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On hyperbolic *k*-Jacobsthal and *k*-Jacobsthal–Lucas octonions

Engin Özkan[1] and Mine Uysal[2]

[1] Department of Mathematics, Faculty of Arts and Sciences,
Erzincan Binali Yıldırım University, Erzincan, Turkey
e-mails: eozkan@erzincan.edu.tr, eozkanmath@gmail.com

[2] Graduate School of Natural and Applied Sciences,
Erzincan Binali Yıldırım University, Erzincan, Turkey.
e-mail: mine.uysal@erzincan.edu.tr

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Abstract: In this work, we investigate the hyperbolic *k*-Jacobsthal and *k*-Jacobsthal–Lucas octonions. We give Binet's Formula, Cassini's identity, Catalan's identity, d'Ocagne identity, generating functions of the hyperbolic *k*-Jacobsthal and *k*-Jacobsthal–Lucas octonions. Also, we present many properties of these octonions.

Keywords: Hyperbolic *k*-Jacobsthal octonions, Hyperbolic *k*-Jacobsthal–Lucas octonions, Binet formula, Cassini identity, Catalan identity.
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1 Introduction

Number sequences have been the subject of many scientists' studies for years, and they have found application in nature and many sciences, as in every branch of mathematics [1, 6, 16]. Of course, the Fibonacci numbers are the best known of the sequences of numbers. Many generalizations of number sequences were then described and studied [4, 5, 12, 15, 19]. One of these generalizations is the Jacobsthal numbers [7, 8, 14, 15, 17, 19]. The Jacobsthal numbers J_n are defined by the relation

$$J_{n+2} = J_{n+1} + 2J_n, n \geq 0$$

with $J_0 = 0$ and $J_1 = 1$.

Similarly, the Jacobsthal–Lucas numbers j_n are given by relation

$$j_{n+2} = j_{n+1} + 2j_n, n \geq 0$$

with $j_0 = 2$ and $j_1 = 1$.

The characteristic equation for J_n and j_n is $x^2 - x - 2 = 0$ whose roots are $a = 2,\ b = 1$. Binet Formulas for J_n and j_n are $J_n = \frac{a^n - b^n}{a-b}$ and $j_n = a + b$, respectively.

In 2016, Uygun and Eldogan [22] defined *k*-Jacobsthal numbers by the recurrence relation

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \text{ for } n \geq 2$$

with $J_{k,0} = 0$ and $J_{k,1} = 1$. They also defined *k*-Jacobsthal–Lucas numbers by the recurrence relation

$$j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}, n \geq 2$$

with $j_{k,0} = 2$ and $j_{k,1} = k$.

The characteristic equation for $J_{k,n}$ and $j_{k,n}$ is $x^2 - kx - 2 = 0$ whose roots are

$$\alpha = \frac{k+\sqrt{k^2+8}}{2},\ \beta = \frac{k-\sqrt{k^2+8}}{2}.$$

The Binet Formulas for $J_{k,n}$ and $j_{k,n}$ are given by $J_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $j_{k,n} = \alpha^n + \beta^n$, respectively.

The quaternion, an algebraic structure, was first described in 1843 by William Rowan Hamilton [11].

In 1963, A. F. Horadam [13] defined the *n*-th Fibonacci and *n*-th Lucas quaternions and gave their some properties.

In 2016, Szynal-Liana, A. and Włoch, I. [21] defined the *n*-th Jacobsthal and *n*-th Jacobsthal–Lucas quaternions. Macfarlane [18] defined the hyperbolic quaternions and studied their properties. A hyperbolic quaternion h has the form:

$$h = h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 i_4 = (h_1, h_2, h_3, h_4),$$

where h_1, h_2, h_3, h_4 are real components and i_1, i_2, i_3, i_4 are hyperbolic quaternion units, which have the rules as given in Table 1.

$\cdot$	$\boldsymbol{i_1}$	$\boldsymbol{i_2}$	$\boldsymbol{i_3}$	$\boldsymbol{i_4}$
i_1	1	i_2	i_3	i_4
i_2	i_2	1	i_4	$-i_3$
i_3	i_3	$-i_4$	1	i_2
i_4	i_4	i_3	$-i_2$	1

Table 1. Multiplication rule for $\{i_1, i_2, i_3, i_4\}$

In [10, 20], the hyperbolic *k*-Fibonacci and *k*-Fibonacci-Lucas, hyperbolic *k*-Jacobsthal and *k*-Jacobsthal–Lucas quaternions were defined and given some of their properties.

In [2, 3], A. Cariow, G. Cariow and J. Knapiski defined the hyperbolic octonions.

A hyperbolic octonion O has the form

$$O = h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7$$
$$= (h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7)$$

where i_1, i_2, i_3, i_4 are quaternion imaginary units, $h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7$ are the real components and e_4 $({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions are defined as in Table 2.

$\cdot$	i_1	i_2	i_3	e_4	e_5	e_6	e_7
i_1	-1	i_3	$-i_2$	e_5	e_4	$-e_7$	e_6
i_2	$-i_3$	-1	i_1	e_6	e_7	e_4	$-e_5$
i_3	i_2	$-i_1$	-1	e_7	$-e_6$	e_5	e_4
e_4	$-e_5$	$-e_6$	$-e_7$	1	i_1	i_2	i_3
e_5	$-e_4$	$-e_7$	e_6	$-i_1$	1	i_3	$-i_2$
e_6	e_7	$-e_4$	$-e_5$	$-i_2$	$-i_3$	1	i_1
e_7	$-e_6$	e_5	$-e_4$	$-i_3$	i_2	$-i_1$	1

Table 2. Multiplication rule for hyperbolic octonions units

In [9], A. Godase defined the hyperbolic *k*-Fibonacci and *k*-Fibonacci–Lucas octonions and gave some of their properties.

In this paper, we introduce hyperbolic *k*-Jacobsthal and *k*-Jacobsthal–Lucas octonions and examine their properties.

2 Hyperbolic *k*-Jacobsthal and *k*-Jacobsthal–Lucas octonions

Definition 2.1. The hyperbolic *k*-Jacobsthal octonions ${O^J}_{k,n}$ and *k*-Jacobsthal–Lucas octonions ${O^j}_{k,n}$ are defined by

$${O^J}_{k,n} = J_{k,n} + J_{k,n+1} i_1 + J_{k,n+2} i_2 + J_{k,n+3} i_3 + J_{k,n+4} e_4 + J_{k,n+5} e_5 + J_{k,n+6} e_6 + J_{k,n+7} e_7$$
$$= (J_{k,n}, J_{k,n+1}, J_{k,n+2}, J_{k,n+3}, J_{k,n+4}, J_{k,n+5}, J_{k,n+6}, J_{k,n+7}),$$
$${O^j}_{k,n} = j_{k,n} + j_{k,n+1} i_1 + j_{k,n+2} i_2 + j_{k,n+3} i_3 + j_{k,n+4} e_4 + j_{k,n+5} e_5 + j_{k,n+6} e_6 + j_{k,n+7} e_7$$
$$= (j_{k,n}, j_{k,n+1}, j_{k,n+2}, j_{k,n+3}, j_{k,n+4}, j_{k,n+5}, j_{k,n+6}, j_{k,n+7}),$$

respectively, $(n \geq 0)$.

Definition 2.2. The conjugate of hyperbolic *k*-Jacobsthal octonions ${\bar{O}^J}_{k,n}$ and *k*-Jacobsthal–Lucas octonions ${\bar{O}^j}_{k,n}$ are defined by

$${\bar{O}^J}_{k,n} = J_{k,n} - J_{k,n+1} i_1 - J_{k,n+2} i_2 - J_{k,n+3} i_3 - J_{k,n+4} e_4 - J_{k,n+5} e_5 - J_{k,n+6} e_6 - J_{k,n+7} e_7$$
$$= (J_{k,n}, -J_{k,n+1}, -J_{k,n+2}, -J_{k,n+3}, -J_{k,n+4}, -J_{k,n+5}, -J_{k,n+6}, -J_{k,n+7}),$$
$${\bar{O}^j}_{k,n} = j_{k,n} - j_{k,n+1} i_1 - j_{k,n+2} i_2 - j_{k,n+3} i_3 - j_{k,n+4} e_4 - j_{k,n+5} e_5 - j_{k,n+6} e_6 - j_{k,n+7} e_7$$
$$= (j_{k,n}, -j_{k,n+1}, -j_{k,n+2}, -j_{k,n+3}, -j_{k,n+4}, -j_{k,n+5}, -j_{k,n+6}, -j_{k,n+7}),$$

respectively, $(n \geq 0)$.

Theorem 2.1. For $n \geq 1$, we have

(i) $O^J{}_{k,n+1} = kO^J{}_{k,n} + 2O^J{}_{k,n-1}$,
(ii) $O^j{}_{k,n+1} = kO^j{}_{k,n} + 2O^j{}_{k,n-1}$,
(iii) $O^j{}_{k,n} = O^J{}_{k,n+1} + 2O^J{}_{k,n-1}$,
(iv) $\bar{O}^J{}_{k,n+1} = k\bar{O}^J{}_{k,n} + 2\bar{O}^J{}_{k,n-1}$,
(v) $\bar{O}^j{}_{k,n+1} = k\bar{O}^j{}_{k,n} + 2\bar{O}^j{}_{k,n-1}$,
(vi) $\bar{O}^j{}_{k,n} = \bar{O}^J{}_{k,n+1} + 2\bar{O}^J{}_{k,n-1}$.

Proof. We prove only equality (i) using Definition 2.1. The proofs of (ii), (iii), (iv), (v) and (vi) are like that of (i).

$$\begin{aligned} kO^J{}_{k,n} + 2O^J{}_{k,n-1} &= k\big(J_{k,n} + J_{k,n+1}i_1 + J_{k,n+2}i_2 + J_{k,n+3}i_3 \\ &\quad + J_{k,n+4}e_4 + J_{k,n+5}e_5 + J_{k,n+6}e_6 + J_{k,n+7}e_7\big) \\ &\quad + 2\big(J_{k,n-1} + J_{k,n}i_1 + J_{k,n+1}i_2 + J_{k,n+2}i_3 \\ &\quad + J_{k,n+3}e_4 + J_{k,n+4}e_5 + J_{k,n+5}e_6 + J_{k,n+6}e_7\big) \\ &= \big(kJ_{k,n} + 2J_{k,n-1}\big) + \big(kJ_{k,n+1} + 2J_{k,n}\big)i_1 \\ &\quad + \big(kJ_{k,n+2} + 2J_{k,n+1}\big)i_2 + \big(kJ_{k,n+3} + 2J_{k,n+2}\big)i_3 \\ &\quad + \big(kJ_{k,n+4} + 2J_{k,n+3}\big)e_4 + \big(kJ_{k,n+5} + 2J_{k,n+4}\big)e_5 \\ &\quad + \big(kJ_{k,n+6} + 2J_{k,n+5}\big)e_6 + \big(kJ_{k,n+7} + 2J_{k,n+6}\big)e_7 \\ &= J_{k,n+1} + J_{k,n+2}i_1 + J_{k,n+3}i_2 + J_{k,n+4}i_3 + J_{k,n+5}e_4 \\ &\quad + J_{k,n+6}e_5 + J_{k,n+7}e_6 + J_{k,n+8}e_7 \\ &= O^J{}_{k,n+1}. \end{aligned}$$

□

Theorem 2.2 (Binet formulas). For $n \geq 1$, the following equations are true.

(i) $O^J{}_{k,n} = \frac{\bar{\alpha}_1 {\alpha_1}^n - \bar{\beta}_1 {\beta_1}^n}{\alpha_1 - \beta_1}$,
(ii) $O^j{}_{k,n} = \bar{\alpha}_1 {\alpha_1}^n + \bar{\beta}_1 {\beta_1}^n$,
(iii) $\bar{O}^J{}_{k,n} = \frac{\tilde{\alpha}_1 {\alpha_1}^n - \tilde{\beta}_1 {\beta_1}^n}{\alpha_1 - \beta_1}$,
(iv) $\bar{O}^j{}_{k,n} = \tilde{\alpha}_1 {\alpha_1}^n + \tilde{\beta}_1 {\beta_1}^n$,

where

$$\begin{aligned} \bar{\alpha}_1 &= 1 + \alpha_1 i_1 + {\alpha_1}^2 i_2 + {\alpha_1}^3 i_3 + {\alpha_1}^4 e_4 + {\alpha_1}^5 e_5 + {\alpha_1}^6 e_6 + {\alpha_1}^7 e_7 \\ &= (1, \alpha_1, {\alpha_1}^2, {\alpha_1}^3, {\alpha_1}^4, {\alpha_1}^5, {\alpha_1}^6, {\alpha_1}^7), \end{aligned}$$

$$\begin{aligned} \bar{\beta}_1 &= 1 + \beta_1 i_1 + {\beta_1}^2 i_2 + {\beta_1}^3 i_3 + {\beta_1}^4 e_4 + {\beta_1}^5 e_5 + {\beta_1}^6 e_6 + {\beta_1}^7 e_7 \\ &= \big(1, \beta_1, {\beta_1}^2, {\beta_1}^3, {\beta_1}^4, {\beta_1}^5, {\beta_1}^6, {\beta_1}^7\big), \end{aligned}$$

$$\begin{aligned} \tilde{\alpha}_1 &= 1 - \alpha_1 i_1 - {\alpha_1}^2 i_2 - {\alpha_1}^3 i_3 - {\alpha_1}^4 e_4 - {\alpha_1}^5 e_5 - {\alpha_1}^6 e_6 - {\alpha_1}^7 e_7 \\ &= (1, -\alpha_1, -{\alpha_1}^2, -{\alpha_1}^3, -{\alpha_1}^4, -{\alpha_1}^5, -{\alpha_1}^6, -{\alpha_1}^7), \end{aligned}$$

$$\begin{aligned} \tilde{\beta}_1 &= 1 - \beta_1 i_1 - {\beta_1}^2 i_2 - {\beta_1}^3 i_3 - {\beta_1}^4 e_4 - {\beta_1}^5 e_5 - {\beta_1}^6 e_6 - {\beta_1}^7 e_7 \\ &= \big(1, -\beta_1, -{\beta_1}^2, -{\beta_1}^3, -{\beta_1}^4, -{\beta_1}^5, -{\beta_1}^6, -{\beta_1}^7\big). \end{aligned}$$

Proof. (i) Using Definition 2.1 and the Binet formula of $J_{k,n}$, we get the following:.

$$O^J{}_{k,n} = J_{k,n} + J_{k,n+1}i_1 + J_{k,n+2}i_2 + J_{k,n+3}i_3 + J_{k,n+4}e_4 + J_{k,n+5}e_5 + J_{k,n+6}e_6 + J_{k,n+7}e_7$$

$$= \left(\frac{{\alpha_1}^n - {\beta_1}^n}{\alpha_1 - \beta_1}\right) + \left(\frac{{\alpha_1}^{n+1} - {\beta_1}^{n+1}}{\alpha_1 - \beta_1}\right)i_1 + \left(\frac{{\alpha_1}^{n+2} - {\beta_1}^{n+2}}{\alpha_1 - \beta_1}\right)i_2 + \left(\frac{{\alpha_1}^{n+3} - {\beta_1}^{n+3}}{\alpha_1 - \beta_1}\right)i_3$$
$$+ \left(\frac{{\alpha_1}^{n+4} - {\beta_1}^{n+4}}{\alpha_1 - \beta_1}\right)e_4 + \left(\frac{{\alpha_1}^{n+5} - {\beta_1}^{n+5}}{\alpha_1 - \beta_1}\right)e_5 + \left(\frac{{\alpha_1}^{n+6} - {\beta_1}^{n+6}}{\alpha_1 - \beta_1}\right)e_6$$
$$+ \left(\frac{{\alpha_1}^{n+7} - {\beta_1}^{n+7}}{\alpha_1 - \beta_1}\right)e_7$$

$$= \frac{{\alpha_1}^n}{\alpha_1 - \beta_1}(1 + \alpha_1 i_1 + {\alpha_1}^2 i_2 + {\alpha_1}^3 i_3 + {\alpha_1}^4 e_4 + {\alpha_1}^5 e_5 + {\alpha_1}^6 e_6 + {\alpha_1}^7 e_7)$$
$$- \frac{{\beta_1}^n}{\alpha_1 - \beta_1}(1 + \beta_1 i_1 + {\beta_1}^2 i_2 + {\beta_1}^3 i_3 + {\beta_1}^4 e_4 + {\beta_1}^5 e_5 + {\beta_1}^6 e_6 + {\beta_1}^7 e_7)$$
$$= \frac{\bar{\alpha}_1 {\alpha_1}^n - \bar{\beta}_1 {\beta_1}^n}{\alpha_1 - \beta_1}.$$

(iii) We have,

$$\bar{O}^J{}_{k,n} = J_{k,n} - J_{k,n+1}i_1 - J_{k,n+2}i_2 - J_{k,n+3}i_3 - J_{k,n+4}e_4 - J_{k,n+5}e_5 - J_{k,n+6}e_6 - J_{k,n+7}e_7$$
$$= \left(\frac{{\alpha_1}^n - {\beta_1}^n}{\alpha_1 - \beta_1}\right) - \left(\frac{{\alpha_1}^{n+1} - {\beta_1}^{n+1}}{\alpha_1 - \beta_1}\right)i_1 - \left(\frac{{\alpha_1}^{n+2} - {\beta_1}^{n+2}}{\alpha_1 - \beta_1}\right)i_2 - \left(\frac{{\alpha_1}^{n+3} - {\beta_1}^{n+3}}{\alpha_1 - \beta_1}\right)i_3$$
$$- \left(\frac{{\alpha_1}^{n+4} - {\beta_1}^{n+4}}{\alpha_1 - \beta_1}\right)e_4 - \left(\frac{{\alpha_1}^{n+5} - {\beta_1}^{n+5}}{\alpha_1 - \beta_1}\right)e_5 - \left(\frac{{\alpha_1}^{n+6} - {\beta_1}^{n+6}}{\alpha_1 - \beta_1}\right)e_6$$
$$- \left(\frac{{\alpha_1}^{n+7} - {\beta_1}^{n+7}}{\alpha_1 - \beta_1}\right)e_7$$
$$= \frac{{\alpha_1}^n}{\alpha_1 - \beta_1}(1 - \alpha_1 i_1 - {\alpha_1}^2 i_2 + {\alpha_1}^3 i_3 - {\alpha_1}^4 e_4 - {\alpha_1}^5 e_5 - {\alpha_1}^6 e_6 - {\alpha_1}^7 e_7)$$
$$- \frac{{\beta_1}^n}{\alpha_1 - \beta_1}(1 - \beta_1 i_1 - {\beta_1}^2 i_2 - {\beta_1}^3 i_3 - {\beta_1}^4 e_4 - {\beta_1}^5 e_5 - {\beta_1}^6 e_6 - {\beta_1}^7 e_7)$$
$$= \frac{\tilde{\alpha}_1 {\alpha_1}^n - \tilde{\beta}_1 {\beta_1}^n}{\alpha_1 - \beta_1}.$$

The proofs of (ii), (iv) are similar to (i) and (iii) by using Definition 2.1. □

The next lemma gives some properties between the roots $\bar{\alpha}_1, \bar{\beta}_1, \tilde{\alpha}_1$ and $\tilde{\beta}_1$.

Lemma 2.1. The following equations are satisfied:

(1) $\bar{\alpha}_1 - \bar{\beta}_1 = \sqrt{\rho} O^J{}_{k,0}$,

(2) $\bar{\alpha}_1 + \bar{\beta}_1 = O^j{}_{k,0}$,

(3) $\bar{\alpha}_1 \bar{\beta}_1 + \bar{\beta}_1 \bar{\alpha}_1 = 2(-75 + O^j{}_{k,0})$,

(4) $\bar{\alpha}_1{}^2 - \bar{\beta}_1{}^2 = \sqrt{\rho}(2O^J{}_{k,0} + k^{13} + 25k^{11} + 241k^9 + 1137k^7 + 2559k^5 + 2543k^3 + 735k) = \bar{v}_6$,

(5) $\bar{\alpha}_1 + \tilde{\alpha}_1 = 2$ and $\bar{\beta}_1 + \tilde{\beta}_1 = 2$,

(6) $\bar{\alpha}_1\tilde{\alpha}_1 = \tilde{\alpha}_1\bar{\alpha}_1 = 1 + {\alpha_1}^2 + {\alpha_1}^4 + {\alpha_1}^6 - {\alpha_1}^8 - {\alpha_1}^{10} - {\alpha_1}^{12} - {\alpha_1}^{14} = \bar{v}_8$,

(7) $\bar{\beta}_1\tilde{\beta}_1 = \tilde{\beta}_1\bar{\beta}_1 = 1 + {\beta_1}^2 + {\beta_1}^4 + {\beta_1}^6 + {\beta_1}^8 - {\beta_1}^{10} - {\beta_1}^{12} - {\beta_1}^{14} = \bar{v}_9$,

(8) $\bar{\alpha}_1\tilde{\beta}_1 + \tilde{\beta}_1\bar{\alpha}_1 = \bar{v}_{10} + \bar{v}_{11} = 2\left(75 + \sqrt{\rho}O^J{}_{k,0}\right)$,

(9) $\bar{\alpha}_1\tilde{\beta}_1 - \tilde{\beta}_1\bar{\alpha}_1 = \bar{v}_{10} - \bar{v}_{11} = -\bar{v}_3 = \bar{\beta}_1\bar{\alpha}_1 - \bar{\alpha}_1\bar{\beta}_1$

$$= -\sqrt{\rho}(0, -168, -100k, 68 - 32(k^2+2), 12(k^3+4k), 8(k^4+6k^2+4) + 20(k^2+2), 4(k^5+8k^3+12k) - 24k, -4(k^4+6k^2+4) - 8(k^2+2) + 16)$$

(10) $\bar{\beta}_1\tilde{\alpha}_1 + \tilde{\alpha}_1\bar{\beta}_1 = \Big((150, -2(\alpha_1 - \beta_1), -2\left({\alpha_1}^2 - {\beta_1}^2\right), -2\left({\alpha_1}^3 - {\beta_1}^3\right), -2\left({\alpha_1}^4 - {\beta_1}^4\right), -2\left({\alpha_1}^5 - {\beta_1}^5\right), -2\left({\alpha_1}^6 - {\beta_1}^6\right), -2\left({\alpha_1}^7 - {\beta_1}^7\right)\Big) = \bar{v}_{14}$

(11) $\bar{v}_{10} + \bar{v}_{12} = (150,0,0,0,0,0,0,0) = 150$,

(12) $\tilde{\alpha}_1\tilde{\beta}_1 - \tilde{\beta}_1\tilde{\alpha}_1 = \bar{v}_1 - \bar{v}_2 = \bar{v}_3$,

(13) $\tilde{\alpha}_1{}^2 - \tilde{\beta}_1{}^2 = \sqrt{\rho}\left(-2O^J{}_{k,0} + k^{13} + 25k^{11} + 241k^9 + 1137k^7 + 2559k^5 + 2543k^3 + 735k\right) = -4\sqrt{\rho}O^J{}_{k,0} + \bar{v}_6$,

Proof.

(1)
$$\begin{aligned}\bar{\alpha}_1 - \bar{\beta}_1 &= (1, \alpha_1, {\alpha_1}^2, {\alpha_1}^3, {\alpha_1}^4, {\alpha_1}^5, {\alpha_1}^6, {\alpha_1}^7) - \left(1, \beta_1, {\beta_1}^2, {\beta_1}^3, {\beta_1}^4, {\beta_1}^5, {\beta_1}^6, {\beta_1}^7\right)\\
&= (0, \alpha_1 - \beta_1, {\alpha_1}^2 - {\beta_1}^2, {\alpha_1}^3 - {\beta_1}^3, {\alpha_1}^4 - {\beta_1}^4, {\alpha_1}^5 - {\beta_1}^5, {\alpha_1}^6 - {\beta_1}^6, {\alpha_1}^7 - {\beta_1}^7)\\
&= (\alpha_1 - \beta_1)\big(0, 1, \alpha_1 + \beta_1, {\alpha_1}^2 + \alpha_1\beta_1 + {\beta_1}^2,\\
&\qquad (\alpha_1 + \beta_1)\left({\alpha_1}^2 + {\beta_1}^2\right){\alpha_1}^4 + {\alpha_1}^3\beta_1 + {\alpha_1}^2{\beta_1}^2 + \alpha_1{\beta_1}^3 + {\beta_1}^4,\\
&\qquad \left({\alpha_1}^2 + \alpha_1\beta_1 + {\beta_1}^2\right)(\alpha_1 + \beta_1)\left({\alpha_1}^2 + \alpha_1\beta_1 - {\beta_1}^2\right),\\
&\qquad {\alpha_1}^6 + {\alpha_1}^5\beta_1 + {\alpha_1}^4{\beta_1}^2 + {\alpha_1}^3{\beta_1}^3 + {\alpha_1}^2{\beta_1}^4 + \alpha_1{\beta_1}^5 + {\beta_1}^4\big)\\
&= \sqrt{\rho}\,(0, 1, k, k^2 + 2, k^3 + 4k, k^4 + 6k^2 + 2, k^5 + 8k^3 + 12k,\\
&\qquad k^6 + 10k^4 + 24k^2 + 8)\\
&= \sqrt{\rho}O^J{}_{k,0}.\end{aligned}$$

The proof of the others is similar to that of (1). □

Theorem 2.3. For $p, r \in \mathbb{Z}^+, p \geq r$ and $n \in \mathbb{N}$, the generating functions for $O^J{}_{k,n}$ and $O^j{}_{k,n}$ are as follows:

(i) $\sum_{n=0}^{\infty} O^J{}_{k,rn}x^n = \frac{O^J{}_{k,0} + (O^j{}_{k,0}\, J_{k,r} - O^J{}_{k,r})x}{(1 - j_{k,r}x + (-2)^r x^2)}$,

(ii) $\sum_{n=0}^{\infty} O^j{}_{k,rn}x^n = \frac{O^j{}_{k,0} - (O^j{}_{k,0}\, j_{k,r} - O^j{}_{k,r})x}{(1 - j_{k,r}x + (-2)^r x^2)}$,

(iii) $\sum_{n=0}^{\infty} O^J{}_{k,rn+p}x^n = \frac{O^J{}_{k,p} - (-2)^r\, O^J{}_{k,p-r}x}{(1 - j_{k,r}x + (-2)^r x^2)}$,

(iv) $\sum_{n=0}^{\infty} O^j{}_{k,rn+p}x^n = \frac{O^j{}_{k,p} - (-2)^r\, O^j{}_{k,p-r}x}{(1 - j_{k,r}x + (-2)^r x^2)}$, and the exponential generating functions for $O^J{}_{k,n}$ and $O^j{}_{k,n}$ are as follows

(v) $\sum_{\boldsymbol{n=0}}^{\infty}\frac{O^J{}_{k,rn}}{n!}x^n=\frac{\bar{\alpha}_1e^{\alpha_1{}^r x}-\bar{\beta}_1e^{\beta_1{}^r x}}{\alpha_1-\beta_1}$,

(vi) $\sum_{n=0}^{\infty}\frac{O^j{}_{k,rn}}{n!}x^n=\bar{\alpha}_1e^{\alpha_1{}^r x}+\bar{\beta}_1e^{\beta_1{}^r x}$,

Proof. (i) Using Theorem 2.4, we get the following:

$$\begin{aligned}
\sum_{\boldsymbol{n=0}}^{\infty}O^J{}_{k,rn}x^n &= \left(\frac{\bar{\alpha}_1\alpha_1{}^{rn}-\bar{\beta}_1\beta_1{}^{rn}}{\alpha_1-\beta_1}\right)x^n\\
&=\frac{\bar{\alpha}_1}{\alpha_1-\beta_1}\sum_{n=0}^{\infty}(\alpha_1{}^r x)^n-\frac{\bar{\beta}_1}{\alpha_1-\beta_1}\sum_{n=0}^{\infty}(\beta_1{}^r x)^n\\
&=\frac{\bar{\alpha}_1}{\alpha_1-\beta_1}\left(\frac{1}{1-\alpha_1{}^r x}\right)-\frac{\bar{\beta}_1}{\alpha_1-\beta_1}\left(\frac{1}{1-\beta_1{}^r x}\right)\\
&=\frac{(\bar{\alpha}_1-\bar{\beta}_1)+(\bar{\beta}_1\alpha_1{}^r-\bar{\alpha}_1\beta_1{}^r)x}{(\alpha_1-\beta_1)(1-(\alpha_1{}^r+\beta_1{}^r)x+(\alpha_1\beta_1)^r x^2)}\\
&=\frac{(\bar{\alpha}_1-\bar{\beta}_1)+(\bar{\beta}_1\alpha_1{}^r-\bar{\beta}_1\beta_1{}^r+\bar{\beta}_1\beta_1{}^r-\bar{\alpha}_1\beta_1{}^r+\bar{\alpha}_1\alpha_1{}^r-\bar{\alpha}_1\alpha_1{}^r)x}{(\alpha_1-\beta_1)(1-(\alpha_1{}^r+\beta_1{}^r)x+(\alpha_1\beta_1)^r x^2)}\\
&=\frac{(\bar{\alpha}_1-\bar{\beta}_1)+\left[(\bar{\alpha}_1+\bar{\beta}_1)(\alpha_1{}^r-\beta_1{}^r)+\bar{\beta}_1\beta_1{}^r-\bar{\alpha}_1\alpha_1{}^r\right]x}{(\alpha_1-\beta_1)(1-(\alpha_1{}^r+\beta_1{}^r)x+(\alpha_1\beta_1)^r x^2)}\\
&=\frac{\left(\frac{\bar{\alpha}_1-\bar{\beta}_1}{\alpha_1-\beta_1}\right)+\left[(\bar{\alpha}_1+\bar{\beta}_1)\left(\frac{\alpha_1{}^r-\beta_1{}^r}{\alpha_1-\beta_1}\right)+\left(\frac{\bar{\beta}_1\beta_1{}^r-\bar{\alpha}_1\alpha_1{}^r}{\alpha_1-\beta_1}\right)\right]x}{(1-(\alpha_1{}^r+\beta_1{}^r)x+(\alpha_1\beta_1)^r x^2)}\\
&=\frac{O^J{}_{k,0}+(O^j{}_{k,0}J_{k,r}-O^J{}_{k,r})x}{(1-j_{k,r}x+(-2)^r x^2)}.
\end{aligned}$$

The proofs of $(ii),(iii),(iv),(v)$ and (vi) are like that of (i) by using Theorem 2.4. □

Now let us give some equations that we will need in the next theorems.

Lemma 2.2. The following equations are provided.

(i) $\frac{\bar{v}_1\beta_1{}^t-\bar{v}_2\alpha_1{}^t}{\alpha_1-\beta_1}=\big(-73J_{k,t},-85J_{k,t+1}-166J_{k,t-1},-51J_{k,t+2}+196J_{k,t-2},$
$-17J_{k,t+3}+34J_{k,t+1}+68J_{k,t-1}+120J_{k,t-3},5J_{k,t+4}-112J_{k,t-4},3J_{k,t+5}-10J_{k,t+3}$
$+80J_{k,t-3}+160J_{k,t-5},J_{k,t+6}-12J_{k,t+2}+48J_{k,t-2}-192J_{k,t-6},-J_{k,t+7}$
$+2J_{k,t+5}\ 4J_{k,t+3}8J_{k,t+1}-128J_{k,t-7}-64J_{k,t-5}-32J_{k,t-3}+16J_{k,t-1}\big)=\bar{V}_{k,t}$

(ii) $\frac{\bar{v}_1\alpha_1{}^t-\bar{v}_2\beta_1{}^t}{\alpha_1-\beta_1}=\big(-73J_{k,t},-83J_{k,t+1}-170J_{k,t-1},-49J_{k,t+2}-204J_{k,t-2},$
$-15J_{k,t+3}+34J_{k,t+1}+68J_{k,t-1}-126J_{k,t-3},7J_{k,t+4}-80J_{k,t-4},$
$-5J_{k,t+5}+10J_{k,t+3}+80J_{k,t-3}+96J_{k,t-5},3J_{k,t+6}-12J_{k,t+2}+48J_{k,t-2}-64J_{k,t-6},$
$J_{k,t+7}-2J_{k,t+5},-4J_{k,t+3}+8J_{k,t+1},-128J_{k,t-2}-64J_{k,t-5}-32J_{k,t-3}+16J_{k,t-1}\big)$
$=\bar{U}_{k,t}$

$(iii)\ \bar{\alpha}_1{}^2\,\alpha_1{}^{2t} + \ \bar{\beta}_1{}^2\beta_1{}^{2t} = 2\,O^j{}_{k,t} - j_{k,t} - j_{k,t+4} - j_{k,t+6} + j_{k,t+8}$
$+j_{k,t+10} + j_{k,t+12} + j_{k,t+10} + j_{k,t14} = \bar{R}_{k,t}.$

Proof.

$$(i)\quad \frac{\bar{v}_1\beta_1{}^t - \bar{v}_2\alpha_1{}^t}{\alpha_1-\beta_1} = \frac{1}{\alpha_1-\beta_1}$$
$$\big(-73, -83\alpha_1 + 85\beta_1, -49\alpha_1{}^2 + 51\beta_1{}^2, 34\alpha_1 - 34\beta_1 - 15\alpha_1{}^3 + 17\beta_1{}^3,$$
$$7\alpha_1{}^4 - 5\beta_1{}^4, 5\alpha_1{}^5 - 3\beta_1{}^5 + 10\alpha_1{}^3 - 10\beta_1{}^3, 3\alpha_1{}^6 - \beta_1{}^6 + 12\beta_1{}^2 - 12\alpha_1{}^2,$$
$$\alpha_1{}^7 - 2\alpha_1{}^5 - 4\alpha_1{}^3 + 8\alpha_1 + \beta_1{}^7 + 2\beta_1{}^5 + 4\beta_1{}^3 - 8\beta_1\big)\beta_1{}^t$$
$$-\big(-73, \quad 85\alpha_1 - 83\beta_1, 51\alpha_1{}^2 - 49\beta_1{}^2, -34\alpha_1 + 34\beta_1 + 17\alpha_1{}^3 - 15\beta_1{}^3,$$
$$-5\alpha_1{}^4 + 7\beta_1{}^4, 5\beta_1{}^5 - 3\alpha_1{}^5 - 10\alpha_1{}^3 + 10\beta_1{}^3, -\alpha_1{}^6 + 3\beta_1{}^6 - 12\beta_1{}^2 + 12\alpha_1{}^2,$$
$$\alpha_1{}^7 + 2\alpha_1{}^5 + 4\alpha_1{}^3 - 8\alpha_1 + \beta_1{}^7 - 2\beta_1{}^5 - 4\beta_1{}^3 + 8\beta_1\big)\alpha_1{}^t$$

If the necessary calculations are made, then

$$\big(-73J_{k,t}, -85J_{k,t+1} - 166J_{k,t-1}, -51J_{k,t+2} + 196J_{k,t-2},$$
$$-17J_{k,t+3} + 34J_{k,t+1} + 68J_{k,t-1} + 120J_{k,t-3}, 5J_{k,t+4} - 112J_{k,t-4}, 3J_{k,t+5} - 10J_{k,t+3}$$
$$+80J_{k,t-3} + 160J_{k,t-5}, J_{k,t+6} - 12J_{k,t+2} + 48J_{k,t-2} - 192J_{k,t-6}, -J_{k,t+7} +$$
$$2J_{k,t+5}\ 4J_{k,t+3}8J_{k,t+1} - 128J_{k,t-7} - 64J_{k,t-5} - 32J_{k,t-3} + 16J_{k,t-1}\big) = \bar{V}_{k,t}.$$

The proofs of (ii) and (iii) are like that of (i). □

Theorem 2.3 (Catalan's Identity). For any integer n and t, we have

$(i)\quad O^J{}_{k,n-t}\,O^J{}_{k,n+t} - \ O^J{}_{k,n}{}^2 = (-2)^{n-t}J_{k,t}\bar{V}_{k,t}$

$(ii)\quad O^j{}_{k,n-t}\,O^j{}_{k,n+t} - \ O^j{}_{k,n}{}^2 = -\rho\,(-2)^{n-t}J_{k,t}\bar{V}_{k,t}$

Proof. (i) By using the Binet formula of the hyperbolic *k*-Jacobsthal, we have

$$O^J{}_{k,n-t}\,O^J{}_{k,n+t} - \ O^J{}_{k,n}{}^2$$
$$= \left(\frac{\bar{\alpha}_1\alpha_1{}^{n-t} - \bar{\beta}_1\beta_1{}^{n-t}}{\alpha_1-\beta_1}\right)\left(\frac{\bar{\alpha}_1\alpha_1{}^{n+t} - \bar{\beta}_1\beta_1{}^{n+t}}{\alpha_1-\beta_1}\right) - \left(\frac{\bar{\alpha}_1\alpha_1{}^{n} - \bar{\beta}_1\beta_1{}^{n}}{\alpha_1-\beta_1}\right)^{\mathbf{2}}$$
$$= \frac{1}{(\alpha_1-\beta_1)^2}\Big[\bar{\alpha}_1{}^2\alpha_1{}^{2n} - \bar{\alpha}_1\bar{\beta}_1\alpha_1{}^{n-t}\beta_1{}^{n+t} - \bar{\beta}_1\bar{\alpha}_1\beta_1{}^{n-t}\alpha_1{}^{n+t} + \bar{\beta}_1{}^2\beta_1{}^{2n}$$
$$-\bar{\alpha}_1{}^2\alpha_1{}^{2n} + \bar{\alpha}_1\bar{\beta}_1\alpha_1{}^n\beta_1{}^n + \bar{\beta}_1\bar{\alpha}_1\beta_1{}^n\alpha_1{}^n - \bar{\beta}_1{}^2\beta_1{}^{2n}\Big]$$
$$= \frac{1}{(\alpha_1-\beta_1)^2}\left[\bar{\alpha}_1\bar{\beta}_1\alpha_1{}^n\beta_1{}^n\left(1 - \frac{\beta_1{}^t}{\alpha_1{}^t}\right) + \bar{\beta}_1\bar{\alpha}_1\beta_1{}^n\alpha_1{}^n\left(1 - \frac{\alpha_1{}^t}{\beta_1{}^t}\right)\right]$$
$$= \frac{(\alpha_1\beta_1)^n\big(\alpha_1{}^t - \beta_1{}^t\big)}{(\alpha_1-\beta_1)^2}\left[\frac{\bar{\alpha}_1\bar{\beta}_1}{\alpha_1{}^t} - \frac{\bar{\beta}_1\bar{\alpha}_1}{\beta_1{}^t}\right] = \frac{(\alpha_1\beta_1)^n\big(\alpha_1{}^t - \beta_1{}^t\big)}{(\alpha_1-\beta_1)^2}\left[\frac{\bar{\alpha}_1\bar{\beta}_1\beta_1{}^t - \bar{\beta}_1\bar{\alpha}_1\alpha_1{}^t}{(\alpha_1\beta_1)^t}\right]$$
$$= (\alpha_1\beta_1)^{n-t}J_{k,t}\left[\frac{\bar{v}_1\beta_1{}^t - \bar{v}_2\alpha_1{}^t}{\alpha_1-\beta_1}\right], \qquad \text{(from } \alpha_1\beta_1 = -2 \text{ and Lemma 2.2)}$$
$$= (-2)^{n-t}J_{k,t}\bar{V}_{k,t}$$

The proof of (ii) is like that of (i). □

Theorem 2.4 (Cassini's Identity). For $n \geq 1$, we get

$(i)\quad O^J{}_{k,n-t}\, O^J{}_{k,n+t} - O^J{}_{k,n}{}^2 = (-2)^{n-t} J_{k,t} \bar{V}_{k,t}$

$(ii)\quad O^j{}_{k,n-t}\, O^j{}_{k,n+t} - O^j{}_{k,n}{}^2 = -\rho\, (-2)^{n-t} J_{k,t} \bar{V}_{k,t}$

Proof. If we substitute 1 for t in the Catalan identity, we get the Cassini identity. □

Theorem 2.5 (D'Ocagne Identity). Let n be any non-negative integer and $t \geq n$, then we have:

$(i)\quad O^J{}_{k,t}\, O^J{}_{k,n+1} - O^J{}_{k,t+1}\, O^J{}_{k,n} = (-2)^n \bar{U}_{k,t-n}$

$(ii)\quad O^j{}_{k,t}\, O^j{}_{k,n+1} - O^j{}_{k,t+1}\, O^j{}_{k,n} = -\rho(-2)^n \bar{U}_{k,t-n}.$

Proof. (i) By using The Binet formula of the hyperbolic k-Jacobsthal, we have

$$
\begin{aligned}
&O^J{}_{k,t}\, O^J{}_{k,n+1} - O^J{}_{k,t+1}\, O^J{}_{k,n} \\
&= \left(\frac{\bar{\alpha}_1 \alpha_1{}^t - \bar{\beta}_1 \beta_1{}^t}{\alpha_1 - \beta_1}\right)\left(\frac{\bar{\alpha}_1 \alpha_1{}^{n+1} - \bar{\beta}_1 \beta_1{}^{n+1}}{\alpha_1 - \beta_1}\right) - \left(\frac{\bar{\alpha}_1 \alpha_1{}^{t+1} - \bar{\beta}_1 \beta_1{}^{t+1}}{\alpha_1 - \beta_1}\right)\left(\frac{\bar{\alpha}_1 \alpha_1{}^n - \bar{\beta}_1 \beta_1{}^n}{\alpha_1 - \beta_1}\right) \\
&= \frac{1}{(\alpha_1 - \beta_1)^2}\Big[\bar{\alpha}_1{}^2 \alpha_1{}^{n+t+1} - \bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^t \beta_1{}^{n+1} - \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^t \alpha_1{}^{n+1} + \bar{\beta}_1{}^2 \beta_1{}^{n+1+t} \\
&\qquad - \bar{\alpha}_1{}^2 \alpha_1{}^{n+t+1} + \bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^{t+1} \beta_1{}^n + \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^{t+1} \alpha_1{}^n - \bar{\beta}_1{}^2 \beta_1{}^{n+t+1}\Big] \\
&= \frac{1}{(\alpha_1 - \beta_1)^2}\left[\bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^t \beta_1{}^n (\alpha_1 - \beta_1) + \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^t \alpha_1{}^n (\beta_1 - \alpha_1)\right] \\
&= \frac{\alpha_1{}^n \beta_1{}^n (\alpha_1 - \beta_1)}{(\alpha_1 - \beta_1)^2}\left[\bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^{t-n} - \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^{t-n}\right] \\
&= (-2)^n \left[\frac{\bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^{t-n} - \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^{t-n}}{\alpha_1 - \beta_1}\right] \\
&= (-2)^n \left[\frac{\bar{v}_1 \beta_1{}^t - \bar{v}_2 \alpha_1{}^t}{\alpha_1 - \beta_1}\right] = (-2)^n \bar{U}_{k,t-n}
\end{aligned}
$$

The proof of (ii) is like to (i). □

Theorem 2.6 (Vajda Identity). If i and j are any two natural numbers then we have

$(i)\quad O^J{}_{k,n+i}\, O^J{}_{k,n+j} - O^J{}_{k,n}\, O^J{}_{k,n+i+j} = -(-2)^n J_{k,i} \bar{V}_{k,j}$

$(ii)\quad O^j{}_{k,n+i}\, O^j{}_{k,n+j} - O^j{}_{k,n}\, O^j{}_{k,n+i+j} = \rho\, (-2)^n J_{k,i} \bar{V}_{k,t}.$

Proof. The proof is done using Theorem 2.2, similar to Theorem 2.5. □

Theorem 2.7. For any integer t, we have

$(i)\quad O^J{}_{k,t}{}^2 + O^j{}_{k,t}{}^2 = \dfrac{1}{\rho}\Big((1+\rho)\bar{R}_{k,2t} + (-2)^{t+1}\big(73\bar{O}^j{}_{k,0}\big)(\rho - 1)\Big)$

$(ii)\quad O^J{}_{k,t}{}^2 - O^j{}_{k,t}{}^2 = \dfrac{1}{\rho}\Big((1-\rho)\bar{R}_{k,2t} + \big(-75 + O^j{}_{k,0}\big)\, (-2)^{t+1}(1+\rho)\Big)$

.

Proof.

$(i)\quad O^J{}_{k,t}{}^2 + O^j{}_{k,t}{}^2 = \left(\dfrac{\bar{\alpha}_1 \alpha_1{}^k - \bar{\beta}_1 \beta_1{}^k}{\alpha_1 - \beta_1}\right)^{\mathbf{2}} + \left(\bar{\alpha}_1 \alpha_1{}^t + \bar{\beta}_1 \beta_1{}^t\right)^{\mathbf{2}}$

$$= \left(\frac{\bar{\alpha}_1{}^2 \alpha_1{}^{2t} - \bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^t \beta_1{}^t - \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^t \alpha_1{}^t + \bar{\beta}_1{}^2 \beta_1{}^{2t}}{\rho} \right)$$
$$+ \left(\bar{\alpha}_1{}^2 \alpha_1{}^{2t} + \bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^t \beta_1{}^t + \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^t \alpha_1{}^t + \bar{\beta}_1{}^2 \beta_1{}^{2t} \right)$$
$$= \left(\frac{(1+\rho)\left(\bar{\alpha}_1{}^2 \alpha_1{}^{2t} + \bar{\beta}_1{}^2 \beta_1{}^{2t} \right) + (\rho - 1)\left(\alpha_1{}^t \beta_1{}^t \right)\left(\bar{\alpha}_1 \bar{\beta}_1 + \bar{\beta}_1 \bar{\alpha}_1 \right)}{\rho} \right)$$

(from Lemma 2.1 and Lemma 2.2)

$$= \frac{1}{\rho}\left((1+\rho)\bar{R}_{k,2t} + (\rho - 1)\,(-2)^{t+1}\left(73\bar{O}^j{}_{k,0}\right) \right)$$

The proof (ii) is similar to that of (i). □

Theorem 2.8. For every integers $r, s \geq t$, there is the following equation.

$$O^J{}_{k,r+s}\, O^j{}_{k,r+t} - O^J{}_{k,r+t}\, O^j{}_{k,r+s} = 2\left(-75 + O^j{}_{k,0}\right)(-2)^{r-t} J_{k,s-t}.$$

Proof.

$$O^J{}_{k,r+s}\, O^j{}_{k,r+t} - O^J{}_{k,r+t}\, O^j{}_{k,r+s}$$
$$= \left(\frac{\bar{\alpha}_1 \alpha_1{}^{r+s} - \bar{\beta}_1 \beta_1{}^{r+s}}{\alpha_1 - \beta_1} \right)\left(\bar{\alpha}_1 \alpha_1{}^{r+t} + \bar{\beta}_1 \beta_1{}^{r+t} \right) - \left(\frac{\bar{\alpha}_1 \alpha_1{}^{r+t} - \bar{\beta}_1 \beta_1{}^{r+t}}{\alpha_1 - \beta_1} \right)\left(\bar{\alpha}_1 \alpha_1{}^{r+s} + \bar{\beta}_1 \beta_1{}^{r+s} \right)$$
$$= \left(\frac{\bar{\alpha}_1{}^2 \alpha_1{}^{2r+s+t} + \bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^{r+s} \beta_1{}^{r+t} - \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^{r+s} \alpha_1{}^{r+t} - \bar{\beta}_1{}^2 \beta_1{}^{2r+s+t}}{\alpha_1 - \beta_1} \right.$$
$$\left. - \frac{\bar{\alpha}_1{}^2 \alpha_1{}^{2r+s+t} + \bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^{r+t} \beta_1{}^{r+s} - \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^{r+t} \alpha_1{}^{r+s} - \bar{\beta}_1{}^2 \beta_1{}^{2r+s+t}}{\alpha_1 - \beta_1} \right)$$
$$= \left(\frac{\bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^r \beta_1{}^r \left(\alpha_1{}^s \beta_1{}^t - \alpha_1{}^t \beta_1{}^s \right) + \bar{\beta}_1 \bar{\alpha}_1 \alpha_1{}^r \beta_1{}^r \left(\alpha_1{}^s \beta_1{}^t - \alpha_1{}^t \beta_1{}^s \right)}{\alpha_1 - \beta_1} \right)$$
$$= \left(\frac{\left(\bar{\alpha}_1 \bar{\beta}_1 + \bar{\beta}_1 \bar{\alpha}_1 \right) \alpha_1{}^r \beta_1{}^r \left(\alpha_1{}^s \beta_1{}^t - \alpha_1{}^t \beta_1{}^s \right)}{\alpha_1 - \beta_1} \right)$$
$$= \left(\bar{\alpha}_1 \bar{\beta}_1 + \bar{\beta}_1 \bar{\alpha}_1 \right)(\alpha_1 \beta_1)^{r-t} \left(\frac{\alpha_1{}^{s-t} - \beta_1{}^{s-t}}{\alpha_1 - \beta_1} \right)$$
$$= 2\left(-75 + O^j{}_{k,0}\right)(-2)^{r-t} J_{k,s-t}.$$

This completes the proof. □

Theorem 2.9. For any integer s and t, we prove that

(i) $\quad O^J{}_{k,s} j_{k,t} = O^J{}_{k,s+t} + (-2)^t O^J{}_{k,s-t}$

(ii) $\quad O^j{}_{k,s} J_{k,t} = O^j{}_{k,s+t} + (-2)^t O^j{}_{k,s-t}$.

Proof. (i) If we use Theorem 2.2, then we have

$$O^J{}_{k,s} j_{k,t} = \left(\frac{\bar{\alpha}_1 \alpha_1{}^s - \bar{\beta}_1 \beta_1{}^s}{\alpha_1 - \beta_1} \right)\left(\alpha_1{}^t + \beta_1{}^t \right)$$
$$= \left(\frac{\bar{\alpha}_1 \alpha_1{}^{s+t} + \bar{\alpha}_1 \alpha_1{}^s \beta_1{}^t - \bar{\beta}_1 \beta_1{}^s \alpha_1{}^t - \bar{\beta}_1 \beta_1{}^{s+t}}{\alpha_1 - \beta_1} \right)$$

$$= \left(\frac{\bar{\alpha}_1 \alpha_1{}^{s+t} - \bar{\beta}_1 \beta_1{}^{s+t}}{\alpha_1 - \beta_1}\right) + \left(\frac{\alpha_1{}^t \beta_1{}^t \left(\bar{\alpha}_1 \alpha_1{}^{s-t} - \bar{\beta}_1 \beta_1{}^{s-t}\right)}{\alpha_1 - \beta_1}\right)$$

$$= O^J{}_{k,s+t} + (-2)^t O^J{}_{k,s-t}.$$

The proof (ii) is similar to that of (i). □

Theorem 2.10. For any integer $s \leq t$, the following equations are true.

(i) $O^J{}_{k,s}\, O^J{}_{k,t} - O^J{}_{k,t} O^J{}_{k,s} = \frac{1}{\sqrt{\rho}}\, (-2)^s \bar{v}_3\, J_{k,t-s}$,

(ii) $O^j{}_{k,s}\, O^j{}_{k,t} - O^j{}_{k,t} O^j{}_{k,s} = \sqrt{\rho}\, (-2)^s \bar{v}_3\, J_{k,t-s}$,

(iii) $O^J{}_{k,t}\, O^j{}_{k,s} - O^J{}_{k,s} O^j{}_{k,t} = (-2)^{s+1}\left(75 - O^j{}_{k,0}\right) J_{k,t-s}$,

(iv) $O^J{}_{k,t}\, O^j{}_{k,s} - O^j{}_{k,t} O^J{}_{k,s} = -(-2)^{s+1} \bar{V}_{k,t-s}$.

Proof. (i)

$O^J{}_{k,s}\, O^J{}_{k,t} - O^J{}_{k,t} O^J{}_{k,s}$

$$= \left(\frac{\bar{\alpha}_1 \alpha_1{}^s - \bar{\beta}_1 \beta_1{}^s}{\alpha_1 - \beta_1}\right)\left(\frac{\bar{\alpha}_1 \alpha_1{}^t - \bar{\beta}_1 \beta_1{}^t}{\alpha_1 - \beta_1}\right) - \left(\frac{\bar{\alpha}_1 \alpha_1{}^t - \bar{\beta}_1 \beta_1{}^t}{\alpha_1 - \beta_1}\right)\left(\frac{\bar{\alpha}_1 \alpha_1{}^s - \bar{\beta}_1 \beta_1{}^s}{\alpha_1 - \beta_1}\right)$$

$$= \frac{1}{(\alpha_1 - \beta_1)^2}\Big(\bar{\alpha}_1{}^2 \alpha_1{}^{s+t} - \bar{\alpha}_1 \overline{\bar{\beta}_1 \beta} \alpha_1{}^s \beta_1{}^t - \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^s \alpha_1{}^t + \bar{\beta}_1{}^2 \beta_1{}^{s+t}$$

$$-\bar{\alpha}_1 \alpha_1{}^{s+t} + \bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^s \beta_1{}^t + \bar{\beta}_1 \bar{\alpha}_1 \beta_1{}^s \alpha_1{}^t - \bar{\beta}_1 \beta_1{}^{s+t}\Big)$$

$$= \frac{\bar{\alpha}_1 \bar{\beta}_1 \alpha_1{}^s \beta_1{}^s \left(\alpha_1{}^{t-s} - \beta_1{}^{t-s}\right) - \bar{\beta}_1 \bar{\alpha}_1 \alpha_1{}^s \beta_1{}^s \left(\alpha_1{}^{t-s} - \beta_1{}^{t-s}\right)}{(\alpha_1 - \beta_1)^2}$$

$$= \frac{\left(\bar{\alpha}_1 \bar{\beta}_1 - \bar{\beta}_1 \bar{\alpha}_1\right) \alpha_1{}^s \beta_1{}^s \left(\alpha_1{}^{t-s} - \beta_1{}^{t-s}\right)}{(\alpha_1 - \beta_1)^2}$$

$$= \frac{1}{\sqrt{\rho}}\, (-2)^s \bar{v}_3\, J_{k,t-s}.$$

The other equations are proved similarly to that of (i). □

3 Conclusion

We defined the hyperbolic k-Jacobsthal and k-Jacobsthal–Lucas octonions. We presented Binet's Formula, Cassini's identity, Catalan's identity, d'Ocagne identity, generating functions of the hyperbolic k-Jacobsthal and k-Jacobsthal–Lucas octonions. Also, it was given many properties of these octonions.

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