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On certain equations and inequalities involving the arithmetical functions $\varphi(n)$ and $d(n)$

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Abstract: By using the results and methods of [1], we will study the equation $\varphi(n) + d(n) = \frac{n}{2}$ and the related inequalities. The equation $\varphi(n) + d^2(n) = 2n$ will be solved, too.
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1 Introduction

Let $\varphi(n)$ and $d(n)$ denote Euler's totient function and the number of divisors function, respectively. It is well-known that $\varphi(1) = d(1) = 1$ and for $n = p_1^{a_1} \dots p_r^{a_r}$ (prime factorization of $n > 1$) one has

$$\varphi(n) = p_1^{a_1-1} \cdots p_r^{a_r-1} \cdots (p_1 - 1) \cdots (p_r - 1) \text{ and } d(n) = (a_1 + 1) \cdots (a_r + 1) \qquad (1)$$

with p_i $(i = 1, 2, \dots, r)$ distinct primes and a_i $(i = 1, 2, \dots r)$ positive integers. In paper [1], we have determined the solutions of equation $\varphi(n) + d(n) = n$, and proved certain related inequalities.

The aim of this note is to study certain new equations, the first one being

$$\varphi(n) + d(n) = \frac{n}{2} \qquad (2)$$

and the related inequalities. Our methods will be based on some methods and results of [1], as well as certain other inequalities, combined with computer computations.

2 Main results

First we state some auxiliary results. The first one is proved in [1]:

Lemma 1. *For all odd and composite integer M one has*

$$\varphi(M) + d(M) \leq M \tag{3}$$

with equality only for $M = 9$.

The following inequality was first proved by W. Sierpiński [4]:

Lemma 2. *If M is composite, then*

$$\varphi(M) \leq M - \sqrt{M}. \tag{4}$$

The next inequality can be found in [3]:

Lemma 3. *For any integer $n \geq 1$ one has*

$$d(n) < 4 \cdot \sqrt[3]{n}. \tag{5}$$

The main result of this paper is contained in:

Theorem 1. *The only solution of equation* (2) *is $n = 72$. If n is even and not of the form 2^k $(k \geq 1)$; or $2^m \cdot p$ $(m = 1, 2, 3, 4)$ p odd prime; or $n = 18, 30, 36, 50,$ then one has*

$$\varphi(n) + d(n) < \frac{n}{2}. \tag{6}$$

If n even is of the form 2^k or $2^m \cdot p$ $(m = 1, 2, 3, 4)$, p prime or $n = 18, 30, 36, 50,$ then one has

$$\varphi(n) + d(n) > \frac{n}{2}. \tag{7}$$

Proof. First remark that if $n = 2^k$, then $\varphi(n) + d(n) = 2^{k-1} + k + 1 > 2^{k-1} = \frac{n}{2}$. Let now $n = 2^k \cdot M$, where $M > 1$ is composite and odd. Then we can write:

$$\varphi(n) + d(n) = 2^{k+1} \cdot \varphi(M) + (k+1)d(M),$$

as φ and d are multiplicative functions. Now, remark that for all $k \geq 3$ one has the inequality $2^{k-1} \geq k + 1$; with equality only for $k = 3$. Thus, by using this inequality, combined with Lemma 1, we get

$$\varphi(n) + d(n) \leq 2^{k-1} \cdot [\varphi(M) + d(M)] \leq 2^{k-1} \cdot M = \frac{n}{2}.$$

There is equality here only for $k = 3$ and $M = 9$, thus $n = 2^3 \cdot 9 = 72$ is a solution of equation (2).

Now, let $n = 2^k \cdot M$ with $k \in \{1, 2\}$. If $n = 2 \cdot M$, then

$$\varphi(n) + d(n) = \varphi(M) + 2d(M) < M - \sqrt{M} + 8 \cdot \sqrt[3]{M},$$

by Lemmas 2 and 3. Thus, for $M - \sqrt{M} + 8 \cdot \sqrt[3]{M} < \frac{n}{2} = M$ we have $\varphi(n) + d(n) < \frac{n}{2}$. The above inequality is satisfied only if $8 \cdot \sqrt[3]{M} < \sqrt{M}$, i.e., $M > 8^6 = 262144$. A computer verification shows that the only values $n = 2 \cdot M$, with $M < 262144$ for which $\varphi(n) + d(n) < \frac{n}{2}$ are $n \neq 18, 30, 50$ in which cases the inequality is reversed. Also, there is no any solution to (2) for these values.

Let now $n = 4 \cdot M$. Now, by the same argument as above,

$$\varphi(n) + d(n) = 2\varphi(M) + 3d(M) < \frac{n}{2} = 2M$$

is satisfied if $2 \cdot (M - \sqrt{M}) + 12 \cdot \sqrt[3]{M} < 2M$, i.e., $6\sqrt[3]{M} < \sqrt{M}$, or $M > 6^6 = 46656$. A computer verification shows again that for $M \leq 46656$ there is no solution to (2) and that for all $n \neq 36$ one has $\varphi(n) = d(n) < \frac{n}{2}$ for these values.

Let now $n = 2^k \cdot p$, where p is an odd prime. Then

$$\varphi(n) + d(n) = 2^{k-1} \cdot (p - 1) + 2 \cdot (k + 1) \leq 2^{k-1} \cdot p = \frac{n}{2} \Leftrightarrow 2 \cdot (k + 1) \leq 2^{k-1}$$

or $k + 1 \leq 2^{k-2}$. It is immediate that, this inequality is valid for all $k \geq 5$, with strict inequality. Thus, for these values of n one has $\varphi(n) + d(n) < \frac{n}{2}$. Finally, for $n = 2p, 4p, 8p, 16p$ simple verifications show that in all cases we have the reverse inequality. For example, for $n = 8p$ one has

$$\varphi(n) + d(n) = 4(p - 1) + 8 = 4p + 4 > 4p = \frac{n}{2}.$$

These prove the validity of inequalities (6) and (7) in the considered cases. □

Remark 1. *The fact that the equation* (2) *is not solvable for* $n = 2M$ $(M > 1)$ *follows without computer verifications. Indeed, in this case the equation becomes*

$$\varphi(M) + 2d(M) = M \tag{8}$$

and as $M \geq 3$ *it is known from* (1) *that* $\varphi(M)$ *is an even number. So, the left-hand side of equation* (8) *is even, while the right-hand side is odd; which is impossible.*

Remark 2. *The study of inequalities*

$$\varphi(n) + d(n) < \frac{n}{2} \tag{9}$$

and

$$\varphi(n) + d(n) > \frac{n}{2} \tag{10}$$

for n *odd, is a difficult open problem.*

Relation (10) *is satisfied clearly for any* $n = p = \text{prime}$. *On the oher hand, infinitely many solutions for* (9) *are provided, e.g., by* $n = 1155 \cdot p$, *where* $p \geq 13$ *is a prime. Indeed, as* $1155 = 3 \cdot 5 \cdot 7 \cdot 11$, *one has* $\varphi(n) = 2 \cdot 4 \cdot 6 \cdot 10 \cdot (p - 1) = 480(p - 1)$, *and the inequality* $480 \cdot (p - 1) + 32 < \frac{1155}{2} \cdot p$ *holds true, as it becomes* $195p > -896$, *which is trivially true.*

Remark 3. *Other related equations and inequalities can be found in the recent book [2].*

Remark 4. *In [1] it was shown that only a verification is necessary for even* $n < 16$ *for the even solutions of* $\varphi(n) + d(n) = n$*. It was stated that only* $n = 8$ *is acceptable. But* $n = 6$ *is acceptable, too; so that we make a correction here, namely that all solutions to the equation are* $n = 6, 8$ *and* 9*.*

Remark 5. *By the methods of [1] we can study also the equation*

$$\varphi(n) + d^2(n) = 2n. \tag{11}$$

Clearly if $n = p$ *is prime, then* $p - 1 + 4 = 2p$ *only if* $p = 3$*. If* n *is composite, and* $n \geq 1262$*, then by* $d(n) < \sqrt{n}$ *(see [1]) and* $\varphi(n) \leq n - \sqrt{n}$ *we get* $\varphi(n) + d^2(n) < 2n - \sqrt{n} < 2n$*, so for* $n \leq 1261$ *a computer verification is necessary to study* (11)*. We get the following:*

Theorem 2. *All solutions to equation* (11) *are* $n = 3, 10, 40, 84$*.*

In a forthcoming paper, more similar equations involving $\varphi(n)$ and $d(n)$ will be studied.

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