \leq a_i \leq P$ and $0 < q \leq b_i \leq Q$, for $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^{n} a_i^{\,2} \sum_{i=1}^{n} b_i^{\,2} - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{1}{4} n^2 (PQ - pq)^2.$$

If $b_i = 1$ for all $i = 1, \ldots, n$ and $q = Q = 1$ in Ozeki's inequality, then we obtain the following particular case which is often used in our results.

Corollary 2.1. *Let $\alpha = (a_1, \ldots, a_n)$ be a positive n-tuples with $0 < p \leq a_i \leq P$ for $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^{n} a_i \geq \sqrt{n \sum_{i=1}^{n} a_i^{\,2} - \frac{1}{4} n^2 (P - p)^2}.$$

Izumino et. al. proved the following remarkable result on Ozeki's inequality.

Lemma 2.6. [17] *Let* $\alpha=(a_1,\dots,a_n)$ *and* $\beta=(b_1,\dots,b_n)$ *be two* n*-tuples with* $0\le p\le a_i\le P$ *and* $0\le q\le b_i\le Q$*, and* $PQ\neq 0$*, for* $i=1,\dots,n$*. Let* $\alpha=\dfrac{p}{P}$ *and* $\beta=\dfrac{q}{Q}$*. If* $(1+\alpha)(1+\beta)\ge 2$*, then Ozeki's inequality holds.*

We also recall the following well-known bounds on the Laplacian-energy-like invariant for an r-regular graph.

Lemma 2.7. [3] *If* G *is an* r*-regular graph of order* n*, then*

$$\frac{nr}{\sqrt{r+1}}\le LEL(G)\le\sqrt{r+1}+\sqrt{(n-2)(nr-r-1)}$$

with both equalities if and only if $G\cong K_n$*.*

3 Laplacian-energy-like invariant

In this section, we give the bounds on the Laplacian-energy-like invariant of $G_1\circ G_2$ and $G_1\diamond G_2$. We observe that the bounds are sharp. First we consider the Laplacian-energy-like invariant of $G_1\circ G_2$.

Theorem 3.1. *Let* G_1 *and* G_2 *be any graphs of order* n_1 *and* n_2 *with* m_1 *and* m_2 *edges, respectively. Let* $G=G_1\circ G_2$*. Then we have the following.*

(i) $LEL(G)\le n_1\sqrt{\dfrac{2m_1}{n_1}+n_2+1+\sqrt{\dfrac{M_1(G_1)+2m_1(2n_2+7)}{n_1}}}+n_1(n_2-1)\sqrt{\dfrac{2m_2}{n_2-1}+1}$,

where the equality holds if and only if $G_1\cong K_{n_1}$ *and* $G_2\cong K_{n_2}$*.*

(ii) $$LEL(G)>n_1\sqrt{\left(\frac{2}{n_1}-1\right)m_1+n_2+1+\sqrt{\frac{M_1(G_1)+2m_1(2n_2+7)}{n_1}-m_1(m_1+n_2+3)}}$$
$$+n_1(n_2-1)\sqrt{\frac{2m_2}{n_2-1}+1-\frac{m_2}{2}}.$$

Proof. Let the Laplacian eigenvalues of G_1 and G_2 be $0=\mu_1\le\mu_2\le\cdots\le\mu_{n_1}$ and $0=\eta_1\le\eta_2\le\cdots\le\eta_{n_2}$, respectively. Then from Theorem 2.1 we have

$$\begin{aligned}
LEL(G)&=\sum_{i=1}^{n_1}\left(\sqrt{\frac{\mu_i+n_2+1+\sqrt{(n_2+1)^2-4\mu_i}}{2}}+\sqrt{\frac{\mu_i+n_2+1-\sqrt{(n_2+1)^2-4\mu_i}}{2}}\right)\\
&\quad+n_1\sum_{j=2}^{n_2}\sqrt{\eta_j+1}\\
&=\sum_{i=1}^{n_1}\sqrt{\left(\sqrt{\frac{\mu_i+n_2+1+\sqrt{(n_2+1)^2-4\mu_i}}{2}}+\sqrt{\frac{\mu_i+n_2+1-\sqrt{(n_2+1)^2-4\mu_i}}{2}}\right)^2}\\
&\quad+n_1\sum_{j=2}^{n_2}\sqrt{\eta_j+1}\\
&=\sum_{i=1}^{n_1}\sqrt{\mu_i+n_2+1+\sqrt{\mu_i{}^2+2(n_2+3)\mu_i}}+n_1\sum_{j=2}^{n_2}\sqrt{\eta_j+1} \qquad (1)
\end{aligned}$$

Note that $\sum_{i=1}^{n_1} \mu_i = 2m_1$ and $\sum_{j=2}^{n_2} \eta_j = 2m_2$. Also by Remark 2.5, $\sum_{i=1}^{n_1} {\mu_i}^2 = 2m_1 + M_1(G_1)$. Applying the Cauchy–Bunyakovsky–Schwarz inequality in (1), we have

$$LEL(G) \leq \sqrt{n_1 \sum_{i=1}^{n_1} \left(\mu_i + n_2 + 1 + \sqrt{{\mu_i}^2 + 2(n_2+3)\mu_i}\right)} + n_1 \sqrt{(n_2 - 1) \sum_{j=2}^{n_2} (\eta_j + 1)}$$

$$\leq \sqrt{n_1 \sum_{i=1}^{n_1} \mu_i + {n_1}^2(n_2+1) + n_1 \sqrt{n_1 \sum_{i=1}^{n_1} [{\mu_i}^2 + 2(n_2+3)\mu_i]}} + n_1 \sqrt{(n_2-1)(2m_2 + n_2 - 1)}$$

$$= \sqrt{2m_1 n_1 + {n_1}^2(n_2+1) + n_1 \sqrt{n_1 M_1(G_1) + 2m_1 n_1 (2n_2 + 7)}} + n_1(n_2 - 1)\sqrt{\frac{2m_2}{n_2 - 1} + 1}$$

$$= n_1 \sqrt{\frac{2m_1}{n_1} + n_2 + 1 + \sqrt{\frac{M_1(G_1) + 2m_1(2n_2+7)}{n_1}}} + n_1(n_2-1)\sqrt{\frac{2m_2}{n_2-1} + 1},$$

where the above is an equality if and only if $\mu_2 = \cdots = \mu_{n_1}$ and $\eta_2 = \cdots = \eta_{n_2}$ and so Lemma 2.3 implies that the equality holds if and only if $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$. This completes the proof of (i).

Next we prove (ii). Let $a_i = \sqrt{\mu_i + n_2 + 1 + \sqrt{{\mu_i}^2 + 2(n_2+3)\mu_i}}$ for $i = 1, 2, \ldots, n_1$. Let $P = \sqrt{2m_1 + n_2 + 1 + \sqrt{4{m_1}^2 + 4(n_2+3)m_1}}$ and $p = \sqrt{n_2 + 1}$. Since $0 \leq \mu_i \leq 2m_1$ we have $0 < p \leq a_i \leq P$. Notice that $2 + P^2 < P^2 + 2Pp + n_2 + 1 = (P+p)^2$. So

$$(P-p)^2 = \frac{(P^2 - p^2)^2}{(P+p)^2} = \frac{4m_1(2+P^2)}{(P+p)^2} < 4m_1.$$

Thus by Corollary 2.1, we have

$$\sum_{i=1}^{n_1} \sqrt{\mu_i + n_2 + 1 + \sqrt{{\mu_i}^2 + 2(n_2+3)\mu_i}} \geq \sqrt{n_1 \sum_{i=1}^{n_1} \left(\mu_i + n_2 + 1 + \sqrt{{\mu_i}^2 + 2(n_2+3)\mu_i}\right) - {n_1}^2 m_1} \tag{2}$$

Now let $a_i = \sqrt{{\mu_i}^2 + 2(n_2+3)\mu_i}$ and $b_i = 1$ for $i = 1, \ldots, n_1$. Let $P = \sqrt{4{m_1}^2 + 4(n_2+3)m_1}$, $p = 0$ and $q = Q = 1$. Since $0 \leq \mu_i \leq 2m_1$ we have $0 \leq p \leq a_i \leq P$ and $0 \leq q \leq b_i \leq Q$, $PQ \neq 0$. Notice that $(PQ - pq)^2 = 4m_1(m_1 + n_2 + 3))$. Since $\left(1 + \frac{p}{P}\right)\left(1 + \frac{q}{Q}\right) = 2$, by Lemma 2.6, we have

$$\sum_{i=1}^{n_1} \sqrt{{\mu_i}^2 + 2(n_2+3)\mu_i} \geq \sqrt{n_1 M_1(G_1) + 2m_1 n_1(2n_2+7) - {n_1}^2 m_1(m_1 + n_2 + 3)} \tag{3}$$

Finally let $a_j = \sqrt{\eta_j + 1}$ for $j = 2, 3, \ldots, n_2$. We choose $P = \sqrt{2m_2 + 1}$ and $p = 1$. Since $0 \leq \eta_j \leq 2m_2$ we have $0 < p \leq a_j \leq P$. Notice that $2m_2 < (P+p)^2$. So

$$(P-p)^2 = \frac{(P^2-p^2)^2}{(P+p)^2} = \frac{4{m_2}^2}{(P+p)^2} < 2m_2.$$

Thus by Corollary 2.1, we have:

$$\begin{aligned}\sum_{j=2}^{n_2}\sqrt{\eta_j+1} &\geq \sqrt{(n_2-1)\sum_{j=2}^{n_2}(\eta_j+1)-\frac{1}{4}(n_2-1)^2(P-p)^2}\\ &> \sqrt{2m_2(n_2-1)+(n_2-1)^2-(n_2-1)^2\frac{m_2}{2}}\\ &= n_1(n_2-1)\sqrt{\frac{2m_2}{n_2-1}+1-\frac{m_2}{2}} \qquad (4)\end{aligned}$$

From (1), (2), (3) and (4) we obtain the required result (ii). □

Corollary 3.1. *Let G_1 and G_2 be r_1- and r_2- regular graphs $(r_1, r_2 \geq 2)$ of order $n_1(\geq 3)$ and n_2 with m_1 and m_2 edges, respectively. Let $G = G_1 \circ G_2$. Then we have the following.*

$$(i)\ LEL(G) \leq n_1\sqrt{r_1+n_2+1+\sqrt{r_1(2n_2+7)+\frac{m_1}{n_1}\left(\frac{r_1n_1}{n_1-1}+n_1-2\right)}} + n_1(n_2-1)\sqrt{\frac{r_2n_2}{n_2-1}+1},$$

where the equality holds if and only if $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$.

$$(ii)\ LEL(G) > n_1\sqrt{n_2+1+\sqrt{r_1(n_2+4)}} + n_1(n_2-1)\sqrt{\frac{r_2n_2}{n_2-1}+1-\frac{r_2n_2}{4}}.$$

Proof. Since G_1 and G_2 are r_1 and r_2 regular graphs respectively, $2m_1 = r_1n_1$ and $2m_2 = r_2n_2$; the proof of (i) directly follows from Lemma 2.1. For the proof of (ii) we use the fact that $0 \leq \mu_i \leq 2r_1$ and $0 \leq \eta_j \leq 2r_2$ while applying Corollary 2.1 as in the Theorem 3.1. The proof of (ii) then follows immediately from Lemma 2.2. □

Next we consider the Laplacian-energy-like invariant of $G_1 \diamond G_2$.

Theorem 3.2. *Let G_1 be an r_1-regular $(r_1 \geq 2)$ graph of order n_1 with m_1 edges and let G_2 be any graph of order n_2 with m_2 edges. Let $G = G_1 \diamond G_2$. Then we have the following.*

$$(i)\ LEL(G) \leq n_1\sqrt{r_1n_2+r_1+2+\frac{2\sqrt{n_2+2}}{n_1}LEL(G_1)+\sqrt{2}(m_1-n_1)} + m_1(n_2-1)\sqrt{\frac{2m_2}{n_2-1}+2},$$

where the equality holds if and only if $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$.

$$(ii)\ LEL(G) > n_1\sqrt{r_1n_2+\frac{r_1}{2}+2+\frac{2\sqrt{n_2+2}}{n_1}\left(LEL(G_1)-\frac{\sqrt{r_1}n_1}{2\sqrt{2}}\right)+\sqrt{2}(m_1-n_1)} + m_1(n_2-1)\sqrt{\frac{2m_2}{n_2-1}+2-\frac{m_2}{2}}.$$

Proof. Let the Laplacian eigenvalues of G_1 and G_2 be respectively $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n_1}$ and $0 = \eta_1 \leq \eta_2 \leq \cdots \leq \eta_{n_2}$. Let

$$\beta_i, \overline{\beta_i} = \frac{r_1n_2+\mu_i+2 \pm \sqrt{(r_1n_2+\mu_i+2)^2-4(n_2+2)\mu_i}}{2}$$

for every μ_i. Then from Theorem 2.2 and a simple computation, we have

$$LEL(G)=\sum_{i=1}^{n_1}\left(\sqrt{\beta_i}+\sqrt{\overline{\beta_i}}\right)+(m_1-n_1)\sqrt{\eta_1+2}+m_1\sum_{j=2}^{n_2}\sqrt{\eta_j+2}$$
$$=\sum_{i=1}^{n_1}\sqrt{\left(\sqrt{\beta_i}+\sqrt{\overline{\beta_i}}\right)^2}+(m_1-n_1)\sqrt{\eta_1+2}+m_1\sum_{j=2}^{n_2}\sqrt{\eta_j+2}$$
$$=\sum_{i=1}^{n_1}\sqrt{r_1n_2+\mu_i+2+2\sqrt{(n_2+2)\mu_i}}+(m_1-n_1)\sqrt{\eta_1+2}+m_1\sum_{j=2}^{n_2}\sqrt{\eta_j+2} \quad (5)$$

Note that $\sum_{i=1}^{n_1}\mu_i=2m_1=r_1n_1$, $\sum_{j=2}^{n_2}\eta_j=2m_2$ and $\eta_1=0$.

Applying Cauchy–Bunyakovsky–Schwarz inequality in (5), we have

$$LEL(G)\leq\sqrt{n_1\sum_{i=1}^{n_1}\left(r_1n_2+\mu_i+2+2\sqrt{(n_2+2)\mu_i}\right)}+(m_1-n_1)\sqrt{2}+m_1\sum_{j=2}^{n_2}\sqrt{\eta_j+2}$$
$$\leq n_1\sqrt{r_1n_2+r_1+2+\frac{2\sqrt{n_2+2}}{n_1}LEL(G_1)}+\sqrt{2}(m_1-n_1)$$
$$+m_1(n_2-1)\sqrt{\frac{2m_2}{n_2-1}+2},$$

where the above is an equality if and only if $\mu_2=\cdots=\mu_{n_1}$ and $\eta_2=\cdots=\eta_{n_2}$ and so Lemma 2.3 implies that the equality holds if and only if $G_1\cong K_{n_1}$ and $G_2\cong K_{n_2}$. This completes the proof of (i).

Next we prove (ii). Let $a_i=\sqrt{r_1n_2+\mu_i+2+2\sqrt{(n_2+2)\mu_i}}$ for $i=1,2,\ldots,n_1$. We choose $P=\sqrt{r_1(n_2+2)+2+2\sqrt{2r_1(n_2+2)}}$ and $p=\sqrt{r_1n_2+2}$. Since $0\leq\mu_i\leq 2r_1$ we have $0<p\leq a_i\leq P$. Notice that $(P-p)^2=[2r_1+2\sqrt{2r_1(n_2+2)}]^2/(P+p)^2<2\left(r_1+\sqrt{2r_1(n_2+2)}\right)$. Thus by Corollary 2.1, we have

$$\sum_{i=1}^{n_1}\sqrt{r_1n_2+\mu_i+2+2\sqrt{(n_2+2)\mu_i}}$$
$$>\sqrt{n_1\sum_{i=1}^{n_1}\left(r_1n_2+\mu_i+2+2\sqrt{(n_2+2)\mu_i}\right)-\frac{{n_1}^2}{2}\left(r_1+\sqrt{2r_1(n_2+2)}\right)}$$
$$=n_1\sqrt{r_1n_2+\frac{r_1}{2}+2+\frac{2\sqrt{n_2+2}}{n_1}\left(LEL(G_1)-\frac{\sqrt{r_1}n_1}{2\sqrt{2}}\right)} \quad (6)$$

Next let $a_j=\sqrt{\eta_j+2}$ for $j=2,3,\ldots,n_2$. We choose $P=\sqrt{2m_2+2}$ and $p=\sqrt{2}$. Since $0\leq\eta_j\leq 2m_2$ we have $0<p\leq a_j\leq P$. Notice that $(P-p)^2=\frac{4{m_2}^2}{(P+p)^2}<2m_2$. Thus by Corollary 2.1, we have

$$\sum_{j=2}^{n_2} \sqrt{\eta_j + 2} > \sqrt{2m_2(n_2-1) + (n_2-1)^2\left(2 - \frac{m_2}{2}\right)}$$
$$= (n_2-1)\sqrt{\frac{2m_2}{n_2-1} + 2 - \frac{m_2}{2}} \tag{7}$$

From (5), (6) and (7) we obtain the required result (ii). □

Corollary 3.2. *Let G_1 and G_2 be r_1- and r_2- regular graphs $(r_1, r_2 \geq 2)$ of order n_1 and n_2 with m_1 and m_2 edges, respectively. Let $G = G_1 \circ G_2$. Then we have the following.*

i) $$LEL(G) \leq n_1\sqrt{r_1n_2 + r_1 + 2 + \frac{2\sqrt{n_2+2}}{n_1}\left(\sqrt{r_1+1} + \sqrt{(n_1-2)(r_1n_1 - r_1 - 1)}\right)} + \sqrt{2}(m_1 - n_1) + m_1(n_2-1)\sqrt{\frac{r_2n_2}{n_2-1} + 2},$$

where the equality holds if and only if $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$.

ii) $$LEL(G) > n_1\sqrt{r_1n_2 + \frac{r_1}{2} + 2 + 2\sqrt{n_2+2}\left(\frac{r_1}{\sqrt{r_1+1}} - \frac{\sqrt{r_1}}{2\sqrt{2}}\right)} + \sqrt{2}(m_1 - n_1) + m_1(n_2-1)\sqrt{\frac{r_2n_2}{n_2-1} + 2 - \frac{r_2}{2}}.$$

Proof. Since $2m_2 = r_2n_2$ the proof follows from Lemma 2.7. □

4 Incidence energy

In this section, we give the bounds on the incidence energy of $G_1 \circ G_2$ and $G_1 \diamond G_2$. We observe that the bounds are sharp. Now we consider the incidence energy of $G_1 \circ G_2$.

Theorem 4.1. *Let G_1 be any graph of order n_1 with m_1 edges and let G_2 be an r_2-regular graph of order n_2 with m_2 edges. Let $G = G_1 \circ G_2$. Then we have the following.*

(i) $$IE(G) \leq n_1\sqrt{\frac{2m_1}{n_1} + n_2 + 1 + 2r_2 + 2\sqrt{2r_2n_2 + \frac{2m_1(2r_2+1)}{n_1}}} + n_1(n_2-1)\sqrt{\frac{r_2(n_2-2)}{n_2-1} + 1},$$

where the equality holds if and only if $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$.

(ii) $$IE(G) > n_1\sqrt{\frac{2m_1}{n_1} + n_2 + 1 + 2r_2 - {m_1}^2 + 2\sqrt{2r_2n_2 + \frac{2m_1(2r_2+1)}{n_1} - \frac{1}{2}m_1(2r_2+1)}} + n_1(n_2-1)\sqrt{\frac{r_2(n_2-3)}{2(n_2-1)} + 1}.$$

Proof. Let the signless Laplacian eigenvalues of G_1 and G_2 be respectively $0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_{n_1}$ and $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n_2}$. Let

$$\gamma_i, \overline{\gamma_i} = \frac{\theta_i + n_2 + 1 + 2r_2 \pm \sqrt{(\theta_i + n_2 + 1 + 2r_2)^2 - 4[2r_2n_2 + (2r_2+1)\theta_i]}}{2}$$

for every θ_i. Then from Theorem 2.3 we have

$$IE(G) = \sum_{i=1}^{n_1}\left(\sqrt{\gamma_i}+\sqrt{\bar{\gamma}_i}\right) + n_1\sum_{j=1}^{n_2-1}\sqrt{\tau_j+1} = \sum_{i=1}^{n_1}\sqrt{\left(\sqrt{\gamma_i}+\sqrt{\bar{\gamma}_i}\right)^2} + n_1\sum_{j=1}^{n_2-1}\sqrt{\tau_j+1}$$

$$= \sum_{i=1}^{n_1}\sqrt{\theta_i+n_2+1+2r_2+2\sqrt{2r_2n_2+(2r_2+1)\theta_i}} + n_1\sum_{j=1}^{n_2-1}\sqrt{\tau_j+1} \qquad (8)$$

Note that $\sum_{i=1}^{n_1}\theta_i = 2m_1$. Since $\sum_{j=1}^{n_2}\tau_j = r_2n_2$ and $\tau_{n_2} = 2r_2$, $\sum_{j=1}^{n_2-1}\tau_j = \sum_{j=1}^{n_2}\tau_j - \tau_{n_2} = (n_2-2)r_2$. Applying the Cauchy–Bunyakovsky–Schwarz inequality in (8), we have

$$IE(G) \le \sqrt{n_1\sum_{i=1}^{n_1}\left(\theta_i+n_2+1+2r_2+2\sqrt{2r_2n_2+(2r_2+1)\theta_i}\right)} + n_1\sqrt{(n_2-1)\sum_{j=1}^{n_2-1}(\tau_j+1)}$$

$$= \sqrt{n_1\sum_{i=1}^{n_1}(\theta_i+n_2+1+2r_2) + 2n_1\sum_{i=1}^{n_1}\sqrt{2r_2n_2+(2r_2+1)\theta_i}}$$

$$+ n_1\sqrt{(n_2-1)\sum_{j=1}^{n_2-1}(\tau_j+1)}$$

$$\le \sqrt{n_1[2m_1+n_1(n_2+1+2r_2)] + 2n_1\sqrt{n_1\sum_{i=1}^{n_1}[2r_2n_2+(2r_2+1)\theta_i]}}$$

$$+ n_1\sqrt{(n_2-1)\sum_{j=1}^{n_2-1}(\tau_j+1)}$$

$$= n_1\sqrt{\frac{2m_1}{n_1}+n_2+1+2r_2+2\sqrt{2r_2n_2+\frac{2m_1(2r_2+1)}{n_1}}} + n_1(n_2-1)\sqrt{\frac{r_2(n_2-2)}{n_2-1}+1},$$

where the above is an equality if and only if $\theta_1 = \cdots = \theta_{n_1}$ and $\tau_1 = \cdots = \tau_{n_2-1}$ and so Lemma 2.4 implies that the equality holds if and only if $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$. This completes the proof of (i).

We now prove (ii). Let $a_i = \sqrt{\theta_i+n_2+1+2r_2+2\sqrt{2r_2n_2+(2r_2+1)\theta_i}}$ for $i = 1, 2, \ldots, n_1$. We choose $P = \sqrt{2m_1+n_2+1+2r_2+2\sqrt{2r_2n_2+2m_1(2r_2+1)}}$ and $p = \sqrt{n_2+1+2r_2+2\sqrt{2r_2n_2}}$. Since $0 \le \theta_i \le 2m_1$ we have $0 < p \le a_i \le P$. Notice that

$$(P-p)^2 = \frac{(P^2-p^2)^2}{(P+p)^2} = \frac{4}{(P+p)^2}\left[m_1+\sqrt{2r_2n_2+2m_1(2r_2+1)}-\sqrt{2r_2n_2}\right]^2$$

$$= \frac{4}{(P+p)^2}\left[m_1+\frac{2m_1(2r_2+1)}{\sqrt{2r_2n_2+2m_1(2r_2+1)}+\sqrt{2r_2n_2}}\right]^2$$

Since $(P+p)^2\left[\sqrt{2r_2n_2+2m_1(2r_2+1)}+\sqrt{2r_2n_2}\right] > \sqrt{2r_2n_2+2m_1(2r_2+1)}+\sqrt{2r_2n_2}+2(2r_2+1)$, we have $(P-p)^2 < 4m_1^2$. Thus by Corollary 2.1, we have

$$\sum_{i=1}^{n_1}\sqrt{\theta_i+n_2+1+2r_2+2\sqrt{2r_2n_2+(2r_2+1)\theta_i}}$$
$$\geq\sqrt{n_1\sum_{i=1}^{n_1}\left(\theta_i+n_2+1+2r_2+2\sqrt{2r_2n_2+(2r_2+1)\theta_i}\right)-{n_1}^2{m_1}^2} \tag{9}$$

Let $a_i=\sqrt{2r_2n_2+(2r_2+1)\theta_i}$ for $i=1,\ldots,n_1$. Let $P=\sqrt{2r_2n_2+2m_1(2r_2+1)}$ and $p=\sqrt{2r_2n_2}$. Since $0\leq\theta_i\leq 2m_1$ we have $0<p\leq a_i\leq P$. Notice that since $(P+p)^2>2m_1(2r_2+1)$ we have $(P-p)^2=\dfrac{[2m_1(2r_2+1)]^2}{(P+p)^2}<2m_1(2r_2+1)$. Thus by Corollary 2.1, we have

$$\sum_{i=1}^{n_1}\sqrt{2r_2n_2+(2r_2+1)\theta_i}>\sqrt{n_1\sum_{i=1}^{n_1}[2r_2n_2+(2r_2+1)\theta_i]-\frac{1}{2}{n_1}^2m_1(2r_2+1)} \tag{10}$$

Finally let $a_j=\sqrt{\tau_j+1}$ for $j=1,2,\ldots,n_2-1$. We choose $P=\sqrt{2r_2+1}$ and $p=1$. Since $0\leq\tau_j\leq 2r_2$ we have $0<p\leq a_j\leq P$. Notice that $(P-p)^2=\dfrac{4{r_2}^2}{(P+p)^2}<2r_2$. Thus by Corollary 2.1, we have

$$\sum_{j=1}^{n_2-1}\sqrt{\tau_j+1}>\sqrt{(n_2-1)\sum_{j=1}^{n_2-1}(\tau_j+1)-\frac{1}{2}(n_2-1)^2r_2}=(n_2-1)\sqrt{\frac{r_2(n_2-3)}{2(n_2-1)}+1} \tag{11}$$

From (8), (9), (10) and (11) we obtain the required result (ii). □

The following corollary is then immediate if both G_1 and G_2 are regular.

Corollary 4.1. *Let G_1 and G_2 be r_1- and r_2- regular graphs $(r_1,r_2\geq 2)$ of order n_1 and n_2 with m_1 and m_2 edges, respectively. Let $G=G_1\circ G_2$. Then we have the following.*

(i) $IE(G)\leq n_1\sqrt{r_1+n_2+1+2r_2+2\sqrt{2r_2n_2+r_1(2r_2+1)}}+n_1(n_2-1)\sqrt{\dfrac{r_2(n_2-2)}{n_2-1}+1}$, *where the equality holds if and only if $G_1\cong K_{n_1}$ and $G_2\cong K_{n_2}$.*

(ii) $$IE(G)>n_1\sqrt{r_1+n_2+1+2r_2-{r_1}^2+2\sqrt{2r_2n_2+\frac{1}{2}r_1(2r_2+1)}}$$
$$+n_1(n_2-1)\sqrt{\frac{r_2(n_2-3)}{2(n_2-1)}+1}.$$

Finally, we consider the incidence energy of $G_1\diamond G_2$.

Theorem 4.2. *Let G_1 be a connected r_1-regular $(r_1\geq 2)$ graph of order n_1 with m_1 edges and let G_2 be an r_2-regular graph of order n_2 with m_2 edges. Let $G=G_1\diamond G_2$. Then we have the following.*

(i) $$IE(G)\leq n_1\sqrt{r_1n_2+2+2r_2+r_1+2\sqrt{r_1(n_2+2r_2n_2+2+2r_2)}}$$
$$+(m_1-n_1)\sqrt{2(2r_2+1)}+m_1(n_2-1)\sqrt{\frac{r_2(n_2-2)}{n_2-1}+2},$$
where the equality holds if and only if $G_1\cong K_{n_1}$ and $G_2\cong K_{n_2}$.

(ii) $IE(G) > n_1\sqrt{r_1n_2+2+2r_2+r_1-{r_1}^2+2\sqrt{r_1\left(\frac{n_2}{2}+2r_2n_2+1+r_2\right)}}$

$$+(m_1-n_1)\sqrt{2(2r_2+1)}+m_1(n_2-1)\sqrt{\frac{r_2(n_2-2)}{n_2-1}+2-\frac{r_2}{2}}.$$

Proof. Let the signless Laplacian eigenvalues of G_1 and G_2 be respectively $0\le\theta_1\le\theta_2\le\ldots\le\theta_{n_1}$ and $0\le\tau_1\le\tau_2\le\cdots\le\tau_{n_2}$. Let

$$\delta_i,\bar{\delta_i}=\frac{r_1n_2+2+2r_2+\theta_i\pm\sqrt{(r_1n_2+2+2r_2+\theta_i)^2-4[(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i]}}{2}$$

for every θ_i. Then from Theorem 2.4 and a simple computation, we have

$$\begin{aligned}IE(G)&=\sum_{i=1}^{n_1}\left(\sqrt{\delta_i}+\sqrt{\bar{\delta_i}}\right)+m_1\sum_{j=1}^{n_2-1}\sqrt{\tau_j+2}+(m_1-n_1)\sqrt{\tau_{n_2}+2}\\&=\sum_{i=1}^{n_1}\sqrt{r_1n_2+2+2r_2+\theta_i+2\sqrt{(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i}}\\&\quad+m_1\sum_{j=1}^{n_2-1}\sqrt{\tau_j+2}+(m_1-n_1)\sqrt{2(\tau_{n_2}+1)}\end{aligned}\tag{12}$$

Note that $\sum_{i=1}^{n_1}\theta_i=r_1n_1$ and $\tau_{n_2}=2r_2$. Also $\sum_{j=1}^{n_2-1}\tau_j=(n_2-2)r_2$. Applying Cauchy–Bunyakovsky–Schwarz inequality in (12), we have

$$\begin{aligned}IE(G)&\le\sqrt{n_1\sum_{i=1}^{n_1}(r_1n_2+2+2r_2+\theta_i)+2n_1\sqrt{n_1\sum_{i=1}^{n_1}[(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i]}}\\&\quad+m_1\sqrt{(n_2-1)\sum_{j=1}^{n_2-1}(\tau_j+2)}+(m_1-n_1)\sqrt{2(2r_2+1)}\\&=n_1\sqrt{r_1n_2+2+2r_2+r_1+2\sqrt{r_1(n_2+2r_2n_2+2+2r_2)}}+(m_1-n_1)\sqrt{2(2r_2+1)}\\&\quad+m_1(n_2-1)\sqrt{\frac{r_2(n_2-2)}{n_2-1}+2},\end{aligned}$$

where the above is an equality if and only if $\theta_1=\cdots=\theta_{n_1}$ and $\tau_1=\cdots=\tau_{n_2-1}$ and so Lemma 2.4 implies that the equality holds if and only if $G_1\cong K_{n_1}$ and $G_2\cong K_{n_2}$. This completes the proof of (i).

Next we prove (ii). Let $a_i=\sqrt{r_1n_2+2+2r_2+\theta_i+2\sqrt{(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i}}$ for $i=1,2,\ldots,n_1$. We choose $P=\sqrt{r_1n_2+2+2r_2+2r_1+2\sqrt{(r_1n_2+2r_1)(2+2r_2)}}$ and $p=\sqrt{r_1n_2+2+2r_2+2\sqrt{2r_1r_2n_2}}$. Since $0\le\theta_i\le 2r_1$ we have $0<p\le a_i\le P$. Notice that

$$(P-p)^2=\frac{4}{(P+p)^2}\left[r_1+\frac{2r_1(n_2+2r_2+2)}{\sqrt{(r_1n_2+2r_1)(2+2r_2)}+\sqrt{2r_1r_2n_2}}\right]^2<4{r_1}^2.$$

Thus by Corollary 2.1, we have

$$\sum_{i=1}^{n_1}\sqrt{r_1n_2+2+2r_2+\theta_i+2\sqrt{(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i}}$$
$$\geq\sqrt{n_1\sum_{i=1}^{n_1}\left(r_1n_2+2+2r_2+\theta_i+2\sqrt{(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i}\right)-{r_1}^2{n_1}^2} \tag{13}$$

Now let $a_i=\sqrt{(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i}$ for $i=1,2,\ldots,n_1$. We choose $p=\sqrt{2r_1r_2n_2}$ and $P=\sqrt{(r_1n_2+2r_1)(2+2r_2)}$. Since $0\leq\theta_i\leq 2r_1$ we have $0<p\leq a_i\leq P$. Since $2r_1(2+2r_2+n_2)<(P+p)^2$ we have $(P-p)^2=\dfrac{[2r_1(2+2r_2+n_2)]^2}{(P+p)^2}<2r_1(2+2r_2+n_2)$. Thus by Corollary 2.1, we have

$$\sum_{i=1}^{n_1}\sqrt{(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i}$$
$$>\sqrt{n_1\sum_{i=1}^{n_1}[(r_1n_2+\theta_i)(2+2r_2)-n_2\theta_i]-\frac{r_1{n_1}^2}{2}(2+2r_2+n_2)} \tag{14}$$

Finally let $a_j=\sqrt{\tau_j+2}$ for $j=1,2,\ldots,n_2-1$. We choose $P=\sqrt{2r_2+2}$ and $p=\sqrt{2}$. Since $0\leq\tau_j\leq 2r_2$ we have $0<p\leq a_j\leq P$. Notice that $(P-p)^2=\dfrac{(2r_2)^2}{(P+p)^2}<2r_2$. Thus by Corollary 2.1, we have

$$\sum_{j=1}^{n_2-1}\sqrt{\tau_j+2}\geq(n_2-1)\sqrt{\frac{r_2(n_2-2)}{n_2-1}+2-\frac{r_2}{2}} \tag{15}$$

From (12), (13), (14) and (15) we obtain the required result (ii). □

The following corollary immediately follows from Theorem 2.4 and Theorem 4.2.

Corollary 4.2. *Let $G_1=K_2$ and let G_2 be an r_2-regular graph of order n_2 with m_2 edges. Let $G=G_1\diamond G_2$. Then we have the following.*

i) $IE(G)\leq\sqrt{n_2+2+2r_2+\theta_2+2\sqrt{(n_2+\theta_2)(2+2r_2)-n_2\theta_2}}+\sqrt{n_2}+(n_2-1)\sqrt{\dfrac{r_2(n_2-2)}{n_2-1}+2},$

where θ_2 is as in Theorem 2.4 and the equality holds if and only if $G_2\cong K_{n_2}$.

ii) $IE(G)>\sqrt{n_2+2+2r_2+\theta_2+2\sqrt{(n_2+\theta_2)(2+2r_2)-n_2\theta_2}}+\sqrt{n_2}+(n_2-1)\sqrt{\dfrac{r_2(n_2-2)}{n_2-1}+2-\dfrac{r_2}{2}}.$

5 Conclusion

We consider some energy-like invariants of a finite simple graph, namely the Laplacian-energy-like invariant and the incidence energy, for important graph operations called corona and edge corona. The corona and edge corona can be used to construct pairs of nonisomorphic graphs with the same spectrum for the adjacency, Laplacian and signless Laplacian matrices. In this paper, we give the bounds on the Laplacian-energy-like invariant and incidence energy of the corona and edge corona of two graphs. We also observe that the bounds on the Laplacian-energy-like invariant and incidence energy of the corona and edge corona are sharp when all graphs considered are the complete graph K_n.

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