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About the theorem that partially solves the Navarrete–Orellana Conjecture

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Abstract: The Navarrete–Orellana Conjecture states that *"given a large prime number a sequence is generated, in such a way that all odd prime numbers, except the given prime, are fixed points of that sequence"*. In this work, we formulated a theorem that partially confirms the veracity of this conjecture, more specifically, all prime numbers of a given line segment are fixed points of this sequence.

Keywords: Prime numbers, Triangular numbers, Fixed points, Sequence family, Conjecture.
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1 Introduction

Prime numbers fascinate mathematicians, physicists, computer scientists, programmers, cryptologists, number-loving people, among others. They are the subject of extensive research and are responsible for many open problems in mathematics. We are going to mention some examples of the most common themes involving prime numbers:

(a) **The distribution of prime numbers:** The open problem: "The Riemann Hypothesis". The Fractional Differential Equation that models the distribution of prime numbers, [1].

(b) **Primality testing:** The "Wilson's Theorem", the "Miller–Rabin Test", [7], the "Goldwasser–Kilian Test", [3], which uses elliptical curves. Here, the non-Primality Testing can also be included. A classic non-primality testing is the counterpositive of Fermat's Little Theorem.

(c) Prime-Generating Polynomial: The classic “Euler Polynomial” that generates 40 prime numbers, [6], the “Dress, Laudreau and Gupta Polynomial” that generates 57 prime numbers, [9], and the “Prime numbers in arithmetic progression”.

(d) Graphical representation of prime numbers: The “Ulam Spiral” and “Image Analysis”, [4], show patterns of prime numbers.

(e) Formulas that generate prime numbers: The classic formula for “Mersenne Primes”. The “Mills Theorem” and the “Wright Theorem” are two beautiful examples of formulas that generate prime numbers, [2].

(f) Natural numbers expressed by prime numbers: The open problem: “The Goldbach conjecture” and the Fundamental Theorem of Arithmetic.

Navarrete and Orellana, [5], created four conjectures that involve prime numbers and fixed points of a given sequence. For the last conjecture, the most important, they performed some numerical tests, before enunciating their formulation. First, they considered a prime number p, with that number, they generated the sequence $A(p) = \{a(n)\}_{n=1}^{10^4}$ (it is known that from 1 to $10\,000$ there are $1\,229$ prime numbers). From the total of prime numbers in the sequence, two were excluded: the even prime and the one that generates the sequence, that is, $1\,227$ prime numbers were considered as reference for the tests. They realized that the greater the value of the prime number p, the greater is the quantity of prime numbers that are fixed points in this sequence. To verify this, they calculated a rate, for example, for $p = 3$ the sequence $A(3) = \{a(n)\}_{n=1}^{10^4}$ has $1\,160$ prime numbers that are fixed points, so the success rate is $(1\,160/1\,227)100\% = 94.53\%$, for $p = 199$ the success rate was $(1\,296/1\,227)100\% = 99.92\%$, and for $p = 541$ the success rate was 100%. These results led to the following conjecture: *“Given a large prime number, any odd prime number, other than the one used to generate the sequence, is a fixed point of that sequence”*.

In this work, we present a partial proof of the Navarrete–Orellana Conjecture. The proof of the main Theorem, as shown in this work, affirms the veracity of the conjecture for a certain line segment of prime numbers, roughly said, all prime numbers that belong to the line segment are fixed points. The counterpositive of this theorem generates a non-primality test. Furthermore, we demonstrate two other conjectures from the same author.

The partial proof of the main Theorem may be a first step towards a definitive proof, or perhaps, for a larger line segment, but in any case it represents a possible path for a more efficient and computable understanding of prime numbers.

2 Preliminary results

In this Section, we will define some concepts and enunciate useful results for the next section.

Definition 2.1 (Set of natural numbers). $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$

Notation 2.1. $\mathbb{N}^\star = \mathbb{N} - \{0\}$

Definition 2.2 (Set of integer numbers). $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$

Definition 2.3 (Set of prime numbers). $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$

Notation 2.2. $\mathcal{P}^\star = \mathcal{P} - \{2\}$

Observation 2.1. The prime numbers as an ordered sequence:

$$\mathcal{P} \;=\; \{p_k\}_{k\in\mathbb{N}^\star} = \{p_1, p_2, p_3, p_4 \ldots\} \tag{1}$$

where, $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7, \ldots$ Similarly, $\mathcal{P}^\star = \{p_k\}_{k\geqslant 2}$.

Definition 2.4 (Divisor of a number n). Let $d, n \in \mathbb{Z},\ d \neq 0$. We define "$d$ divides n", denoted by $d|n$, as

$$d|n \iff \exists\, k \in \mathbb{Z};\ n = kd \tag{2}$$

Definition 2.5 (The set of divisors of a number n). Let $n \in \mathbb{N}$. We define, the set of divisors of n, denoted by $D[n]$, as

$$D\,[n] \;=\; \{d \in \mathbb{N}^\star :\ d|n\} \tag{3}$$

Definition 2.6 (Sequence of triangular numbers).

$$\{t(n)\}_{n\in\mathbb{N}^\star} \;=\; \left\{\frac{n(n-1)}{2}\right\}_{n\in\mathbb{N}^\star} \;=\; \{0, 1, 3, 6, 10, 15, 21, \ldots\} \tag{4}$$

Definition 2.7 (Sequence of p-multiples of triangular numbers). Let $p \in \mathcal{P}$. We define,

$$\{t_p(n)\}_{n\in\mathbb{N}^\star} \;=\; \left\{\frac{n(n-1)p}{2}\right\}_{n\in\mathbb{N}^\star} \;=\; \{0, p, 3p, 6p, 10p, 15p, 21p, \ldots\} \tag{5}$$

Definition 2.8 (The set of divisors of $t_p(n)$). Let $p \in \mathcal{P}$, and for each $n \in \mathbb{N}^\star$, we define the set of divisors of $t_p(n)$, as

$$D\,[t_p(n)] \;=\; \{d_p \in \mathbb{N}^\star :\ d_p \mid t_p(n)\} \tag{6}$$

Example 2.1. Given $p = 3$. We show, in the last column of Table 1, the first seven sets $D\,[t_p(n)]\,,\ n = 1, 2, \ldots, 7$.

n	$t(n)$	$t_3(n)$	$D\,[t_3(n)]$
1	0	0	$\mathbb{N}^*$
2	1	3	$\{1, 3\}$
3	3	9	$\{1, 3, 9\}$
4	6	18	$\{1, 2, 3, 6, 9, 18\}$
5	10	30	$\{1, 2, 3, 5, 6, 10, 15, 30\}$
6	15	45	$\{1, 3, 5, 9, 15, 45\}$
7	21	63	$\{1, 3, 7, 9, 21, 63\}$

Table 1. The set of divisors of $t_3(n)$ for $n = 1, 2, \ldots, 7$.

Definition 2.9 (Fixed point). Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function. We say that $n \in \mathbb{Z}$ is a fixed point of f, if $f(n) = n$.

Theorem 2.1 (Bertrand–Chebyshev Theorem). *Let $p_k \in \mathcal{P}$, $k \geqslant 1$. Then, $p_{k+1} < 2p_k$.*

Proof. The proof of this Theorem can be found at [8]. □

Example 2.2. Let $p_9 = 23$ and $p_{10} = 29$. Then, by the Bertrand–Chebyshev Theorem, $p_{10} = 23$ $< 2p_9 = 58$.

3 The family of sequence $\mathcal{A}$

In this Section, we will define and study a special family of sequences, denoted by $\mathcal{A} = \{A(p)\}_{p\in\mathcal{P}}$.

Definition 3.1 (Sequence $A(p)$). Let $p \in \mathcal{P}$. We define the sequence $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star}$ by

$$a(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ p & \text{if} \quad n = 2 \\ \min D\,[t_p(n)] & \text{if} \quad n \geqslant 3,\ a(n) \neq a(n'),\ \forall n' < n \end{cases} \tag{7}$$

where, $a(n)$ is the n-th term of the sequence $A(p)$, and a is a function from $\mathbb{N}^\star$ onto $\mathbb{N}^\star$.

Observation 3.1.

(i) The sequence $A(p)$ is formed by the smallest element of each set of $t_p(n)$ divisors, which is not yet an element of the sequence.

(ii) Note that the nth term of the sequence $A(p)$ has the property of $a(1) = 1$ and $a(2) = p$, a consequence of 1 being the smallest divisor of the set $\mathbb{N}^\star$ and p being the smallest divisor of the set $\{1, p\}$, which is not yet an element of the sequence .

(iii) It is important to note that for each $p \in \mathcal{P}$, there is a sequence $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star} = \{1, p, \ldots\}$.

(iv) When the prime number p is increasing, the calculation process for obtaining the sets $D[t_p(n)]$ increases, this increase is even greater when the value of n grows, making it difficult to obtain without the sets using a computer program.

Example 3.1. In Table 2 we show that for $p = 3$, the first seven elements of the sequence $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star}$ are $A(3) = \{1, p, 9, 2, 5, 15, 7, \ldots\}$. The third element of $A(3)$ is 9, because it is the smallest element of the set of divisors $D[t_3(3)] = \{1, 3, 9\}$ which is not yet part of the sequence $A(3)$.

Proposition 3.1. *The elements of $A(p)$ satisfy a biunivocal relation, that is, if $n \neq n'$, then $a(n) \neq a(n')$.*

Proof. The proof is an immediate consequence of the definition of $A(p)$, given in (7). □

n	$t(n)$	$t_3(n)$	$D[t_3(n)]$	$a(n)$
1	0	0	$\mathbb{N}^*$	1
2	1	3	$\{1,3\}$	p
3	3	9	$\{1,3,9\}$	9
4	6	18	$\{1,2,3,6,9,18\}$	2
5	10	30	$\{1,2,3,5,6,10,15,30\}$	5
6	15	45	$\{1,3,5,9,15,45\}$	15
7	21	63	$\{1,3,7,9,21,63\}$	7

Table 2. For $p = 3$, as shown in the last column, the first seven elements of the sequence $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star}$ are $A(3) = \{1, p, 9, 2, 5, 15, 7, \ldots\}$.

In order to reduce the process of calculating the elements of the sequence $A(p)$, we define the following sequence.

Definition 3.2 (Sequence $\tilde{A}(p)$). Let $p \in \mathcal{P}^\star$. We define the sequence $\tilde{A}(p) = \{\tilde{a}(n)\}_{n\in\mathbb{N}^\star}$ by

$$\tilde{a}(n) \quad = \quad \begin{cases} 1 & \text{if } \ n = 1 \\ p & \text{if } \ n = 2 \\ \min D\,[t(n)] & \text{if } \ n \geqslant 3, \ \tilde{a}(n) \neq \tilde{a}(n'), \ \forall n' < n \end{cases} \tag{8}$$

where, $\tilde{a}(n)$ is the n-th term of the sequence $\tilde{A}(p)$, and $\tilde{a}$ is a function from $\mathbb{N}^\star$ onto $\mathbb{N}^\star$.

Observation 3.2.

(i) The computational time used to calculate the elements of the sequence $\tilde{A}(p)$ is not as high as the time consumed on the sequence $A(p)$, because, in the first sequence, we will be calculating the divisors of (4) and, in the second, the divisors of (5). This fact is significant for large values of p.

(ii) Taking advantage of the definition of $\tilde{A}(p)$, we define sequence $\tilde{A}(1)$ by (8), where $p = 1$.

The following proposition ensures that from a certain prime number, the $A(p)$ and $\tilde{A}(p)$ sequences are the same.

Proposition 3.2. *Let $A(p)$ and $\tilde{A}(p)$ be two sequences such that $p \geqslant 5$ for all $p \in \mathcal{P}$. If $p' \in \mathcal{P}^\star$, $p' < 2p$ and $p' \neq p$, then $a(p') = \tilde{a}(p')$.*

Proof. Let $a(p') = m$ and $\tilde{a}(p') = m'$. First, let's prove that m' is not a multiple of p. Suppose that m' is a multiple of p. As $a(2) = p$, it follows that $m' \geqslant 2p$. We have $p' \mid \left[\frac{p'(p'-1)}{2}\right]$ and $\tilde{a}(n) \neq p', n < p'$, since $p' \nmid \left[\frac{n(n-1)}{2}\right]$, and as $p' < 2p \leqslant m'$ then $\tilde{a}(p') = p'$, which is absurd. Therefore m' is not a multiple of p. Similarly, we have that m is not a multiple of p.

Let $m' < m$. As $m' \mid \left[\frac{p'(p'-1)}{2}\right]$, and hence it follows that $m' \mid \left[\frac{p'(p'-1)p}{2}\right]$. So, $\exists q;\ a(q) = m',\ q < p'$. Thus, $m' \mid \left[\frac{q(q-1)p}{2}\right]$, and as m' is not a multiple of p, it follows that $m' \mid \left[\frac{q(q-1)}{2}\right]$, and as $q < p'$, it follows that $\tilde{a}(q) = m'$; absurd.

Let $m < m'$. As $m \mid \left[\frac{p'(p'-1)p}{2}\right]$, which implies that $m \mid \left[\frac{p'(p'-1)}{2}\right]$, since m is not a multiple of p, so $\exists q;\ \tilde{a}(q) = m,\ q < p'$. Thus, $m \mid \left[\frac{q(q-1)}{2}\right]$, and as $m \mid \left[\frac{q(q-1)p}{2}\right]$ and $q < p'$; it follows that $a(q) = m$; absurd.

Therefore $m = m'$, i.e., $a(p') = \tilde{a}(p')$. □

Notation 3.1. In the next tables, where the acronym "n.a." appears, it means that for this value of n, the example is not applicable.

Example 3.2. For $p = 5$ we show, in Table 3, the values of $\tilde{a}(n)$ and $a(n)$ when $p' < 2p = 10$. Note that $\tilde{A}(5) = A(5) = \{1, p, 3, 7\}$.

n	p'	$t(n)$	$t_5(n)$	$D[t(n)]$	$D[t_5(n)]$	$\tilde{a}(n)$	$a(n)$	$\tilde{a}(n) = a(n)$
1		0	0	$\mathbb{N}^*$	$\mathbb{N}^*$	1	1	✓
2		1	5	$\{1\}$	$\{1, p\}$	p	p	✓
3	3	3	15	$\{1, 3\}$	$\{1, 3, 5, 15\}$	3	3	✓
4		6	30	$\{1, 2, 3, 6\}$	$\{1, 2, 3, 5, 6, 10, 15, 30\}$	2	2	n.a.
5	5	10	50	$\{1, 2, 5, 10\}$	$\{1, 2, 5, 10, 25, 50\}$	10	10	n.a.
6		15	75	$\{1, 3, 5, 15\}$	$\{1, 3, 5, 15, 25, 75\}$	15	15	n.a.
7	7	21	105	$\{1, 3, 7, 21\}$	$\{1, 3, 5, 7, 15, 21, 35, 105\}$	7	7	✓
8		28	140	$\{1, 2, 4, 7, 14, 28\}$	$\{1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, 140\}$	4	4	n.a.
9		36	180	$\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$	$\{1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180\}$	6	6	n.a.

Table 3. The elements of $\tilde{A}(5) = \{\tilde{a}(n)\}_{n\in\mathbb{N}^\star}$ and $A(5) = \{a(n)\}_{n\in\mathbb{N}^\star}$.

Example 3.3. For $p = 7$ we show, in Table 4, the values of $\tilde{a}(n)$ and $a(n)$ when $p' < 2p = 14$. Note that $\tilde{A}(7) = A(7) = \{1, p, 3, 5, 11, 13\}$.

n	p'	$t(n)$	$t_7(n)$	$D[t(n)]$	$D[t_7(n)]$	$\tilde{a}(n)$	$a(n)$	$\tilde{a}(n) = a(n)$
1		0	0	$\mathbb{N}^*$	$\mathbb{N}^*$	1	1	✓
2		1	7	$\{1\}$	$\{1, p\}$	p	p	✓
3	3	3	21	$\{1, 3\}$	$\{1, 3, 7, 21\}$	3	3	✓
4		6	42	$\{1, 2, 3, 6\}$	$\{1, 2, 3, 6, 7, 14, 21, 42\}$	2	2	n.a.
5	5	10	70	$\{1, 2, 5, 10\}$	$\{1, 2, 5, 7, 10, 14, 35, 70\}$	5	5	✓
6		15	105	$\{1, 3, 5, 15\}$	$\{1, 3, 5, 7, 15, 21, 35, 105\}$	15	15	n.a.
7	7	21	147	$\{1, 3, 7, 21\}$	$\{1, 3, 7, 21, 49, 147\}$	21	21	n.a.
8		28	196	$\{1, 2, 4, 7, 14, 28\}$	$\{1, 2, 4, 7, 14, 28, 49, 98, 196\}$	4	4	n.a.
9		36	252	$\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$	$\{1, 2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 28, 36, 42, 63, 84, 126, 252\}$	6	6	n.a.
10		45	315	$\{1, 3, 5, 9, 15, 45\}$	$\{1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315\}$	9	9	n.a.
11	11	55	385	$\{1, 5, 11, 55\}$	$\{1, 5, 7, 11, 35, 55, 77, 385\}$	11	11	✓
12		66	462	$\{1, 2, 3, 6, 11, 22, 33, 66\}$	$\{1, 2, 3, 6, 7, 11, 14, 21, 22, 33, 42, 66, 77, 154, 231, 462\}$	22	14	n.a.
13	13	78	546	$\{1, 2, 3, 6, 13, 26, 39, 78\}$	$\{1, 2, 3, 6, 7, 13, 14, 21, 26, 39, 42, 78, 91, 182, 273, 546\}$	13	13	✓

Table 4. The elements of $\tilde{A}(7) = \{\tilde{a}(n)\}_{n\in\mathbb{N}^\star}$ and $A(7) = \{a(n)\}_{n\in\mathbb{N}^\star}$.

4 Patterns in the sequence $A(p)$

In this Section we will present the patterns involved in the prime numbers and fixed points of the sequence $A(p)$.

Next, we will define a fixed point in the sequence $A(p)$.

Definition 4.1 (Fixed point of the sequence $A(p)$). Let $p \in \mathcal{P}$ and $m \in \mathbb{N}^{\star}$. We say that m is a fixed point of the sequence $A(p)$, if $a(m) = m$, where $a(m) \in A(p)$.

Let us see how the patterns came about. Consider a prime number, for example, $p = 7$. In the eighth column of Table 4 we see the first thirteen elements of $A(7)$, that is,

$$A(7) = \{a(n)\}_{n=1}^{13} = \{1, p, 3, 2, 5, 15, 21, 4, 6, 9, 11, 14, 13\} \tag{9}$$

If we continue to calculate more elements of $A(7)$, we would have

$$\begin{aligned} A(7) = \{a(n)\}_{n=1}^{24} &= \{1, p, 3, 2, 5, 15, 21, 4, 6, 9, 11, 14, 13, 49, 35, 8, 17, \\ &\quad 51, 19, 10, 30, 33, 23, 12\} \end{aligned} \tag{10}$$

From (10) we can obtain the graph of $a(n)$, that is,

$$\begin{aligned} G(a) &= \{(1,1), (2,p), (3,3), (4,2), (5,5), (6,15), (7,21), (8,4), (9,6), (10,9), \\ &\quad (11,11), (12,14), (13,13), (14,49), (15,35), (16,8), (17,17), (18,51), \\ &\quad (19,19), (20,10), (21,30), (22,33), (23,23), (24,12)\} \end{aligned} \tag{11}$$

On the other hand, consider the graph of the identity function $i(n)$, given by

$$\begin{aligned} G(i) &= \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (9,9), (10,10), \\ &\quad (11,11), (12,12), (13,13), (14,14), (15,15), (16,16), (17,17), (18,18), \\ &\quad (19,19), (20,20), (21,21), (22,22), (23,23), (24,24)\} \end{aligned} \tag{12}$$

The intersection of these two graphs is shown in Figure 1. Note that, with the exception of $p = 7$, the intersection occurs in all odd prime numbers: $\{3, 5, 11, 13, 17, 19, 23\}$, that is, $a(3) = 3$, $a(5) = 5$, $a(11) = 11$, $a(13) = 13$, $a(17) = 17$, $a(19) = 19$ e $a(23) = 23$. This latter relationship also appears in columns 2 and 8 of Table 4. Then, naturally, the following questions arise:

1ª *Are all the odd prime numbers in sequence $A(7)$, with the exception of 7, fixed points in that sequence?*

or, conversely,

2ª *Are all the fixed points of $A(7)$, with the exception of 7, odd primes numbers?*

We would like to answer it affirmatively, but first let us look at Table 5; it presents the first 48 terms of the sequence $A(7) = \{a(n)\}_{n=1}^{48}$. Note that $a(27) = 27$, so 27 is a fixed point in the sequence that is not a prime number, answering negatively to the second question. Also note that $a(41) = 28$ where 41 is a prime number, but it is not a fixed point in the sequence, so the answer, too, is negative for the first question. However, Navarrete and Orellana, [5], stated a conjecture, which answers affirmatively to the first question for large prime numbers p.

n	p'	$a(n)$	n	p'	$a(n)$	n	p'	$a(n)$
1		1	17	17	17	33		22
2		p	18		51	34		77
3	3	3	19	19	19	35		85
4		2	20		10	36		42
5	5	5	21		30	37	37	37
6		15	22		33	38		133
7	7	21	23	23	23	39		39
8		4	24		12	40		26
9		6	25		20	41	41	28
10		9	26		25	42		41
11	11	11	27		27	43	43	43
12		14	28		18	44		86
13	13	13	29	29	29	45		45
14		49	30		87	46		63
15		35	31	31	31	47	47	47
16		8	32		16	48		24

Table 5. The first 48 terms of the sequence $A(7)$.

In this work, we will partially demonstrate this conjecture (that is, in a line segment of primes). On the other hand, we also present a theorem whose proof states that the second question is false, for all sequences $A(p)$, where $p > 3$. The details of these two facts will be shown in the next section.

To better understand the above, we need to define a line segment of primes.

Definition 4.2 (The $2p$ line segment of prime numbers). Let $p \in \mathcal{P}^{\star}$. The $2p$ line segment of prime numbers, denoted by $\mathcal{L}_{2p}$, is defined as

$$\mathcal{L}_{2p} \quad = \quad \{p' \in \mathcal{P}^{\star} : \, p' < 2p\} \tag{13}$$

Definition 4.3 (The $2p-$star line segment of prime numbers).

$$\mathcal{L}^{\star}_{2p} \quad = \quad \mathcal{F}_{2p} - \{p\} \tag{14}$$

Example 4.1. In Figure 2, we show on the Cartesian plane $\mathbb{Z} \times \mathbb{Z}$ six $2p$ line segment of prime numbers, when $p = 3, 5, 7, 11, 13, 17$. For the particular case $p = 17$, the $2p$ and $2p-$star line segment of prime numbers are given, respectively, by:

(i) $\mathcal{L}_{2(17)} = \{3, 5, 7, 11, 13, \mathbf{17}, 19, 23, 29, 31\}$

(ii) $\mathcal{L}^{\star}_{2(17)} = \{3, 5, 7, 11, 13, 19, 23, 29, 31\}$

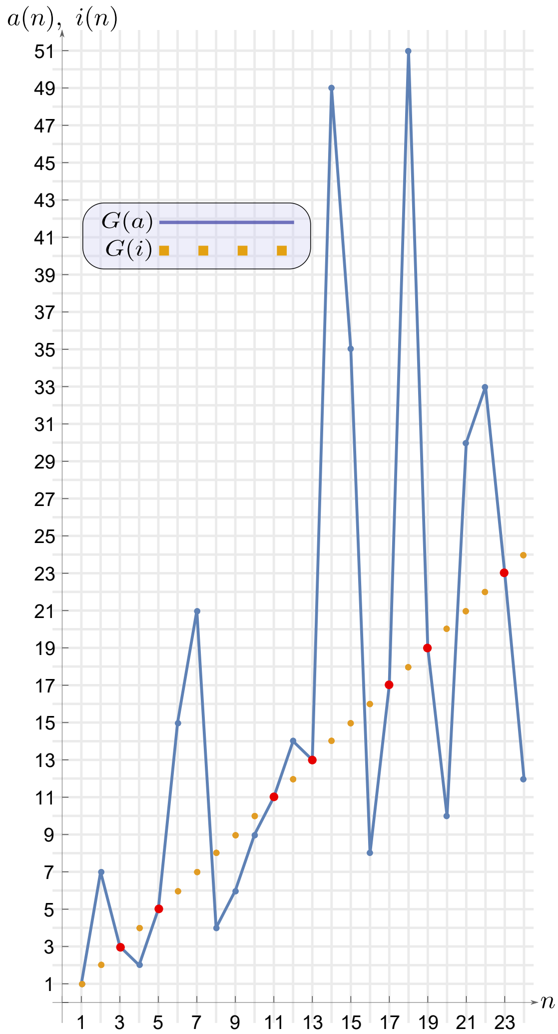

Figure 1. Intersection of the graphs of functions $a(n)$ and $i(n)$.

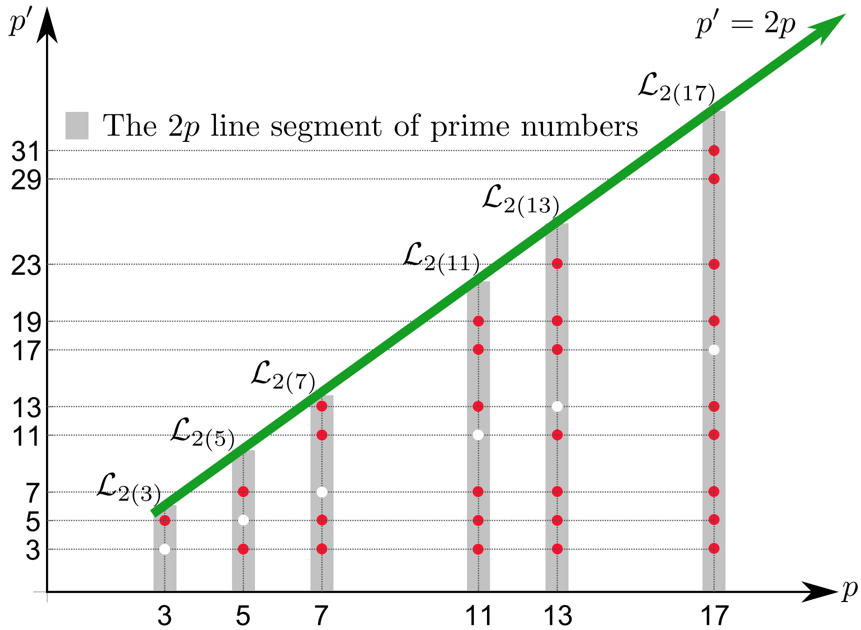

Figure 2. The $2p$ line segment of prime numbers, $p = 3, 5, 7, 11, 13, 17$.

5 The Main Theorem

In this Section, we will present the Navarrete–Orellana Conjecture (N–OC), (the last conjecture, the "Conjecture 6.1" in [5]), whose study motivated us to formulate our theorem, the main result of this work. This theorem expresses that in a line of prime numbers the conjecture is true.

N–OC provides us with a *Non-Primality Test*. First, we know that given a prime number p there is a sequence $A(p)$, see Definition 3.1. On the other hand, let p be a large prime number and $m \in \mathbb{N}$ with $m > 2$, if $m \neq p$ is not a fixed point in the sequence $A(p)$, then m is not a prime number.

Next, we will state the N–OC adapted to the proper notation of this work.

Conjecture 5.1 (The Navarrete–Orellana Conjecture). *Let $p \in \mathcal{P}$ be a large number and $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star}$. If $p' \in \mathcal{P}^\star$ and $p' \neq p$, then $a(p') = p'$.*

Additional information about this conjecture is found in [5].

The following Lemma is useful for demonstrating the Main Theorem.

Lemma 5.1. *Let $p \in \mathcal{P}$, $m \in \mathbb{N}^\star$ and $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star}$. If $a(n) = m$ then $n \leqslant 2m$.*

Proof. We are going to demonstrate through the absurd. Suppose that there exists an $n \in \mathbb{N}^\star$, such that $n > 2m$. Let n' be the smallest of these elements, and let $a(n') = m'$. Set $a(2m') = q$. As $2m' < n'$, it follows that $2m' \leqslant 2q$, that is, $m' \leqslant q$, since n' is a smallest element so that for $a(n) = m$, then $n > 2m$, and by Proposition 3.1, as $a(n') = m'$ and $a(2m') = q$ we have that $m' \neq q$, since $n' \neq 2m'$, and so $m' < q$. As $m' \mid \left\lceil \frac{2m'(2m'-1)p}{2} \right\rceil$ and $m' < q$, then $a(2m') = m'$, since $n' > 2m'$, which is absurd, so that there exists no $n \in \mathbb{N}^\star$, such that $n > 2m$. □

Observation 5.1. Lemma 5.1 is also valid when $p = 1$, that is, it is valid for the sequence $\tilde{A}(1)$.

Example 5.1. We will apply Lemma 5.1 for the particular case: $p = 3$, $n < 6$ and $m < 6$. These results are shown in Table 6.

n	$m = a(n)$	$n \leqslant 2m$
1	1	✓
2	3	✓
3	9	✓
4	2	✓
5	5	✓
6	15	✓
7	7	✓

Table 6. Application of the Lemma 5.1, when $p = 3$.

The following Theorem is the main result of this work; it states that N–OC is true for a set of primes, that is, it ensures that all prime numbers that are in the line segment $2p-$star, $\mathcal{L}^\star_{2p}$, are fixed points of the sequence $A(p)$.

Theorem 5.1 (**Main Theorem**). *Let $p \in \mathcal{P}^\star$ and $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star}$. If $p' \in \mathcal{P}^\star$, $p' \neq p$ and $p' < 2p$, then $a(p') = p'$.*

Proof. Let $p' = 2q + 1$, for some $q \in \mathbb{N}^*$. Set $a(n) = q$. By Lemma 5.1, we have that $n \leqslant 2q$. Thus, $n < 2q + 1 = p'$, that is, $a(p') \neq q$. Note that $2q < 2q + 1 < 3q < \cdots$, however, $2q \nmid p'qp$, since p' and p **are odd**. In addition, $2q+1 < 2p < 3p < \cdots$ and $(2q+1) \mid p'qp$. Thus, $a(p') = 2q + 1 = p'$. Therefore, p' is a fixed point of the sequence $A(p)$. □

Observation 5.2.

(i) Since the proof of the main theorem occurs in a prime line segment and, not for every prime number, we say that the N–OC is partially solved.

(ii) The main theorem is a **non-primality test**. In fact, given $p \in \mathcal{P}^\star$ and $A(p)$, we have for every $m \in \mathbb{N}$,

$$\text{if } m \neq p,\ 2 < m < 2p \text{ and } a(m) \neq m, \text{ then } m \notin \mathcal{P}^\star \tag{15}$$

Example 5.2. We are going to apply the Main Theorem for the particular cases: $p = 3, 5, 7$ and 11. These results are shown in Tables 7, 8, 9 and 10, respectively. It is observed that, in the last column of all tables, the prime numbers p', respecting the line segment $p' < 2p$, are fixed points in their respective sequences $A(p)$.

n	p'	$a(n)$	$p' < 2p = 6$	$a(p') = p',\ p' \neq p$
1		1	n.a.	n.a.
2		p	n.a.	n.a.
3	3	9	✓	n.a.
4		2	n.a.	n.a.
5	5	5	✓	✓

Table 7. Application of the Main Theorem, when $p = 3$.

n	p'	$\tilde{a}(n)$	$a(n)$	$p' < 2p = 10$	$a(p') = p',\ p' \neq p$
1		1	1	n.a.	n.a.
2		p	p	n.a.	n.a.
3	3	3	3	✓	✓
4		2	2	n.a.	n.a.
5	5	10	10	✓	n.a.
6		15	15	n.a.	n.a.
7	7	7	7	✓	✓
8		4	4	n.a.	n.a.
9		6	6	n.a.	n.a.

Table 8. Application of the Main Theorem, when $p = 5$.

n	p'	$\tilde{a}(n)$	$a(n)$	$p' < 2p = 14$	$a(p') = p',\ p' \neq p$
1		1	1	n.a.	n.a.
2		p	p	n.a.	n.a.
3	3	3	3	✓	✓
4		2	2	n.a.	n.a.
5	5	5	5	✓	✓
6		15	15	n.a.	n.a.
7	7	21	21	✓	n.a.
8		4	4	n.a.	n.a.
9		6	6	n.a.	n.a.
10		9	9	n.a.	n.a.
11	11	11	11	✓	✓
12		**22**	**14**	n.a.	n.a.
13	13	13	13	✓	✓

Table 9. Application of the Main Theorem, when $p = 7$.

n	p'	$\tilde{a}(n)$	$a(n)$	$p' < 2p = 22$	$a(p') = p',\ p' \neq p$
1		1	1	n.a.	n.a.
2		p	p	n.a.	n.a.
3	3	3	3	✓	✓
4		2	2	n.a.	n.a.
5	5	5	5	✓	✓
6		15	15	n.a.	n.a.
7	7	7	7	✓	✓
8		4	4	n.a.	n.a.
9		6	6	n.a.	n.a.
10		9	9	n.a.	n.a.
11	11	55	55	✓	n.a.
12		22	22	n.a.	n.a.
13	13	13	13	✓	✓
14		**91**	**77**	n.a.	n.a.
15		21	21	n.a.	n.a.
16		8	8	n.a.	n.a.
17	17	17	17	✓	✓
18		**51**	**33**	n.a.	n.a.
19	19	19	19	✓	✓
20		10	10	n.a.	n.a.
21		14	14	n.a.	n.a.

Table 10. Application of the Main Theorem, when $p = 11$.

The following corollary expresses that any prime number greater than 3 is a fixed point in at least one sequence of the family $\mathcal{A}$.

Corollary 5.1. $\forall p_k \in \mathcal{P},\ k \geqslant 3,\ a(p_k) = p_k$ *for some* $a(p_k) \in A(p_{k-1})$.

Proof. Given $p_k \in \mathcal{P}$, consider $a(p_k) \in A(p_{k-1})$. By Bertrand–Chebyshev Theorem (Theorem 2.1) we have $p_k < 2p_{k-1}$. As $p_k \neq p_{k-1}$, we have, by the Main Theorem, that $a(p_k) = p_k$. □

In the following, we present two examples of application of the Corollary 5.1 for the cases $k = 3$ and $k = 4$, respectively. The prime numbers to be considered are given in (1).

Example 5.3. Consider the prime number $p_3 = 5$ and the sequence

$$A(p_2) = A(3) = \{1, 3, 9, 2, 5, 15, 7, \ldots\}$$

Therefore, by Corollary 5.1, $a(p_3) = a(5) = 5 = p_3$.

Example 5.4. Consider the prime number $p_4 = 7$ and the sequence

$$A(p_3) = A(5) = \{1, 5, 3, 2, 10, 15, 7, 4, 6, 9, 11, 22, 13, \ldots\}$$

Therefore, by Corollary 5.1, $a(p_4) = a(7) = 7 = p_4$.

Next, we present the Proposition that demonstrates the third conjecture of Navarrete and Orellana ("Conjecture 5.1" in [5]).

Proposition 5.1. *Let* $\tilde{A}(1) = \{\tilde{a}(n)\}_{n\in\mathbb{N}^\star}$. *If* $p \in \mathcal{P}^\star$, *then* $\tilde{a}(p) = p$.

Proof. Let $\tilde{a}(p) = k$ and $p = 2q + 1$. Then, by (7), we have $k \mid pq$. By Lemma 5.1, we have $2k \geqslant 2q + 1 > 2q$. This implies that $k > q$, then $k \neq q$, and as $2q \nmid pq$, it follows that $k = p$, since $p \leqslant 3q$ and $\tilde{a}(n) \neq p$, $n < p$, because p only divides multiples of p. Thus, $\tilde{a}(p) = p$, for every odd p. □

In the following, we will announce the theorem that demonstrates that the second question, from the previous section, is false.

Theorem 5.2. *Let* $p \in \mathcal{P}$, $p \geqslant 11$ *and* $A(p) = \{a(n)\}_{n\in\mathbb{N}^\star}$. *Then there exists* $n_0 \in \mathbb{N}^\star$ *such that, if* $a(n_0) = n_0$, *then* $n_0 \notin \mathcal{P}^\star$.

Proof. Let $a(27) = q$. Then by Lemma 5.1, we have $2q \geqslant 27 > 26$. This implies that $q > 13$. And, we also have that $q \mid \dfrac{27 \cdot (27-1)p}{2}$, so $q \mid 27 \cdot 13p$, and as $3p > 27$, it follows that 27 is the smallest divisor of $27 \cdot 13p$. Now, we state that there exists no number k such that $a(k) = 27$, $k < 27$. In fact, if $9 \mid k$, then $3 \nmid k - 1$, and if $3 \mid 3$, then $9 \nmid k - 1$. Therefore, for every $p \geqslant 11$, there exists $n_0 = 27$ such that $a(27) = 27$. □

Observation 5.3. For the cases $p = 5$ and 7, too, it is true that 27 is a fixed point in the sequence $A(p)$. Only, for the case $p = 3$, we have that 27 is not a fixed point of $A(3)$, since $a(27) = 39$.

6 Application of the Main Theorem to larger primes

In this section we will apply the Main Theorem to larger prime numbers. Six particular cases were chosen: $p = 7,\ 97,\ 997,\ 9\,973,\ 19\,997$ and $29\,989$, which made the calculation complex. In this sense, it was necessary to implement a computational code. For this work, the language Python 3.8 was chosen. No more particular cases were considered because of the computational cost, and the lack of a high-performance computer. The computer used was a notebook with an Intel i5 sixth generation 2.3 GHz CPU.

In order to understand the application of these particular cases, we need to define some concepts of quantity of prime numbers.

Definition 6.1 (**Prime-counting function**). Let $x \in \mathbb{R}$, $x > 0$. The prime-counting function $\pi(x)$ is defined as the numbers of primes not greater than x, that is,

$$\pi(x) \ = \ \#\{p \in \mathcal{P} : p \leqslant x\}\,, \tag{16}$$

where, the symbol $\#$ denotes the number of elements in a set.

Notation 6.1.

(i) Let $p \in \mathcal{P}$, $p > 3$. We denote **the number of prime numbers that are fixed points of the sequence $\boldsymbol{A(p)}$**, in the $2p-$star prime line segment, by

$$\Pi_F(2p) \ = \ \#\left\{p' \in \mathcal{L}^{\star}_{2p} :\ a(p') = p',\ a(p') \in A(p)\right\}.$$

(ii) Let $p \in \mathcal{P}$. We denote **the number of prime of the $\boldsymbol{2p-}$star** line segment, by

$$\Pi(2p) \ = \ \#\mathcal{L}^{\star}_{2p} = \ \pi(2p) - 2.$$

6.1 Application of the particular cases

To know if the amount of prime numbers that exist in the $2p-$star line segment coincides with the amount of prime numbers that are fixed points of sequence $A(p)$, in this line segment, as the Main Theorem indicates, six particular cases (or numerical tests) were performed. The results are shown in Table 11, where the last test needed approximately five hours to run. The first column expresses the test number. The second column shows the prime numbers that were chosen for the tests; the first 4 tests were performed with the prime numbers closest to 10, 100, $1\,000$ and $10\,000$, respectively; the fifth and sixth tests were done with the prime numbers closest to $20\,000$ and $30\,000$, respectively. The third column expresses twice the prime numbers found in the second column. In the fourth column, we present the amount of prime numbers that are fixed points of the sequence $A(p)$, i.e., $\Pi_F(2p)$, this value was obtained using our code. The fifth column indicates the amount of prime numbers that exist in the $2p-$star line segment, i.e., $\Pi(2p)$, this value was obtained using the software "Wolfram Mathematica 12" by the command: `PrimePi`$[2p] - 2$". And finally, the last column expresses whether the amount of prime numbers that are fixed points of sequence $A(p)$ in the $2p-$star line segment, according to the Main Theorem of this work, is equal to all prime numbers that exist in that line segment.

Test	p	$2p$	$\Pi_F(2p)$	$\Pi(2p)$	$\Pi_F(2p) = \Pi(2p)$
1º	7	14	4	4	✓
2º	97	194	42	42	✓
3º	997	1 994	299	299	✓
4º	9 973	19 946	2 252	2 252	✓
5º	19 997	39 994	4 201	4 201	✓
6º	29 989	59 978	6 503	6 053	✓

Table 11. Application of the Main Theorem for six particular cases of p, where the sequence $A(p) = \{a(n)\}_{1\leqslant n<2p}$.

7 Conclusion

In this work, we partially demonstrate the Navarrete–Orellana Conjecture through the Main Theorem (Theorem 5.1). The proof of this Theorem covers for all prime numbers p' that are in the line segment of prime numbers $2p$, that is, given the prime number p, the prime numbers $p' < 2p$ are fixed points of sequence $A(p)$. This Theorem is also formulated as a non-primality test, as indicated in (15).

The future works that can happen as a result of this paper are:

(a) Expansion of the line segment of prime numbers $2p$ to another that is a multiple of p. In case it is possible, we would have the complete demonstration of N–OC.

(b) Identification of a pattern for fixed points that are not prime numbers. For example, from Theorem 5.2 and Observation 5.3 we know that in all sequences $A(p), p > 3$, we have that $a(27) = 27$. In case it is possible, we could select only the fixed points that are prime and, by Item (a), we would know that all fixed points are prime numbers.

If items (a) and (b) can be proved, we would have necessary and sufficient condition for all fixed points in a sequence to be prime numbers.

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