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Two 2-Fibonacci sequences generated by a mixed scheme. Part 1

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Abstract: A new scheme of 2-Fibonacci sequences is introduced and the explicit formulas for its n-th members are given. For difference of all previous sequences from Fibonacci type, the present 2-Fibonacci sequences are obtained by a new way. It is proved that the new sequences have bases with 48 elements about function φ and modulo 9.

Keywords: Fibonacci sequence, 2-Fibonacci sequence.
2020 Mathematics Subject Classification: 11B39.

1 Introduction

In a series of papers, starting with [10], the author introduces so called by him n-Fibonacci sequences ($n \geq 2$, natural number) and study their basic properties (see, e.g. [11]). After [3, 24], a lot of other authors started interested in these sequences (see 1, [13–36]). In some of his papers, the author discussed different 2- and 3-Fibonacci sequences, some of which, he called "combined" (see [4, 5, 7–9, 12]), because the next members of these sequences are results of operations over some of the previous members.

In the present paper, we return to the idea for 2-Fibonacci sequences, but use a new way, different form all other ways, for their construction.

2 Main results

Let everywhere below, a, b, c, d be arbitrary real numbers.

The first one of the new 2-Fibonacci sequences have the form: $\alpha_0 = a$, $\beta_0 = b$, $\alpha_1 = c$, $\beta_1 = d$ and for each natural number $n \geq 0$:

$$\alpha_{2n+2} = \alpha_{2n+1} + \alpha_{2n},$$
$$\beta_{2n+2} = \beta_{2n+1} + \beta_{2n},$$
$$\alpha_{2n+3} = \alpha_{2n+2} + \beta_{2n+1},$$
$$\beta_{2n+3} = \beta_{2n+2} + \alpha_{2n+1}.$$

The first several values of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are as listed in Table 1.

n	α_n	β_n
0	a	b
1	c	d
2	$a+c$	$b+d$
3	$a+c+d$	$b+c+d$
4	$2a+2c+d$	$2b+c+2d$
5	$2a+b+3c+2d$	$a+2b+2c+3d$
6	$4a+b+5c+3d$	$a+4b+3c+5d$
7	$5a+3b+7c+6d$	$3a+5b+6c+7d$
8	$9a+4b+12c+9d$	$4a+9b+9c+12d$
9	$12a+9b+18c+16d$	$9a+12b+16c+18d$
10	$21a+13b+30c+25d$	$13a+21b+25c+30d$
	$\vdots$	$\vdots$

Graphically, we can represent the scheme in Figure 1 below.

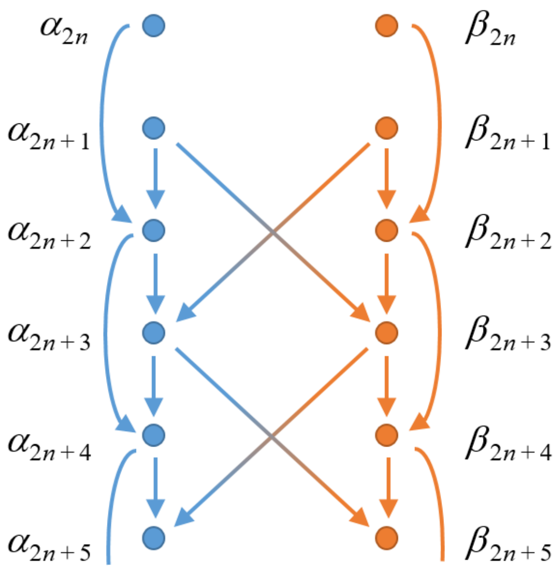

Figure 1. Scheme of constructing the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$

Let $\{F_n\}_{n=0}^{\infty}$ be the standard Fibonacci sequence, where $F_0=0$, $F_1=1$, and $F_{n+2}=F_{n+1}+F_n$ for each natural number $n \geq 0$.

Theorem 1. *For each natural number $n \geq 0$:*

$$\alpha_{2n} = \frac{F_{2n-1}+F_{n+1}}{2}a + \frac{F_{2n-1}-F_{n+1}}{2}b + \frac{F_{2n}-F_n}{2}c + \frac{F_{2n}+F_n}{2}d,$$
$$\beta_{2n} = \frac{F_{2n-1}-F_{n+1}}{2}a + \frac{F_{2n-1}+F_{n+1}}{2}b + \frac{F_{2n}+F_n}{2}c + \frac{F_{2n}-F_n}{2}d,$$
$$\alpha_{2n+1} = \frac{F_{2n}+F_n}{2}a + \frac{F_{2n}-F_n}{2}b + \frac{F_{2n+1}-F_{n-1}}{2}c + \frac{F_{2n+1}+F_{n-1}}{2}d,$$
$$\beta_{2n+1} = \frac{F_{2n}-F_n}{2}a + \frac{F_{2n}+F_n}{2}b + \frac{F_{2n+1}+F_{n-1}}{2}c + \frac{F_{2n+1}-F_{n-1}}{2}d.$$

Proof. We can prove the theorem, for example, by induction.

First, we must mention that for each natural number k, the numbers $F_{2k-1}+F_{k+1}, F_{2k}-F_k$ are even.

Really, for $k = 1$ both numbers are even. If we assume that the assertion is valid for all numbers smaller or equal to some number k, then, obviously

$$F_{2k+1}+F_{k+2} = F_{2k}+F_{k+1}+F_{2k-1}+F_k$$

is an even number as a sum of even numbers.

Second, for $n = 0$, the validity of the theorem is checked directly from the above table. Let us assume that the theorem is valid for some natural number $n \geq 0$. Then:

$$\begin{aligned}
\alpha_{2n+2} &= \alpha_{2n+1}+\alpha_{2n} \\
&= \tfrac{F_{2n}+F_n}{2}a + \tfrac{F_{2n}-F_n}{2}b + \tfrac{F_{2n+1}-F_{n-1}}{2}c + \tfrac{F_{2n+1}+F_{n-1}}{2}d, \\
&\quad + \tfrac{F_{2n-1}+F_{n+1}}{2}a + \tfrac{F_{2n-1}-F_{n+1}}{2}b + \tfrac{F_{2n}-F_n}{2}c + \tfrac{F_{2n}+F_n}{2}d \\
&= \tfrac{F_{2n+1}+F_{n+2}}{2}a + \tfrac{F_{2n+1}-F_{n+2}}{2}b + \tfrac{F_{2n+2}-F_{n+1}}{2}c + \tfrac{F_{2n+2}+F_{n+1}}{2}d.
\end{aligned}$$

The check for β_{2n+2} is similar.

$$\begin{aligned}
\alpha_{2n+3} &= \alpha_{2n+2}+\beta_{2n+1} \\
&= \tfrac{F_{2n+1}+F_{n+2}}{2}a + \tfrac{F_{2n+1}-F_{n+2}}{2}b + \tfrac{F_{2n+2}-F_{n+1}}{2}c + \tfrac{F_{2n+2}+F_{n+1}}{2}d \\
&\quad + \tfrac{F_{2n}-F_n}{2}a + \tfrac{F_{2n}+F_n}{2}b + \tfrac{F_{2n+1}+F_{n-1}}{2}c + \tfrac{F_{2n+1}-F_{n-1}}{2}d \\
&= \tfrac{F_{2n+2}+F_{n+2}-F_n}{2}a + \tfrac{F_{2n+2}-F_{n+2}+F_n}{2}b + \tfrac{F_{2n+3}-F_{n+1}+F_{n-1}}{2}c + \tfrac{F_{2n+2}+F_{n+1}-F_{n-1}}{2}d \\
&= \tfrac{F_{2n+2}+F_{n+1}}{2}a + \tfrac{F_{2n+2}-F_{n+1}}{2}b + \tfrac{F_{2n+3}-F_n}{2}c + \tfrac{F_{2n+3}+F_n}{2}d.
\end{aligned}$$

The check for β_{2n+3} is similar. $\square$

In [2,6], a digital arithmetic function was defined as follows. Let $n = \sum_{i=1}^{k} a_i.10^{k-i} \equiv \overline{a_1 a_2 \cdots a_k}$, where a_i is a natural number and $0 \leq a_i \leq 9$ $(1 \leq i \leq k)$. Let for $n = 0 : \varphi(n) = 0$ and for $n > 0$: $\varphi(n) = \sum_{i=1}^{k} a_i$. Let us define a sequence of functions $\varphi_0, \varphi_1, \varphi_2, \ldots,$ (where l is a natural number) by $\varphi_0(n) = n$, $\varphi_{l+1} = \varphi(\varphi_l(n))$. Then for each natural number n there exists a natural number l so that $\varphi_l(n) = \varphi_{l+1}(n) \in \{0, 1, 2, \ldots, 9\}$.

Let function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

where $\varphi_{l+1}(n) = \varphi_l(n)$.

Let a sequence with natural numbers $a_1, a_2, \dots$ be given and let $c_i = \psi(a_i)$ $(i = 1, 2, \dots)$. Hence, we deduce the sequence $c_1, c_2, \dots$ from the former sequence. If k and l exist so that $l \geq 0$, $c_{i+l} = c_{k+i+l} = c_{2k+i+l} = \cdots$ for $1 \leq i \leq k$, then we shall say that

$$[c_{l+1}, c_{l+2}, \dots, c_{l+k}]$$

is *base* of the sequence $c_1, c_2, \dots$ with length of k with respect to function ψ.

Theorem 2. *For every four natural numbers a, b, c, d, the mixed 2-combined Fibonacci sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ have bases with length of 48 with respect to function ψ.*
Proof. Using the property of ψ function: for every two natural number x and y: $\psi(x+9y) = \psi(x)$, we obtain sequentially the sequences:

n	α_n	β_n
0	$\varphi(a)$	$\varphi(b)$
1	$\varphi(c)$	$\varphi(d)$
2	$\varphi(a+c)$	$\varphi(b+d)$
3	$\varphi(a+c+d)$	$\varphi(b+c+d)$
4	$\varphi(2a+2c+d)$	$\varphi(2b+c+2d)$
5	$\varphi(2a+b+3c+2d)$	$\varphi(a+2b+2c+3d)$
6	$\varphi(4a+b+5c+3d)$	$\varphi(a+4b+3c+5d)$
7	$\varphi(5a+3b+7c+6d)$	$\varphi(3a+5b+6c+7d)$
8	$\varphi(4b+3c)$	$\varphi(4a+3d)$
9	$\varphi(3a+7d)$	$\varphi(3b+7c)$
10	$\varphi(3a+4b+3c+7d)$	$\varphi(4a+3b+7c+3d)$
11	$\varphi(3a+7b+c+7d)$	$\varphi(7a+3b+7c+d)$
12	$\varphi(6a+2b+4c+5d)$	$\varphi(2a+6b+5c+4d)$
13	$\varphi(4a+5b+2c+6d)$	$\varphi(5a+4b+6c+2d)$
14	$\varphi(a+7b+6c+2d)$	$\varphi(7a+b+2c+6d)$
15	$\varphi(6a+2b+3c+4d)$	$\varphi(2a+6b+4c+3d)$
16	$\varphi(7a+6d)$	$\varphi(7b+6c)$
17	$\varphi(6b+4c)$	$\varphi(6a+4d)$
18	$\varphi(7a+6b+4c+6d)$	$\varphi(6a+7b+6c+4d)$
19	$\varphi(4a+6b+4c+d)$	$\varphi(6a+4b+c+4d)$
20	$\varphi(2a+3b+8c+7d)$	$\varphi(3a+2b+7c+8d)$
21	$\varphi(8a+7b+2d)$	$\varphi(7a+8b+2c)$
22	$\varphi(a+b+8c)$	$\varphi(a+b+8d)$
23	$\varphi(8a+c)$	$\varphi(8b+d)$
24	$\varphi(b)$	$\varphi(a)$

Contd.

n	α_n	β_n
25	$\varphi(d)$	$\varphi(c)$
26	$\varphi(b+d)$	$\varphi(a+c)$
27	$\varphi(b+c+d)$	$\varphi(a+c+d)$
28	$\varphi(2b+c+2d)$	$\varphi(2a+2c+d)$
29	$\varphi(a+2b+2c+3d)$	$\varphi(2a+b+3c+2d)$
30	$\varphi(a+4b+3c+5d)$	$\varphi(4a+b+5c+3d)$
31	$\varphi(3a+5b+6c+7d)$	$\varphi(5a+3b+7c+6d)$
32	$\varphi(4a+3d)$	$\varphi(4b+3c)$
33	$\varphi(3b+7c)$	$\varphi(3a+7d)$
34	$\varphi(4a+3b+7c+3d)$	$\varphi(3a+4b+3c+7d)$
35	$\varphi(7a+3b+7c+d)$	$\varphi(3a+7b+c+7d)$
36	$\varphi(2a+6b+5c+4d)$	$\varphi(6a+2b+4c+5d)$
37	$\varphi(5a+4b+6c+2d)$	$\varphi(4a+5b+2c+6d)$
38	$\varphi(7a+b+2c+6d)$	$\varphi(a+7b+6c+2d)$
39	$\varphi(2a+6b+4c+3d)$	$\varphi(6a+2b+3c+4d)$
40	$\varphi(7b+6c)$	$\varphi(7a+6d)$
41	$\varphi(6a+4d)$	$\varphi(6b+4c)$
42	$\varphi(6a+7b+6c+4d)$	$\varphi(7a+6b+4c+6d)$
43	$\varphi(6a+4b+c+4d)$	$\varphi(4a+6b+4c+d)$
44	$\varphi(3a+2b+7c+8d)$	$\varphi(2a+3b+8c+7d)$
45	$\varphi(7a+8b+2c)$	$\varphi(8a+7b+2d)$
46	$\varphi(a+b+8d)$	$\varphi(a+b+8c)$
47	$\varphi(8b+d)$	$\varphi(8a+c)$
48	$\varphi(a)$	$\varphi(b)$
49	$\varphi(c)$	$\varphi(d)$
	$\vdots$	$\vdots$

From here, the validity of the theorem holds. It is important to note the the base starts with the second members of the sequences. □

3 Conclusion

In the paper, a scheme for constructing of 2-Fibonacci sequences from a new type is given. In the next step of this research, some other similar (but different) schemes will be introduced.

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