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Perfect squares in the sum and difference of balancing-like numbers

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Abstract: In this study, we deal with the existence of perfect powers which are the sum and difference of two balancing numbers. Moreover, as a generalization we explore the perfect squares which are the sum and difference of two balancing-like numbers, where balancing-like sequence is defined recursively as $G_{n+1} = AG_n - G_{n-1}$ with initial terms $G_0 = 0, G_1 = 1$ for $A \geq 3$.

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1 Introduction

Let $\{u_n\}_{n\geq 0}$ be the Lucas sequence satisfying $u_0 = 0$, $u_1 = 1$ and the recurrence $u_{n+1} = Au_n + Bu_{n-1}$ for all integers $n \geq 1$, where A and B are fixed nonzero integers. The corresponding associated Lucas sequence $\{v_n\}_{n\geq 0}$ satisfies the same recurrence relation as of Lucas sequence but with different initial terms $v_0 = 2$ and $v_1 = A$. Both sequences have characteristic equation $x^2 - Ax - B = 0$ with two roots α and β. The Binet formulas for these sequences are given by

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{A+\sqrt{A^2+4B}}{2}$ and $\beta = \frac{A-\sqrt{A^2+4B}}{2}$. The sequences $\{u_n\}$ and $\{v_n\}$ can be extended to negative indices n as $u_{-n} = -(-B)^{-n}u_n$ and $v_{-n} = (-B)^{-n}v_n$, respectively. In this paper, we will assume that $A^2 + 4B > 0$. If $A = 6$ and $B = -1$, then the sequence $\{u_n\}$ reduces to balancing sequence $\{B_n\}$, which is defined by Behera and Panda [2], as the solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + \cdots + (n+r)$$

for some natural number r, known as the balancer corresponding to n. It is well known that $8B_n^2 + 1$ is a perfect square for all n and its positive square root is called as n-th Lucas-balancing number C_n, which satisfies the recurrence relation $C_{n+1} = 6C_n - C_{n-1},\ C_0 = 1, C_1 = 3$. Pell and associated Pell sequences $\{P_n\}$ and $\{Q_n\}$ are defined recursively as $P_{n+1} = 2P_n + P_{n-1}, P_0 = 0, P_1 = 1$ and $Q_{n+1} = 2Q_n + Q_{n-1}, Q_0 = 1, Q_1 = 1$. The characteristic equation of Pell and associated Pell sequences is $x^2 - 2x - 1 = 0$ with distinct roots $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$ and the corresponding Binet forms are

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ Q_n = \frac{\alpha^n + \beta^n}{2}. \tag{1}$$

Twice of the associated Pell numbers are known as Pell-Lucas numbers (or companion Pell numbers). Pell and associated Pell numbers are connected to balancing and Lucas-balancing numbers by the identities $B_n = P_nQ_n$, $P_{2n} = 2B_n,\ P_{2n-1} = B_n - B_{n-1}$ and $Q_{2n} = \frac{B_{n+1}-B_{n-1}}{2}$, $Q_{2n-1} = B_n + B_{n-1}$ [17]. Balancing-like sequence $\{G_n\}$ and the associated balancing-like sequence $\{H_n\}$ are the Lucas and associated Lucas sequences corresponding to $B = -1$ and their Binet forms are

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ H_n = \alpha^n + \beta^n, \tag{2}$$

where $\alpha = \frac{A+\sqrt{A^2-4}}{2}$ and $\beta = \frac{A-\sqrt{A^2-4}}{2}$. Balancing sequence is a particular case of balancing-like sequence with $A = 6$. Generalizing the relations of Pell and associated Pell numbers with balancing and Lucas-balancing numbers, Pell-like sequence $\{p_n\}_{n\geq 0}$ and the associated Pell-like sequence $\{q_n\}_{n\geq 0}$ are defined by $p_{2n} = 2G_n,\ p_{2n-1} = G_n - G_{n-1}$ and $q_{2n} = \frac{G_{n+1}-G_{n-1}}{2},\ q_{2n-1} = G_n + G_{n-1}$ [15]. The sequence $\{s_n\} = \{2q_n\}$ is called Lucas Pell-like sequence [19] and it can be observed that $H_n = s_{2n}$ for all $n \geq 0$.

Exploring different type of the numbers among the terms of binary recurrence sequences have motivated to solve many interesting problems. Alekseyev and Tengely [1] showed that the equation $u_n(k, \pm 1) = ax^2 + b$ has only a finite number of solutions n when $a \neq 0$ and b are integers. In particular, Bugeaud, Mignotte and Siksek [6] solved the Diophantine equation $F_n = y^p$ for $p \geq 2$ using modular approach and classical linear forms in logarithms. Bugeaud, Luca, Mignotte and Siksek [7] found all integer solutions of $F_n \pm 1 = y^p$. Replacing Fibonacci numbers with balancing numbers and balancing-like numbers, Sahukar and Panda [19] solved the equations $B_n \pm 1 = y^p$ and $G_n \pm 1 = y^2$. As an extension of [7], Luca and Patel [10] solved the Diophantine equation $F_n \pm F_m = y^p$ for $p \geq 2$. They found that if $n \equiv m \pmod 2$, this equation has solution either $\max\{|n|, |m|\} \leq 36$ or $y = 0$ and $|n| = |m|$.

In this study, we solve the equation $B_n \pm B_m = y^p$, where B_n is the n-th balancing number. In order to solve this equation, we will also determine all indices N and M such that $P_N Q_M = y^p$ for positive integers N, M, y with $p \geq 2$. We also deal with the equation $G_n \pm G_m = y^2$, where G_n is the n-th balancing-like number. For the solution of this equation, we need the indices N and M satisfying the equation $G_N H_M = y^2$ for natural numbers N, M. In particular, we prove the following theorems.

Theorem 1.1. *For positive integers N, M, the only solutions of the Diophantine equation $P_N Q_M = y^p$ are $(N, M) = (1, 1)$ for $y = 1$ and every positive integer p; $(N, M) = (7, 1)$ for $y = 13$ and $p = 2$; and $(N, M) = (4, 2)$ for $y = 6$ and $p = 2$.*

Theorem 1.2. *If the Diophantine equation $G_N H_M = y^2$ has a solution for some nonnegative integers N, M, then $N \in \{0, 1, 2, 4, 8\}$.*

Theorem 1.3. *Let n and m be non-negative integers. Then the only solutions of the equation $B_n + B_m = y^p$ are $(n, m) = (1, 0)$, $(0, 1)$ for $y = 1$ and every positive integer p; and $(n, m) = (3, 1)$, $(1, 3)$ for $y = 6$ and $p = 2$. The only solutions of the equation $B_n - B_m = y^p$ are $(n, m) = (1, 0)$ for $y = 1$ and every positive integer p; and $(n, m) = (4, 3)$ for $y = 13$ and $p = 2$.*

Theorem 1.4. *Let n and m be non-negative integers and $G_n \pm G_m = y^2$ for some integer y. If n and m have the same parity, then $n \pm m \in \{0, 2, 4, 8, 16\}$, and if n and m have the opposite parity $n \pm m \in \{1, 3\}$.*

Moreover, in this paper, we will also deal with the equations $p_N = x^2$, $p_N = 2x^2, q_M = x^2, q_M = 2x^2$, and $p_N q_M = y^2$ for odd natural numbers N, M.

Throughout the paper, (a, b) and $\left(\frac{a}{b}\right)$ denote the greatest common divisor and the Jacobi symbol of a and b, respectively. Also, $\square$ denotes the perfect square.

2 Preliminaries

In this section, we recall some results and definitions that are useful to prove the main results.

Lemma 2.1. *If m and n are natural numbers, then*

1. $P_{2n} = 2B_n$, $Q_{2n} = C_n$, *and* $B_n = P_n Q_n$.
2. $Q_n^2 - 2P_n^2 = (-1)^n$.
3. $(P_n, Q_n) = 1$,
4. *if n is even, then P_n, B_n are even and Q_n and C_n is odd for all $n \geq 0$,*
5. $u_{2n} = u_n v_n$,

6. $v_n^2 - (A^2+4B)u_n^2 = 4(-B)^n$,
7. $(u_n, u_m) = u_{(m,n)}$,
8. *if* $m = 2^a m'$ *and* $n = 2^b n'$, m' *and* n' *are odd, then*

$$(u_m, v_n) = \begin{cases} v_{(m,n)} & \text{if } a > b, \\ 1 \text{ or } 2 & \text{otherwise,} \end{cases}$$

and

$$(v_m, v_n) = \begin{cases} v_{(m,n)} & \text{if } a = b, \\ 1 \text{ or } 2 & \text{otherwise,} \end{cases}.$$

Proof. The assertions (1)–(6) can be proved by using Binet formulas of corresponding sequences. The proofs of (7) and (8) can be found in [12]. □

For Lucas sequence $\{u_n\}_{n\geq 0}$ and associated Lucas sequence $\{v_n\}_{n\geq 0}$, the following theorem is proved in [20].

Lemma 2.2. *Let* $n \in \mathbb{N} - \{0\}$, $m, r \in \mathbb{Z}$ *and* m *be a non-zero integer. Then*

$$G_{2mn+r} \equiv G_r \pmod{G_m} \tag{3}$$

and

$$G_{2mn+r} \equiv (-1)^n G_r \pmod{H_m}. \tag{4}$$

Moreover, the following properties can be found in [20].

$$u_{m+n} + (-B)^n u_{m-n} = u_m v_n \tag{5}$$

and

$$u_{m+n} - (-B)^n u_{m-n} = u_n v_m. \tag{6}$$

Lemma 2.3. ([13]) *The Diophantine equation* $u^2 - (A^2-4)v^2 = 4$ *has the positive integer solutions* $(u, v) = (H_n, G_n)$ *for each* $n \geq 0$.

Lemma 2.4. ([18], Theorem 1)

1. *If* $v_n = \square$, *then* $n = 1, 3, 5$.
2. $v_3 = \square$ *if and only if* $A = \square$ *and* $A^2 + 3B = \square$, *or* $A = 3\square$ *and* $A^2 + 3B = 3\square$. *Furthermore, if* $v_3 = \square$, *then* $B \equiv 1 (\text{mod } 4)$.
3. $v_5 = \square$, *if and only if* $A = 5\square$ *and* $A^4 + 5A^2B + 5B^2 = 5\square$. *Furthermore, if* $v_5 = \square$, *then* $A \equiv 5 (\text{mod } 8)$ *and* $B \equiv 3 (\text{mod} 8)$.

Lemma 2.5. ([18], Theorem 2)

1. *If* $v_n = 2\square$, *then the possible values of* n *in nonnegative integers are* $n = 0, 3, 6$.
2. $v_3 = 2\square$ *iff* $A = 3\square$ *and* $A^2 + 3B = 6\square$. *Further, if* $v_3 = 2\square$, *then* $A \equiv 3 (\text{mod } 24)$ *and* $B \equiv 5, 7 (\text{mod } 8)$.

3. $v_6 = 2\Box$ *iff* $A^2 + 2B = 3\Box$ *and* $(A^2 + 2B)^2 - 3B^2 = 6\Box$. *Further, if* $v_6 = 2\Box$, *then* $B \equiv 1 \pmod 4$.

Let $\{T_k + U_k\sqrt{d} : k \geq 1\}$ be the set of solutions of the Pell's equation $X^2 - dY^2 = 1$. The following Lemma analyzes the existence of the solutions of $b^2X^4 - dY^2 = 1$.

Lemma 2.6. ([3], Theorem 1.1) *If* $b, d > 1$ *are squarefree integers, then there is at most one index* k *for which* $T_k = bx^2$ *for some* $x \in \mathbb{Z}$ *and consequently, the equation* $b^2X^4 - dY^2 = 1$ *has at most one solution in positive integers* X, Y . *Moreover, if such a solution exists, then* k *is the smallest positive integer such that* $b|T_k$.

From Lemma 2.3, it is clear that when A is even, the equation $4u^4 - ((\frac{A}{2})^2 - 1)v^2 = 1$ gives the perfect squares in the sequence $\{H_n\}$ and is equivalent to $b^2u^4 - dv^2 = 1$ when $b = 2$ and $d = (\frac{A}{2})^2 - 1$. Hence, the following Lemma results combining Lemma 2.4 with Lemma 2.6.

Lemma 2.7. *If A is odd, then* $H_n = \Box$ *implies that* $n = 1$ *and if A is even integer, then* $H_n = \Box$ *implies that* $n = 1$, *otherwise it has no solution.*

Proof. If A is odd, then by Lemma 2.4, $H_n = \Box$ only when $n = 1$. If A is even, then the Diophantine equation given in Lemma 2.3 reduces to $(\frac{u}{2})^2 - ((\frac{A}{2})^2 - 1)v^2 = 1$. When $H_n = u = w^2$, the last equation reduces to $4(\frac{w}{2})^4 - ((\frac{A}{2})^2 - 1)v^2 = 1$, which is of the form $b^2X^4 - DY^2 = 1$. For $b = 2$, the only possible value of n is $n = 1$ by Lemma 2.6. $\Box$

If we consider the Diophantine equation $x^4 - dy^2 = 1$, then the following lemma gives a brief idea on the possible solutions depending on the values of d.

Lemma 2.8. ([21], Theorem 1) *The equation* $x^4 - dy^2 = 1$ *has at most one solution in positive integers except* $d = 1785, 4 \cdot 1785, 16 \cdot 1785$ *in which the given equation has two positive integer solutions* $(x, y) = (13, 4), (239, 1352)$, $(x, y) = (13, 2), (239, 276)$, $(x, y) = (13, 1), (239, 338)$ *respectively. If the given equation has one solution* (x_1, y_1) *in positive integers, then* $x_1^2 = x_0$ *or* $x_1^2 = 2x_0^2 - 1$, *where* $x_0 + y_0\sqrt{d}$ *is the fundamental solution of* $x^2 - dy^2 = 1$.

Lemma 2.9. *If A is odd, then* $H_n = 2\Box$ *hold for* $n = 0$ *or* 3 *and if A is even, then* $H_n = 2\Box$ *has only one solution* $n = 2$ *except* $A = 338$ *in which it has two integer solutions* $n = 1, 2$.

Proof. If A is odd, then it can be seen that $n = 0$ or 3 by Lemma 2.5. When A is even and $H_n = 2w^2$ for some n, the Diophantine equation mentioned in Lemma 2.3 reduces to $w^4 - ((\frac{A}{2})^2 - 1)v^2 = 1$, which is of the form $w^4 - Dv^2 = 1$ with $D = (\frac{A}{2})^2 - 1$. By Lemma 2.8, the equation $w^4 - Dv^2 = 1$ has at most one solution except $D = 1785, 4 \cdot 1785, 16 \cdot 1785$. But, For $A > 2$ and $(\frac{A}{2})^2 - 1$ being an integer, the only possible value of D of the form $(\frac{A}{2})^2 - 1$ is $D = 16 \cdot 1785$ for $A = 338$. Lemma 2.8 results in that for $A = 338$, the given equation ha solutions $w = 13, 239$ and hence $H_n = 2w^2 = 338 = A = H_1$ and $y_n = 114242 = A^2 - 2 = H_2$. Since the fundamental solution of $w^2 - ((\frac{A}{2})^2 - 1)v^2 = 1$ is $(\frac{A}{2}, 1)$, if $D \neq 16 \cdot 1785$, then $w^4 - Dv^2 = 1$ has only one solution (x_1, y_1) such that $x_1^2 = \frac{A}{2}$ or $x_1^2 = 2(\frac{A}{2})^2 - 1$ and hence $H_n = 2x_1^2 = A$ or $A^2 - 2$. $\Box$

The following Lemma can be found in [6].

Lemma 2.10. *For $n \geq 3$, $G_n = \square$ only if $(A, n) = (338, 4), (3, 6)$ and $G_n = 2\square$ has no solution.*

Lemma 2.11. ([14], Lemma 2.6, [19], Lemma 2.6) *The only perfect power terms in Pell sequence are $P_1 = 1$, $P_7 = 169$ and P_2 is the only Pell number which is twice of perfect power.*

Lemma 2.12. ([5], Theorem 2.6) *The only solutions of the Diophantine equation $u_n = 2^\alpha 3^\beta z^m$ in positive integers n, z, m with $m \geq 2$ and non-negative integers α, β are given by $n = 1, 2, 4, 7$.*

Lemma 2.13. ([5], Theorem 2.6) *The only perfect power term in associated Pell sequence is $Q_1 = 1$.*

Lemma 2.14. ([19], Lemma 2.6) *The only perfect power term in balancing sequence is $B_1 = 1$.*

The following theorem gives a useful factorization of sum of two balancing numbers.

Lemma 2.15. *For any natural numbers n and m,*

$$B_n + B_m = \begin{cases} P_{n+m}Q_{n-m}, & \text{if } n \equiv m \pmod 2 \\ P_{n-m}Q_{n+m}, & \text{if } n \not\equiv m \pmod 2. \end{cases}$$

Proof. If n and m are of same parity, i.e., $n \equiv m(\bmod\ 2)$, then $n+m, n-m$ are even and using Lemma 2.1 (1) and (5) we get

$$\begin{aligned} B_n + B_m &= B_{(\frac{n+m}{2})+(\frac{n-m}{2})} + B_{(\frac{n+m}{2})-(\frac{n-m}{2})} \\ &= 2B_{\frac{n+m}{2}}C_{\frac{n-m}{2}} = P_{n+m}Q_{n-m}. \end{aligned}$$

If n and m are of different parity i.e, $n \not\equiv m(\bmod 2)$, then $n+m, n-m$ are odd and

$$\begin{aligned} P_{n-m}Q_{n+m} &= \left(\frac{\alpha^{n-m} - \beta^{n-m}}{2\sqrt{2}}\right)\left(\frac{\alpha^{n+m} + \beta^{n+m}}{2}\right) \\ &= \frac{1}{4\sqrt{2}}\{\alpha^{2n} - \beta^{2n} - (\alpha\beta)^{n-m}(\alpha^{2m} - \beta^{2m})\} \\ &= \frac{P_{2n}}{2} + \frac{P_{2m}}{2} \\ &= B_n + B_m. \end{aligned}$$

□

Similarly, using Lemma 2.1 and (6) the following lemma can be proved.

Lemma 2.16. *For any natural numbers n and m,*

$$B_n - B_m = \begin{cases} P_{n-m}Q_{n+m}, & \text{if } n \equiv m \pmod 2 \\ P_{n+m}Q_{n-m}, & \text{if } n \not\equiv m \pmod 2. \end{cases}$$

Moreover, it is also given in Theorem 2.22 [19] that for $w = \frac{\sqrt{A+2}+\sqrt{A-2}}{2}$ and $z = \frac{\sqrt{A+2}-\sqrt{A-2}}{2}$,

$$p_{2n+1} = \frac{w^{2n+1} + z^{2n+1}}{w+z}, \quad \frac{s_{2n+1}}{2} = \frac{w^{2n+1} - z^{2n+1}}{w-z}, \tag{7}$$
$$s_{2n} = w^{2n} + z^{2n}, \quad \frac{p_{2n}}{2} = \frac{w^{2n} - z^{2n}}{w^2 - z^2}.$$

Recall that Lehmer and associated Lehmer numbers are defined as

$$U_n = \begin{cases} \frac{\lambda^n - \delta^n}{\lambda - \delta} & \text{if } n \text{ is odd,} \\ \frac{\lambda^n - \delta^n}{\lambda^2 - \delta^2} & \text{if } n \text{ is even} \end{cases} \tag{8}$$

and

$$V_n = \begin{cases} \frac{\lambda^n + \delta^n}{\alpha + \delta} & \text{if } n \text{ is odd,} \\ \lambda^n + \delta^n & \text{if } n \text{ is even,} \end{cases} \tag{9}$$

respectively, where λ and δ are complex numbers with $((\lambda+\delta)^2, \lambda\delta) = 1$ and $\frac{\lambda}{\delta}$ not a root of unity. Since $(w+z)^2$ and wz are non-zero coprime integers, the Lehmer and the associated Lehmer numbers corresponding to w and z are

$$U_{2n} = \frac{p_{2n}}{2}, \quad U_{2n+1} = \frac{s_{2n+1}}{2} = q_{2n+1}, \tag{10}$$
$$V_{2n} = s_{2n}, \quad V_{2n+1} = p_{2n+1}.$$

Lehmer and associated Lehmer numbers satisfy the divisibility properties similar to Lucas and associated Lucas numbers discussed in Lemma 2.1. The following two lemmas are the generalization of the Lemmas 2.15 and 2.16.

Lemma 2.17. *For any natural numbers n and m,*

$$G_n + G_m = \begin{cases} \frac{1}{2} p_{n+m} s_{n-m}, & \text{if } n \equiv m \pmod 2) \\ \frac{1}{2} p_{n-m} s_{n+m}, & \text{if } n \not\equiv m \pmod 2). \end{cases}$$

Proof. If n and m are same parity, then $n+m$ and $n-m$ are even. Hence, using (5), we get

$$\begin{aligned} 2(G_n + G_m) &= 2\left(G_{\frac{n+m}{2}+\frac{n-m}{2}} + G_{\frac{n+m}{2}-\frac{n-m}{2}}\right) \\ &= 2G_{\frac{n+m}{2}} H_{\frac{n-m}{2}} = p_{n+m} s_{n-m}. \end{aligned}$$

If n and m are of different parity, then $n+m$ and $n-m$ are odd. Using (7) in the above equation, we have

$$\begin{aligned} \frac{1}{2} p_{n-m} s_{n+m} &= p_{n-m} \frac{s_{n+m}}{2} \\ &= \left(\frac{w^{n-m} + z^{n-m}}{w+z}\right)\left(\frac{w^{n+m} - z^{n+m}}{w-z}\right) \\ &= \frac{1}{w^2 - z^2}(w^{2n} - w^{n-m}z^{n+m} + z^{n-m}w^{n+m} - z^{2n}) \\ &= \frac{1}{w^2 - z^2}(w^{2n} - z^{2n} - z^{2m} + w^{2m}) \\ &= \frac{w^{2n} - z^{2n}}{w^2 - z^2} + \frac{w^{2m} - z^{2m}}{w^2 - z^2} = \frac{p_{2n}}{2} + \frac{p_{2m}}{2} = G_n + G_m. \end{aligned}$$

□

Similarly, using (7), the following lemma can be proved.

Lemma 2.18. *For any natural numbers n and m,*

$$G_n - G_m = \begin{cases} \frac{1}{2}p_{n-m}s_{n+m}, & \text{if } n \equiv m (\bmod 2) \\ \frac{1}{2}p_{n+m}s_{n-m}, & \text{if } n \not\equiv m (\bmod 2). \end{cases}$$

When A is odd, it is seen that $H_{2^r} \equiv 7 \pmod 8$, which gives

$$\left(\frac{-1}{H_{2^r}}\right) = -1 \quad \text{and} \quad \left(\frac{2}{H_{2^r}}\right) = 1 \tag{11}$$

for $r \geq 1$. It can be easily seen that

$$\left(\frac{A+1}{H_{2^r}}\right) = \left(\frac{A-1}{H_{2^r}}\right) = 1. \tag{12}$$

Lemma 2.19. ([19], Theorem 2.22) *All positive integer solutions of $(A+2)u^2 - (A-2)v^2 = 4$ are $(u, v) = (p_{2n+1}, q_{2n+1}) = (G_{n+1} - G_n, G_{n+1} + G_n)$.*

In [8], Keskin and Duman proved the following two lemmas.

Lemma 2.20. *If A is odd and the equation $G_n + G_{n+1} = u^2$ has a solution, then $n = 0$ or 1.*

Lemma 2.21. *If A is odd and the equation $G_{n+1} - G_n = u^2$ has a solution, then $n = 0$ or 1.*

Since $q_{2n+1} = G_{n+1} + G_n$ are the v-coordinates of the solutions of $(A+2)u^2 - (A-2)v^2 = 4$ by Lemma 2.19, the solutions of the equation $(A+2)u^2 - (A-2)v^4 = 4$ give a brief idea on the existence of perfect squares in the sequence $\{q_{2n+1}\}$. Thus, the following lemma can be concluded from [19].

Lemma 2.22. ([19], Theorem 2.26) *For natural number n, the only perfect squares in $\{q_{2n+1}\}$ are q_1 or q_3.*

3 Main theorems

By Lemma 2.19, we know that the positive integer solutions of $(A+2)u^4 - (A-2)v^2 = 4$ are given by $(u^2, v) = (p_{2n+1}, q_{2n+1})$ for $n \geq 0$. This leads us to find the perfect squares in $\{p_{2n+1}\}_{n\geq 0}$. If A is odd, we can see from Lemma 2.21 that the perfect squares in $\{p_{2n+1}\}_{n\geq 0}$ are only p_1 and p_3. If A is even, then we have the equation $(\frac{A}{2}+1)u^4 - (\frac{A}{2}-1)v^2 = 2$ or $\left(\frac{A+2}{4}\right)u^4 - \left(\frac{A-2}{4}\right)v^2 = 1$. The solutions of these equations gives the perfect squares in $\{p_{2n+1}\}_{n\geq 0}$ when A is even. For this, we need the following two lemmas.

Lemma 3.1. ([23], Theorem 1.4) *For any positive odd integers M and N, the equation $Mu^4 - Nv^2 = 2$ has at most one solution in positive integers.*

Lemma 3.2. ([4], Theorem 1.2) *Let m be a positive integer. If $t = m^2 + m$, then positive integer solutions of the equation*

$$(m^2 + m + 1)u^4 - (m^2 + m)v^2 = 1 \tag{13}$$

are $(u, v) = (1, 1)$ and $(u, v) = (2m + 1, 4m^2 + 4m + 3)$.

If $t = m^2 + m$, then (13) is of the form $(t+1)u^4 - tv^2 = 1$. If $t \neq m^2 + m$, Walsh [22] conjectured the following.

Conjecture 3.3. *Let* $t > 1$. *The only positive integer solution to* $(t+1)u^4 - tv^2 = 1$ *is* $(u, v) = (1, 1)$ *if* $t \neq m^2 + m$ *for some positive integer* m.

Thus, we can give the following theorem.

Theorem 3.4. *Let* $A > 2$ *be a natural number and*

$$(A+2)u^4 - (A-2)v^2 = 4 \tag{14}$$

for some integers $u > 1$ *and* $v > 1$. *Then, we have the following cases:*

1. *If* A *is odd and* $A - 1$ *is a perfect square, then the only solution of* (14) *is* $(u, v) = (\sqrt{A-1}, A+1)$.
2. *If* $A \equiv 0 (\text{mod } 4)$, *then* (14) *has no solutions.*
3. *If* $A \equiv 2(\text{mod } 4)$, $\frac{A-2}{4} > 1$ *and* $\frac{A-2}{4} \neq m^2 + m$ *for positive integer* m, *and Conjecture 3.3 is true, then* (14) *has no solutions.*
4. *If* $A \equiv 2(\text{mod } 4)$, $\frac{A-2}{4} > 1$ *and* $\frac{A-2}{4} = m^2 + m$, *then* (14) *has the only solution* $(u, v) = (2m+1, 4m^2+4m+3)$.
5. *If* $A = 6$, *then* $(u, v) = (13, 239)$.

Proof. Assume that the equation (14) holds for some integers $u > 1$ and $v > 1$. Then, by Lemma 2.19, it follows that $(u^2, v) = (G_{n+1} - G_n, G_{n+1} + G_n)$. Thus, if A is odd, then $u^2 = G_{n+1} - G_n$ implies that $n = 1$ by Lemma 2.21. In this case, (14) has only solution $(u, v) = (\sqrt{A-1}, A+1)$, where $A - 1$ is perfect square.

Now, let A is even. Then $A + 2$ and $A - 2$ are even. Hence (14) can be written as

$$(\frac{A}{2} + 1)u^4 - (\frac{A}{2} - 1)v^2 = 2. \tag{15}$$

If $A \equiv 0 \pmod 4$, then $\frac{A}{2} + 1$ and $\frac{A}{2} - 1$ are odd. Thus, by Lemma 3.1, (15) has the only solution $(u, v) = (1, 1)$. But, since $u > 1$ and $v > 1$, this is impossible. If $A \equiv 2 \pmod 4$, then $4|(A+2)$ and $4|(A-2)$. Therefore (14) takes the form

$$(\frac{A+2}{4})u^4 - (\frac{A-2}{4})v^2 = 1. \tag{16}$$

Taking $t = \frac{A-2}{4}$, the above equation can be written as

$$(t+1)u^4 - tv^2 = 1. \tag{17}$$

Assume that $t > 1$ and $t \neq m^2 + m$. If we assume that Conjecture 3.3 is true, then (14) has no solutions since $u > 1$ and $v > 1$. If $t > 1$ and $\frac{A-2}{4} = m^2 + m$, then $(u, v) = (2m+1, 4m^2+4m+3)$ for some positive integer m by Lemma 3.2. On the other hand, if $t = 1$, then we have $A = 6$ and the equation $v^2 - 2u^4 = -1$. It is well known that this equation has solution $(v, u^2) = (Q_{2n+1}, P_{2n+1})$ for $n \geq 0$. Thus, it can be easily seen that $(u, v) = (13, 239)$ by Lemma 2.11. $\square$

Since $p_{2n+1} = G_{n+1} - G_n$, as a consequence of Lemma 2.21 and Theorem 3.4, we can give the following result, which expresses the perfect squares in $\{p_{2n+1}\}_{n\geq 0}$.

Corollary 3.4.1. *Let n be a non-negative integer and $p_{2n+1} = x^2$ for some integer x.*

1. *If A is odd, then $n = 0$ or 1.*
2. *If $4|A$, then $n = 0$.*
3. *If $A \equiv 2 (\text{mod } 4)$, and $A - 1$ is a perfect square or $A = 6$, then $n = 0, 1$ or 3.*
4. *If $A \equiv 2 (\text{mod } 4)$, $A - 1$ is not a perfect square, and Conjecture 3.3 is true, then $n = 0$.*

In the following theorems, we determine the terms, which is twice of a perfect square in the sequences $\{p_{2n+1}\}$ and $\{q_{2n+1}\}$.

Theorem 3.5. *Let $p_{2n+1} = 2x^2$ for some nonnegative integer n. Then $n = 1$.*

Proof. Assume that $p_{2n+1} = 2x^2$ for some nonnegative integer n. Then we have

$$2(G_{n+1} - G_n) = u^2 \tag{18}$$

for some integer u. If A is even, then consecutive terms of $\{G_n\}$ are of opposite parity, which implies that $G_{n+1} - G_n$ is odd for all n. Thus, we get $u^2 \equiv 2 \pmod 8$, which is a contradiction. Thus, A is odd.
If $n = 4q - 1$ for some $q > 0$, then $n = 2 \cdot 2^r \cdot m - 1$ for $r \geq 1$, m odd and Lemma 2.2 reduces (18) to

$$u^2 = 2(G_{4q} - G_{4q-1}) \equiv -2(G_0 - G_{-1}) \equiv -2 (\text{mod } H_{2^r}).$$

But, since $(\frac{-2}{H_{2^r}}) = -1$ by (11), the above congruence is not possible.
If $n = 4q$ for some $q > 0$, then $n = 2 \cdot 2^r \cdot m$ for $r \geq 1$, m odd and applying Lemma 2.2 in (18), we have

$$u^2 = 2(G_{4q+1} - G_{4q}) \equiv -2(G_1 - G_0) \equiv -2 (\text{mod } H_{2^r}),$$

which is not true since $(\frac{-2}{H_{2^r}}) = -1$ by virtue of (11).
If $n = 4q + 1$ for some $q > 0$, then $n = 2 \cdot 2^r \cdot m + 1$ for $r \geq 1$, m odd and by Lemma 2.2, (18) reduces to

$$u^2 = 2(G_{4q+2} - G_{4q+1}) \equiv -2(G_2 - G_1) \equiv -2(A - 1) (\text{mod } H_{2^r}), \tag{19}$$

which is not true since $(\frac{-2(A-1)}{H_{2^r}}) = -1$ by virtue of (11) and (12).
If $n = 4q - 2$ with $q > 0$, then $n = 2 \cdot 2^r \cdot m - 2$ for $r \geq 1$, m odd and an application of Lemma 2.2 in (18) gives

$$u^2 = 2(G_{4q-1} - G_{4q-2}) \equiv -2(G_{-1} - G_{-2}) \equiv -2(A - 1) \pmod{H_{2^r}}, \tag{20}$$

which does not hold since $(\frac{-2(A-1)}{H_{2^r}}) = -1$ by (11) and (12). Hence, the only remaining cases are $n = 0, 1, 2$. If $n = 0$, then $2(G_1 - G_0) = 2$, which is not a perfect square. If $n = 1$, then $2(G_2 - G_1) = 2(A - 1)$, which is a perfect square only if $2(A - 1)$ is so. If $n = 2$, then we get $u^2 = 2(A^2 - A - 1) \equiv 2 \pmod 4$, which is a contradiction. □

Since $\frac{s_{2n+1}}{2} = q_{2n+1} = G_{n+1} + G_n$, the following lemma derives the solutions of the equation $q_{2n+1} = 2u^2$.

Theorem 3.6. *Let $q_{2n+1} = 2x^2$ for some nonnegative integer n. Then $n = 1$.*

Proof. Assume that $q_{2n+1} = 2u^2$ for some nonnegative integer n. Then, we have

$$G_{n+1} + G_n = 2u^2 \tag{21}$$

for some integer u. It is clear that $n \neq 0$. If A is even, then the terms of $\{G_n\}$ are alternatively odd and even integers and hence $G_{n+1} + G_n$ is odd for all n, which implies that (21) is not possible for any natural number n. Hence, A is odd. If we take $n = 12q + r$, $0 \leq r < 12$, for some integers q and r, then we have $G_{n+1} + G_n \equiv G_{r+1} + G_r \pmod 8$ by (2.1) since $8|G_3$. Besides, since $2|G_{n+1} + G_n$, it can be seen that $r = 1, 4, 7$, or 10.
If $n = 12q + 1$ with $q > 0$, then $n = 2 \cdot 2^r \cdot 3m + 1$ for $r \geq 1$ and m odd. In view of Lemma 2.2, (21) results in

$$2u^2 = G_{12q+1} + G_{12q+2} \equiv -(G_2 + G_1) \equiv -(A + 1) \pmod{H_{2^r}}, \tag{22}$$

which holds only if $(\frac{2}{H_{2^r}}) = (\frac{-(A+1)}{H_{2^r}})$. But, by (11) and (12), $1 = (\frac{2}{H_{2^r}}) = (\frac{-(A+1)}{H_{2^r}}) = (\frac{-1}{H_{2^r}})(\frac{A+1}{H_{2^r}}) = -1$, which is a contradiction to (22).
If $n = 12q + 4$, then $n = 2 \cdot 2^r \cdot m$ for $r \geq 1$, m odd and because of Lemma 2.2, (21) gives

$$2u^2 = G_{4q_1} + G_{4q_1+1} \equiv -(G_0 + G_1) \equiv -1 \pmod{H_{2^r}}. \tag{23}$$

This shows that $(\frac{-2}{H_{2^r}}) = 1$, which is not possible since $H_{2^r} \equiv 7 \pmod 8$.
If $n = 12q - 2$ for some $q > 0$, then by virtue of Lemma 2.2, (21) modulo G_2 reduces to

$$2u^2 = G_{12q-2} + G_{12q-1} \equiv G_{-2} + G_{-1} \pmod{G_2} \equiv -(A + 1) \equiv -1 \pmod A, \tag{24}$$

which implies that $(\frac{-2}{A}) = 1$. This holds only if $A \equiv 1, 3 \pmod 8$. As a result of Lemma 2.2, (21) modulo G_3 gives

$$2u^2 = G_{12q-2} + G_{12q-1} \equiv G_{-2} + G_{-1} \pmod{G_3} \equiv -(A + 1) \pmod{G_3}. \tag{25}$$

Since $8|G_3$, (25) implies that $2u^2 \equiv -(A+1) \pmod 8$. If $A \equiv 1, 3 \pmod 8$, then $u^2 \equiv -1, -2 \pmod 4$, which is a contradiction.
If $n = 12q - 5$ for some $q > 0$, then n can be written as $n = 2 \cdot 2^r \cdot 3 \cdot m - 5$ for $r \geq 1$ and m odd. Using (3), we get

$$\begin{aligned} 2u^2 &= G_{n+1} + G_n = G_{12q-5} + G_{12q-4} \equiv G_{-5} + G_{-4} \\ &\equiv -(A^4 + A^3 - 3A^2 - 2A + 1) \\ &\equiv A + 1 (\mathrm{mod} G_3). \end{aligned}$$

Since $8|G_3$, it follows that $2u^2 \equiv A+1 \pmod 8$, which implies that $A \equiv 1, 7 \pmod 8$. Besides, using (4), we get

$$\begin{aligned} 2u^2 &= G_{n+1} + G_n = G_{12q-5} + G_{12q-4} \equiv G_{-5} + G_{-4} \\ &\equiv -(A^4 + A^3 - 3A^2 - 2A + 1) \\ &\equiv -(A + 1)(\mathrm{mod} H_3). \end{aligned} \tag{26}$$

Assume that $A \equiv 7(\text{mod}8)$. From the equavalent (26), it is seen that $2u^2 \equiv -1(\text{mod}A)$. This shows that $\left(\frac{-1}{A}\right) = \left(\frac{2}{A}\right)$, which is impossible since $\left(\frac{-1}{A}\right) = -1$ and $\left(\frac{2}{A}\right) = 1$ for $A \equiv 7(\text{mod}8)$. Assume that $A \equiv 1(\bmod 8)$. From the equavalent (26), it is seen that $2u^2 \equiv -(A+1)(\bmod A^2-3)$, which implies that $u^2 \equiv -\frac{(A+1)}{2}(\bmod \frac{A^2-3}{2})$. Thus, it follows that $\left(\frac{-1}{\frac{A^2-3}{2}}\right)\left(\frac{\frac{(A+1)}{2}}{\frac{A^2-3}{2}}\right) = 1$. On the other hand, since $\frac{A^2-3}{2} \equiv 7(\text{mod}8)$, it is seen that $\left(\frac{-1}{\frac{A^2-3}{2}}\right) = -1$. Moreover, $\left(\frac{\frac{(A+1)}{2}}{\frac{A^2-3}{2}}\right) = \left(\frac{\frac{A^2-3}{2}}{\frac{(A+1)}{2}}\right) = \left(\frac{-1}{\frac{(A+1)}{2}}\right) = 1$ since $\frac{A+1}{2} \equiv 1(\text{mod}4)$ and $\frac{A^2-3}{2} \equiv -1(\text{mod}\frac{A+1}{2})$. Thus, we have $1 = \left(\frac{-1}{\frac{A^2-3}{2}}\right)\left(\frac{\frac{(A+1)}{2}}{\frac{A^2-3}{2}}\right) = (-1)(+1) = -1$, which is a contradiction. □

Thus, we can the following result.

Theorem 3.7. *Let N and M be odd positive integers and $p_N q_M = y^2$ for some integer y.*

1. *If A is odd, then $(N, M) = (1, 1), (1, 3), (3, 1)$.*
2. *If $4|A$, then $(N, M) = (1, 1), (1, 3)$.*
3. *If $A \equiv 2(\text{mod } 4)$, and $A - 1$ is a perfect square or $A = 6$, then $(N, M) = (1, 1), (1, 3)$, or $(7, 1)$.*
4. *If $A \equiv 2(\text{mod } 4)$, $A - 1$ is not a perfect square, and Conjecture 3.3 is true, then $(N, M) = (1, 1)$.*

Proof. Suppose that

$$p_N q_M = y^2 \tag{27}$$

holds for some odd positive integers N and M. In consequence of Lemma 2.1 and (8), $\{p_n\}$ and $\{q_n\}$ are Lehmer and associated Lehmer sequences, which satisfy the divisibility properties analogous to Lucas sequence [12] and hence $(p_N, q_M) = 1$ or 2. As a result, (27) splits into $p_N = y_1^2$ and $q_M = y_2^2$ or $p_N = 2y_1^2$ and $q_M = 2y_2^2$. Thus, by Theorems 3.5 and 3.6, Corollary 3.4.1, and Lemma 2.22, the proof follows. □

3.1 Proof of Theorem 1.1

Suppose that

$$P_N Q_M = y^p \tag{28}$$

has solution for some positive integers N, M and $p \geq 2$. Let $N = 2^a N_1$ and $M = 2^b M_1$, where N_1 and M_1 are odd and a, b are non-negative integers. If $a \leq b$, then $(P_N, Q_M) = 1$ by Lemma 2.1. Hence, from (28), we get $P_N = y_1^p$ and $Q_M = y_2^p$ for some positive integers y_1 and y_2. By Lemmas 2.11 and 2.13, it can be seen that $(N, M) = (1, 1)$ for $y = 1$, and $(N, M) = (7, 1)$ for $y = 13$ and $p = 2$.

Hence, $a > b$. Let $r = a - b \geq 1$, $d = (N, M) = 2^b(N_1, M_1)$ and therefore, $N = 2^a N_1 = 2^r kd$ for some odd k. As a result of Lemma 2.1, (28) can be written as

$$y^p = P_N Q_M = P_{2^r kd} Q_M = 2^r P_{kd} Q_{kd} Q_{2kd} \cdots Q_{2^{r-1}kd} Q_M.$$

In view of Lemma 2.1, $(P_{kd}, Q_{2^i kd}) = 1$ for $0 \leq i \leq r-1$ since $v_2(kd) \leq v_2(2^i kd)$ and $(P_{kd}, Q_M) = 1$ since $v_2(kd) = v_2(M) = b$. Consequently, P_{kd} is relatively prime to the product $Q_{kd}Q_{2kd}\cdots Q_{2^{r-1}kd}Q_M$ and hence

$$2^r P_{kd} = y_3^p \tag{29}$$

and

$$Q_{kd}Q_{2kd}\cdots Q_{2^{r-1}kd}Q_M = y_4^p. \tag{30}$$

Assume that $r \geq 2$. In view of Lemma 2.1, $Q_{2^{r-1}kd}$ is relatively prime to $Q_{kd}Q_{2kd}\cdots Q_{2^{r-2}kd}$ since $v_2(2^i kd) < v_2(2^{r-1}kd)$ and $(Q_M, Q_{2^{r-1}kd}) = 1$ since $v_2(M) < v_2(2^{r-1}kd)$. Therefore, $Q_{2^{r-1}kd} = y_5^p$ for some positive integer y_5. But, Lemma 2.13 results in $2^{r-1}kd = 1$ and hence $r = 1$, which contradicts our assumption that $r \geq 2$. Thus $r = 1$ and $2P_{kd} = y_3^p$ from (29). This implies that $P_{kd} = 2^{p-1}(y_3/2)^p$, which results $kd = 2$ by Lemma 2.12. Therefore, $2^a N_1 = N = 2^r kd = 4$, and hence $a = 2$. Besides, since $v_2(y^p) = v_2(P_N Q_M) = v_2(P_4 Q_M) = 2$, it follows that $p = 2$. Thus, we obtain the equation $12 \cdot Q_M = y^2$ or equavalently, $Q_2 \cdot Q_M = (y/2)^2$ from (28). The last equation has solution only for $M = 2$ by Theorem 2 given in [11]. Thus, the proof is completed. □

3.2 Proof of Theorem 1.2

Suppose that the given equation

$$G_N H_M = y^2 \tag{31}$$

has solution for some positive integers N, M. Let $N = 2^a N_1$ and $M = 2^b M_1$, where N_1 and M_1 are odd and a, b are non-negative integers. If $a \leq b$, then $(G_N, H_M) = 1$ or 2 by Lemma 2.1. If $(G_N, H_M) = 1$, then (31) results in $G_N = y_1^2$ and $H_M = y_2^2$. By Lemma 2.7 and 2.10, we get $(A, N) \in \{(3,6), (338,4)\}$ for $N \geq 3$ and $M = 1$. But, these cases are impossible since $H_M = H_1 = A \neq \square$. Thus, (31) has no solution for $(G_N, H_M) = 1$ and $N \geq 3$. If $(G_N, H_M) = 2$, then $G_N = 2y_1^2$ and $H_M = 2y_2^2$. In view of Lemma 2.10, if $N \geq 3$, then $G_N = 2y_1^2$ has no solution, which implies that (31) has no solution in this case. Hence the only possible values of N for $a \leq b$ are $N = 0, 1, 2$. Now suppose that $a > b$. Let $r = a - b \geq 1$ and $d = (N, M) = 2^b(N_1, M_1)$. N can be written as $N = 2^a N_1 = 2^r kd$ for some odd k. Then, (31) can be written as

$$y^2 = G_N H_M = G_{2^r kd} H_M = G_{kd} H_{kd} H_{2kd} \cdots H_{2^{r-1}kd} H_M.$$

In view of Lemma 2.1, $(G_{kd}, H_{2^i kd}) = 1$ or 2 for $0 \leq i \leq r-1$ and $(G_{kd}, H_M) = 1$ or 2 since $v_2(kd) \leq v_2(2^i kd)$ and $v_2(kd) = v_2(M) = b$, respectively. Thus, greatest common divisor of G_{kd} and $H_{kd}H_{2kd}\cdots H_{2^{r-1}kd}H_M$ is a power of 2, which gives rise to

$$G_{kd} = 2^j y_3^2 \tag{32}$$

and

$$H_{kd}H_{2kd}\cdots H_{2^{r-1}kd}H_M = 2^j y_4^2, \tag{33}$$

where $j \in \{0, 1\}$. Since $v_2(2^i kd) < v_2(2^{r-1}kd)$ for $0 \leq i < r-1$, Lemma 2.1 results in that greatest common divisor of $H_{2^{r-1}kd}$ and $H_{kd}H_{2kd}\cdots H_{2^{r-2}kd}$ is a power of 2. Note that if $r \geq 2$, then $v_2(M) < v_2(2^{r-1}kd)$ and hence $(H_M, H_{2^{r-1}kd}) = 1$ or 2 by Lemma 2.1.

If $r \geq 2$, then it is clear that greatest common divisor of $H_{2^{r-1}kd}$ and $H_{kd}H_{2kd} \cdots H_{2^{r-2}kd}H_M$ is a power of 2, which gives

$$H_{2^{r-1}kd} = y_5^2 \tag{34}$$

or

$$H_{2^{r-1}kd} = 2y_5^2. \tag{35}$$

Suppose that $kd \geq 3$. Then (32) has only solutions $(A, kd) = (3, 6), (338, 4)$ by Lemma 2.10. If $A = 3$ or $A = 338$, then (34) has no solution by Lemma 2.7. In consideration of Lemma 2.9, if $A = 3$, then (35) has only solutions $2^{r-1}kd = 0, 3$, which is not possible since $kd = 6$ in this case. If $A = 338$, then Lemma 2.9 results in that (35) has only solutions $2^{r-1}kd = 1, 2$, which is not true since $kd = 4$ in this case. Hence $kd \in \{1, 2\}$. From (34) and (35), it is clear that $2^{r-1}kd \in \{0, 1, 2, 3\}$. Therefore, $r \in \{1, 2\}$ and hence $2^r kd = N \in \{2, 4, 8\}$ when $a > b$. □

3.3 Proof of Theorem 1.3

If $n = 0$ or $m = 0$, then $B_m = y^p$ or $B_n = y^p$, which has only one solution $m = 1$ or $n = 1$ by Lemma 2.14. Thus, the equation $y^p = B_n + B_m$ has the solutions $(n, m) = (0, 1), (1, 0)$, and the equation $y^p = B_n - B_m$ has the solution $(n, m) = (1, 0)$. Now, let n and m be positive integers. If $n = m$, then we get $2B_n = P_{2n} = y^p$ by Lemma 2.1 (1). The equation $P_{2n} = y^p$ has no solution by Lemma 2.11. Hence, we suppose that $0 < m < n$. For some $\epsilon = \pm 1$, let $N = n + \epsilon m$ and $M = n - \epsilon m$. Then, we can write $y^p = B_n \pm B_m = P_{n+\epsilon m}Q_{n-\epsilon m} = P_N Q_M$ by Lemmas 2.15 and 2.16. By Theorem 1.1, the equation $P_N Q_M = y^p$ implies that $(N, M, y, p) = (1, 1, 1, p), (7, 1, 13, 2)$ or $(4, 2, 6, 2)$. Thus, it can be easily seen that the equation $B_n + B_m = y^p$ has the solution $(n, m) = (3, 1)$ for $y = 6$ and $p = 2$ and the equation $B_n - B_m = y^p$ has the only solution $(n, m) = (4, 3)$ for $y = 13$ and $p = 2$. In the equation $B_n + B_m = y^p$, since B_n and B_m are symmetric, $(n, m) = (1, 3)$ is a solution of this equation. □

3.4 Proof of Theorem 1.4

Suppose that $G_n + G_m = y^2$ has a solution for some nonnegative integers n, m. If n and m are of same parity, then $n + m$ and $n - m$ are even. Thus, we can write

$$y^2 = G_n + G_m = (\frac{p_{n+m}}{2})s_{|n-m|} = G_{\frac{n+m}{2}} H_{\frac{|n-m|}{2}} \tag{36}$$

by Lemma 2.17. By Theorem 1.2, it is seen that $n + m \in \{0, 2, 4, 8, 16\}$.

If n and m are of opposite parity, then $n + m$ and $n - m$ are odd and so, the equation $G_n + G_m = y^2$ can be written as

$$y^2 = G_n + G_m = p_{|n-m|}(\frac{s_{n+m}}{2}) = p_{|n-m|}q_{n+m} \tag{37}$$

by Lemma 2.17. In view of Theorem 3.6, Lemma 2.22 and Theorem 3.7, we get $n + m \in \{1, 3\}$ from (37).

Now assume that $G_n - G_m = y^2$ for some nonnegative integers n and m. Proceeding similar to the above discussion, one can conclude that if n and m are of same parity then the given equation has possible solutions (n, m) such that $n - m \in \{0, 2, 4, 8, 16\}$ by Theorems 1.2 and Lemma 2.18. If n and m are of opposite parity, then the given equation has possible solutions (n, m) such that $n - m \in \{1, 3\}$ by Theorems 3.7 and Lemma 2.18. □

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