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The Hadamard-type k-step Pell sequences

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Abstract: In this paper, we define the Hadamard-type k-step Pell sequence by using the Hadamard-type product of characteristic polynomials of the Pell sequence and the k-step Pell sequence. Also, we derive the generating matrices for these sequences, and then we obtain relationships between the Hadamard-type k-step Pell sequences and these generating matrices. Furthermore, we produce the Binet formula for the Hadamard-type k-step Pell numbers for the case that k is odd integers and $k \geq 3$. Finally, we derive some properties of the Hadamard-type k-step Pell sequences such as the combinatorial representation, the generating function, and the exponential representation by using its generating matrix.
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1 Introduction

It is well-known that Pell sequence is defined by the following equation:

$$P_{n+1} = 2P_n + P_{n-1}$$

for $n > 0$, where $P_0 = 0$, $P_1 = 1$.

Kılıç and Tascı [11] defined k sequences of the generalized order-k Pell numbers as shown:

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i$$

for $n > 0$ and $1 \le i \le k$, with initial conditions

$$P_n^i = \begin{cases} 1, & \text{if } n = 1 - i, \\ 0, & \text{otherwise}, \end{cases} \qquad 1 - k \le n \le 0,$$

where P_n^i is the n-th term of the i-th sequence.

It is clear that the characteristic polynomials of Pell sequence and the generalized order-k Pell sequence are $P(x) = x^2 - 2x - 1$ and $P_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - 1$, respectively.

Aküzüm and Deveci [1] defined the Hadamard-type product of polynomials f and g as follows:

$$f(x) * g(x) = \sum_{i=0}^{\infty} (a_i * b_i)\, x^i, \text{ where } a_i * b_i = \begin{cases} a_i b_i, & \text{if } \; a_i b_i \neq 0 \\ a_i + b_i, & \text{if } \; a_i b_i = 0 \end{cases},$$

such that $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$.

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [10], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \ge 0$.

The study of linear and recurrent sequences has been known for a long time and miscellaneous properties of these sequences have been studied by some authors; see, for example [2,5–9,14–16]. In [1] Akuzum and Deveci defined the Hadamard-type product of two polynomials and then they obtain the Hadamard-type k-step Fibonacci sequence by the aid of this the Hadamard-type product. In this paper, we define the Hadamard-type k-step Pell sequence. Then we obtain the

Binet formula for the Hadamard-type k-step Pell numbers for the case that k is odd integers and $k \geq 3$. Also, we give the permanental representations, the determinantal representations, the combinatorial representations, the generating function, the exponential representation, and the sums of the Hadamard-type k-step Pell numbers.

2 The Hadamard-type k-step Pell sequences

We define the Hadamard-type k-step Pell sequence by using Hadamard-type product of the characteristic polynomials of Pell sequence and the generalized order-k Pell sequence as shown:

$$h_{n+k}^{k} = 2h_{n+k-1}^{k} + h_{n+k-2}^{k} + \cdots + h_{n+2}^{k} - 2h_{n+1}^{k} - h_{n}^{k} \tag{1}$$

for the integers $n \geq 0$ and $k \geq 3$, with initial constants $h_0^k = h_1^k = \cdots = h_{k-2}^k = 0$ and $h_{k-1}^k = 1$.

By (1), we can write the generating matrix for the Hadamard-type k-step Pell sequence as follows:

$$H_k^p = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 & -2 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}_{k\times k}.$$

The matrix H_k^p is said to be a Hadamard-type k-Pell matrix.

By induction on n, we get

(i) . For $k = 3$,

$$(H_3^p)^n = \begin{bmatrix} h_{n+2}^3 & -2h_{n+1}^3 - h_n^3 & -h_{n+1}^3 \\ h_{n+1}^3 & -2h_n^3 - h_{n-1}^3 & -h_n^3 \\ h_n^3 & -2h_{n-1}^3 - h_{n-2}^3 & -h_{n-1}^3 \end{bmatrix},$$

(ii) . For $k = 4$,

$$(H_4^p)^n = \begin{bmatrix} h_{n+3}^4 & h_{n+4}^4 - 2h_{n+3}^4 & -2h_{n+2}^4 - h_{n+1}^4 & -h_{n+2}^4 \\ h_{n+2}^4 & h_{n+3}^4 - 2h_{n+2}^4 & -2h_{n+1}^4 - h_n^4 & -h_{n+1}^4 \\ h_{n+1}^4 & h_{n+2}^4 - 2h_{n+1}^4 & -2h_n^4 - h_{n-1}^4 & -h_n^4 \\ h_n^4 & h_{n+1}^4 - 2h_n^4 & -2h_{n-1}^4 - h_{n-2}^4 & -h_{n-1}^4 \end{bmatrix},$$

(iii) . For $k \geq 5$,

$$(H_k^p)^n = \begin{bmatrix} h_{n+k-1}^k & h_{n+k}^k - 2h_{n+k-1}^k & & -2h_{n+k-2}^k - h_{n+k-3}^k & -h_{n+k-2}^k \\ h_{n+k-2}^k & h_{n+k-1}^k - 2h_{n+k-2}^k & & -2h_{n+k-3}^k - h_{n+k-4}^k & -h_{n+k-3}^k \\ \vdots & \vdots & H_k^{p*} & \vdots & \vdots \\ h_{n+1}^k & h_{n+2}^k - 2h_{n+1}^k & & -2h_n^k - h_{n-1}^k & -h_n^k \\ h_n^k & h_{n+1}^k - 2h_n^k & & -2h_{n-1}^k - h_{n-2}^k & -h_{n-1}^k \end{bmatrix}, \tag{2}$$

where H_k^{p*} is a $(k) \times (k-4)$ matrix as follows:

$$\begin{bmatrix} h^k_{n+k-2}+h^k_{n+k-3}+\cdots+h^k_{n+3}-2h^k_{n+2}-h^k_{n+1} & h^k_{n+k-2}+h^k_{n+k-3}+\cdots+h^k_{n+4}-2h^k_{n+3}-h^k_{n+2} & \cdots & h^k_{n+k-2}-2h^k_{n+k-3}-h^k_{n+k-4} \\ h^k_{n+k-3}+h^k_{n+k-4}+\cdots+h^k_{n+2}-2h^k_{n+1}-h^k_{n} & h^k_{n+k-3}+h^k_{n+k-4}+\cdots+h^k_{n+3}-2h^k_{n+2}-h^k_{n+1} & \cdots & h^k_{n+k-3}-2h^k_{n+k-4}-h^k_{n+k-5} \\ \vdots & \vdots & \ddots & \vdots \\ h^k_{n}+h^k_{n-1}+\cdots+h^k_{n-k+5}-2h^k_{n-k+4}-h^k_{n-k+3} & h^k_{n}+h^k_{n-1}+\cdots+h^k_{n-k+6}-2h^k_{n-k+5}-h^k_{n-k+4} & \cdots & h^k_{n}-2h^k_{n-1}-h^k_{n-2} \\ h^k_{n-1}+h^k_{n-2}+\cdots+h^k_{n-k+4}-2h^k_{n-k+3}-h^k_{n-k+2} & h^k_{n-1}+h^k_{n-2}+\cdots+h^k_{n-k+5}-2h^k_{n-k+4}-h^k_{n-k+3} & \cdots & h^k_{n-1}-2h^k_{n-2}-h^k_{n-3} \end{bmatrix}$$

for $n \geq k-3$. Also, It is easy to see that $\det H_k^p = (-1)^k$.

We consider the Binet formula for the Hadamard-type k-step Pell sequence with the following theorem.

Lemma 2.1. *Let k be an odd integer such that $k \geq 3$. The characteristic equation of the Hadamard-*
type k-step Pell sequence $x^k - 2x^{k-1} - x^{k-2} - \cdots - x^2 + 2x + 1 = 0$ does not have multiple roots.

Proof. Let $f(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x^2 + 2x + 1$. For $k = 3$, we reach the equation $x^3 - 2x^2 + 2x + 1$. Then, we obtain the roots of this equation as follows:

$$\begin{aligned} x_1 &= 1.1766 - 1.2028\, i, \\ x_2 &= 1.1766 + 1.2028\, i, \quad \text{and} \\ x_3 &= -0.35321. \end{aligned}$$

Thus, it is easily seen that the equation $f(x) = 0$ does not have multiple roots for $k = 3$.

Now, we consider the proof for $k \geq 5$ in which case k is an odd integer number. Suppose that $g(x) = (x-1) f(x) = x^{k+1} - 3x^k + x^{k-1} + 3x^2 - x - 1$. So, we obtain

$$x^k = \frac{-3x^3 + x^2 + x}{x^2 - 3x + 1}. \tag{3}$$

Moreover, it can be written $g'(x) = (k+1)x^k - 3kx^{k-1} + (k-1)x^{k-2} + 6x - 1$ and thus, we get

$$x^k = \frac{-6x^3 + x^2}{(k+1)x^2 - 3kx + k - 1}. \tag{4}$$

From (3) and (4), we reach the equation

$$k = 1 + \frac{8x^3 - 8x^2 + 4x}{-3x^4 + 10x^3 - 5x^2 - 2x + 1}$$

Using an appropriate software such as Wolfram Mathematica 10.0 [17], we obtain that there is no solution for $k \geq 5$. Since k are odd integers such that $k \geq 5$, it is a contradiction. Therefore, the equation $f(x) = 0$ does not have multiple roots. $\square$

If $x_1, x_2, \ldots, x_k$ are roots of the equation $x^k - 2x^{k-1} - x^{k-2} - \cdots - x^2 + 2x + 1$, then by Lemma 2.1, it is known that $x_1, x_2, \ldots, x_k$ are distinct. Let V^k be $k \times k$ Vandermonde matrices as follows:

$$V^k = \begin{bmatrix} (x_1)^{k-1} & (x_2)^{k-1} & \cdots & (x_k)^{k-1} \\ (x_1)^{k-2} & (x_2)^{k-2} & \cdots & (x_k)^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Let

$$R^k(i,j) = \begin{bmatrix} x_1^{n+k-i} \\ x_2^{n+k-i} \\ \vdots \\ x_k^{n+k-i} \end{bmatrix}$$

and suppose that $V^k(i,j)$ is a $k \times k$ matrix obtained from V^k by replacing the j-th column of V^k by $R^k(i,j)$.

Theorem 2.1. *Let k be an odd integer such that $k \geq 3$ and let $(H_k^p)^n = \left[h_{i,j}^{p,k,n}\right]$, then*

$$h_{i,j}^{p,k,n} = \frac{\det V^k(i,j)}{\det V^k},$$

for $n \geq 0$.

Proof. Since $x_1, x_2, \ldots, x_k$ are distinct, the matrix H_k^p is diagonalizable. Then, $H_k^p V^k = V^k S^k$, where $S^k = (x_1, x_2, \ldots, x_k)$. Since V^k is invertible, we can write $\left(V^k\right)^{-1} H_k^p V^k = S^k$. Then, the matrix H_k^p is similar to S^k and so $(H_k^p)^n V^k = V^k \left(S^k\right)^n$. We can now easily establish the following linear system of equations:

$$\begin{cases} h_{i,1}^{p,k,n} x_1^{k-1} + h_{i,2}^{p,k,n} x_1^{k-2} + \cdots + h_{i,k}^{p,k,n} = x_1^{n+k-i} \\ h_{i,1}^{p,k,n} x_2^{k-1} + h_{i,2}^{p,k,n} x_2^{k-2} + \cdots + h_{i,k}^{p,k,n} = x_2^{n+k-i} \\ \vdots \\ h_{i,1}^{p,k,n} x_k^{k-1} + h_{i,2}^{p,k,n} x_k^{k-2} + \cdots + h_{i,k}^{p,k,n} = x_k^{n+k-i} \end{cases}$$

Therefore, for each $i, j = 1, 2, \ldots, k$, we obtain

$$h_{i,j}^{p,k,n} = \frac{\det V^k(i,j)}{\det V^k}.$$

□

The following Corollary gives the Binet formula for the Hadamard-type k-step Pell numbers.

Corollary 2.1. *Let k be an odd integer such that $k \geq 3$ and let h_n^k be the n-th the Hadamard-type k-step Pell number, then*

$$h_n^k = \frac{\det V^k(k,1)}{\det V^k} = -\frac{\det V^k(k-1,k)}{\det V^k}$$

for $n \geq 0$.

Now we consider the permanental representations of the Hadamard-type k-step Pell numbers.

Definition 2.1. *A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k-th column (respectively, row) if the k-th column (respectively, row) contains exactly two non-zero entries.*

Suppose that $x_1, x_2, \dots, x_u$ are row vectors of the matrix M. If M is contractible in the k-th column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i-th row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j-th row. The k-th column is called the contraction in the k-th column relative to the i-th row and the j-th row.

In [3], Brualdi and Gibson obtained that $\operatorname{per}(M) = \operatorname{per}(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $r \geq k$ ($k \geq 3$) be an integer and let $M^{r,k} = \left[m_{i,j}^{r,k}\right]$ be the $r \times r$ super-diagonal matrix, defined by

$$m_{i,j}^{r,k} = \begin{cases} 1, & \begin{array}{l} \text{if } i = t \text{ and } j = t+1 \text{ for } 1 \leq t \leq r-1, \\ \quad i = t \text{ and } j = t+2 \text{ for } 1 \leq t \leq r-2, \\ \quad \vdots \\ \quad i = t \text{ and } j = t+k-3 \text{ for } 1 \leq t \leq r-k+3, \text{ and} \\ \quad i = t \text{ and } j = t-1 \text{ for } 2 \leq t \leq r, \end{array} \\ 2, & \text{if } i = t \text{ and } \ j = t \text{ for } 1 \leq t \leq r, \\ -2, & \text{if } i = t \text{ and } \ j = t+k-2 \text{ for } 1 \leq t \leq r-k+2, \\ -1, & \text{if } i = t \text{ and } \ j = t+k-1 \text{ for } 1 \leq t \leq r-k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following Theorem.

Theorem 2.2. *For $r \geq k$ and $k \geq 3$,*

$$\operatorname{per} M^{r,k} = h_{r+k-1}^{k}$$

Proof. We prove this by mathematical induction. Let the equation be hold for $r \geqslant k$, then we show that the equation holds for $r+1$. If we expand the $\operatorname{per} M^{r,k}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\operatorname{per} M^{r+1,k} = 2 \operatorname{per} M^{r,k} + \operatorname{per} M^{r-1,k} + \cdots + \operatorname{per} M^{r-k+3,k} - 2 \operatorname{per} M^{r-k+2,k} - \operatorname{per} M^{r-k+1,k}.$$

Since $\operatorname{per} M^{r,k} = h_{r+k-1}^{k}, \operatorname{per} M^{r-1,k} = h_{r+k-2}^{k}, \dots, \operatorname{per} M^{r-k+3,k} = h_{r+2}^{k}, \operatorname{per} M^{r-k+2,k} = h_{r+1}^{k}$ and $\operatorname{per} M^{r-k+1,k} = h_{r}^{k}$, it is easy to see that $\operatorname{per} M^{r+1,k} = h_{r+k}^{k}$. Thus, the proof is complete. $\square$

Let $r > k$ and let $N^{r,k} = \left[n_{i,j}^{r,k}\right]$ be the $r \times r$ matrix, defined by

$$n_{i,j}^{r,k} = \begin{cases} 1 & \begin{array}{l} \text{if } i = t \text{ and } j = t+1 \text{ for } 1 \le t \le r-2, \\ \quad i = t \text{ and } j = t+2 \text{ for } 1 \le t \le r-3, \\ \quad \vdots \\ \quad i = t \text{ and } j = t+k-3 \text{ for } 1 \le t \le r-k+2, \text{ and} \\ \quad i = t \text{ and } j = t-1 \text{ for } 2 \le t \le r, \end{array} \\ 2 & \text{if } i = t \text{ and } j = t \text{ for } 1 \le t \le r-1, \\ -2 & \text{if } i = t \text{ and } j = t+k-2 \text{ for } 1 \le t \le r-k+1, \\ -1 & \text{if } i = t \text{ and } j = t+k-1 \text{ for } 1 \le t \le r-k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that the $r \times r$ matrix $B^{r,k} = \left[b_{i,j}^{r,k}\right]$ is defined by

$$B^{r,k} = \begin{array}{c} \begin{array}{cccccc} & & (r-k)\text{-th} & & & \\ & & \downarrow & & & \end{array} \\ \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & N^{r-1,k} & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix} \end{array}.$$

Then we have the following interesting results.

Theorem 2.3. (i) *For* $r > k$,

$$\operatorname{per} N^{r,k} = -h_{r-1}^{k}.$$

(ii) *For* $r > k+1$,

$$\operatorname{per} B^{r,k} = -\sum_{i=0}^{r-2} h_i^k.$$

Proof. (i) We will use the induction method on r. Assume that the equation holds for $r > k$, then we show that the equation holds for $r+1$. If we expand the $\operatorname{per} N^{r,k}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\operatorname{per} N^{r+1,k} = 2\operatorname{per} N^{r,k} + \operatorname{per} N^{r-1,k} + \cdots + \operatorname{per} N^{r-k+3,k} - 2\operatorname{per} N^{r-k+2,k} - \operatorname{per} N^{r-k+1,k}.$$

Also, since $\operatorname{per} N^{r,k} = h_{r-1}^k, \operatorname{per} N^{r-1,k} = h_{r-2}^k, \ldots, \operatorname{per} N^{r-k+3,k} = h_{r-k+2}^k, \operatorname{per} N^{r-k+2,k} = h_{r-k+1}^k$ and $\operatorname{per} N^{r-k+1,k} = h_{r-k}^k$, we easily obtain that $\operatorname{per} N^{r+1,k} = h_r^k$. So, the proof of (i) is complete.

(ii) . If we expand the $\operatorname{per} B^{r,k}$ with respect to the first row, we can write

$$\operatorname{per} B^{r,k} = \operatorname{per} B^{r-1,k} + \operatorname{per} N^{r-1,k}.$$

From Theorem 2.2 and Theorem 2.3. (i) and induction on r, the proof follows directly. □

Let the notation $M \circ K$ denote the Hadamard product of M and K. A matrix M is called convertible if there is an $u \times u$ $(1,-1)$-matrix K such that $\operatorname{per} M = \det(M \circ K)$.

Let $r > k+1$ and let Q be the $r \times r$ matrix, defined by

$$Q = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

As an immediate consequence, we have the following.

Corollary 2.2. *For $r > k+1$,*

$$\det\left(M^{r,k} \circ Q\right) = h_{r+k-1}^k,$$
$$\det\left(N^{r,k} \circ Q\right) = -h_{r-1}^k,$$

and

$$\det\left(B^{r,k} \circ Q\right) = -\sum_{i=0}^{r-2} h_i^k.$$

Let $C(c_1, c_2, \ldots, c_v)$ be a $v \times v$ companion matrix as follows:

$$C(c_1, c_2, \ldots, c_v) = \begin{bmatrix} c_1 & c_2 & \cdots & c_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For more details see [12, 13].

Theorem 2.4. (Chen and Louck [4]) *The (i,j) entry $k_{i,j}^{(u)}(k_1, k_2, \ldots, k_v)$ in the matrix $K^u(k_1, k_2, \ldots, k_v)$ is given by the following formula:*

$$k_{i,j}^{(u)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v}, \tag{5}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = u - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (5) are defined to be 1 if $u = i - j$.

Then we have the following Corollary for the Hadamard-type k-step Pell numbers.

Corollary 2.3. *Let h_n^k be the n-th the Hadamard-type k-step Pell number for $k \geq 3$. Then*

(i)

$$h_n^k = \sum_{(t_1, t_2 \ldots, t_k)} \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} 2^{t_1} (-2)^{t_{k-1}} (-1)^{t_k},$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (k)\, t_k = n - k + 1$.

(ii)

$$h_n^k = -\sum_{(t_1, t_2 \ldots, t_k)} \frac{t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} 2^{t_1} (-2)^{t_{k-1}} (-1)^{t_k},$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (k)\, t_k = n + 1$.

Proof. In Theorem 2.4, If we choose $i = k$ and $j = 1$, for case (i) and $i = k-1$, $j = k$, for case (ii), then we can directly see the conclusions from (2). $\square$

It is easy to show that the generating function of the Hadamard-type k-step Pell numbers is as follows:

$$g_k(x) = \frac{x^{k-1}}{1 - 2x - x^2 - \cdots - x^{k-2} + 2x^{k-1} + x^k}.$$

Then we can give an exponential representation for the Hadamard-type k-step Pell numbers by the aid of the generating function with the following theorem.

Theorem 2.5. *The Hadamard-type k-step Pell numbers have the following exponential representation:*

$$g_k(x) = x^{k-1} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(2 + x + \cdots + x^{k-3} - 2x^{k-2} - x^{k-1}\right)^i \right),$$

where $k \geq 3$.

Proof. It is clear that

$$\ln \frac{g_k(x)}{x^{k-1}} = -\ln\left(1 - 2x - x^2 - \cdots - x^{k-2} + 2x^{k-1} + x^k\right).$$

Also, we have

$$\begin{aligned} -\ln\left(1 - 2x - x^2 - \cdots - x^{k-2} + 2x^{k-1} + x^k\right) = -[&-x\left(2 + x + \cdots + x^{k-3} - 2x^{k-2} - x^{k-1}\right) - \\ &\frac{1}{2}x^2\left(2 + x + \cdots + x^{k-3} - 2x^{k-2} - x^{k-1}\right)^2 - \cdots - \\ &\frac{1}{n}x^n\left(2 + x + \cdots + x^{k-3} - 2x^{k-2} - x^{k-1}\right)^n - \cdots] \end{aligned}$$

Therefore, we obtain

$$\ln \frac{g_k(x)}{x^{k-1}} = \sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(2 + x + \cdots + x^{k-3} - 2x^{k-2} - x^{k-1}\right)^i.$$

Thus we have the conclusion. $\square$

Now we give the sums of the Hadamard-type k-step Pell numbers.

Let

$$S_n = \sum_{i=0}^{n} h_n^k$$

for $n \geq 0$ and $k \geq 3$, and suppose that L_k^h is the $(k+1) \times (k+1)$ matrix, such that

$$L_k^h = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & H_k^p & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then by the inductive argument, we write

$$\left(L_k^h\right)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{n+k-2} & & & \\ S_{n+k-3} & & \left(H_k^p\right)^n & \\ \vdots & & & \\ S_{n-1} & & & \end{bmatrix}.$$

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