Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 2, Pages 281–285
DOI: 10.7546/nntdm.2022.28.2.281-285

On certain rational perfect numbers

József Sándor

Department of Mathematics, Babeş-Bolyai University
Cluj-Napoca, Romania
e-mail: jsandor@math.ubbcluj.ro

Received: 12 March 2022 **Revised:** 11 May 2022
Accepted: 11 May 2022 **Online First:** 12 May 2022

Abstract: We study equations of type $\sigma(n) = \frac{k+1}{k} \cdot n + a$, where $a \in \{0, 1, 2, 3\}$, where k and n are positive integers, while $\sigma(n)$ denotes the sum of divisors of n.
Keywords: Sum of divisors, Perfect numbers.
2020 Mathematics Subject Classification: 11A25.

1 Introduction

Let $\sigma(n)$ denote the sum of distinct positive divisors of $n > 1$. Let $k \geq 1$ be a positive integer. The equation

$$\sigma(n) = \frac{k+1}{k} \cdot n, \tag{1}$$

where $k = 1$, has a long history in mathematics, as these are the famous perfect numbers n for which $\sigma(n) = 2n$ (see [5, 6]).

For $k > 1$, however, these are certain "rational perfect" numbers n. For $k = p - 1$, with $p \geq 3$ a prime, we have introduced Equation (1) in [3]. A similar equation, namely

$$\sigma(n) = \frac{k+1}{k} \cdot n + 1 \tag{2}$$

for $k = 2$ has been considered in our note [4].

In this paper we will solve completely these equations, as well as the following new ones:

$$\sigma(n) = \frac{k+1}{k} \cdot n + 2, \tag{3}$$

$$\sigma(n) = \frac{k+1}{k} \cdot n + 3. \tag{3'}$$

2 Main results

The following auxiliary result will be used more times:

Lemma 1. *$\sigma(n)$ is a multiplicative function, i.e., $\sigma(mn) = \sigma(m)\sigma(n)$ for all $(m, n) = 1$. One has the inequality*

$$\sigma(a \cdot b) \geq a \cdot \sigma(b) \tag{4}$$

with equality only for $a = 1$.

For a proof of (4), see our paper [1], as well as [2].

Lemma 2. *Let $x > 1$ be a positive integer. Then*

$$\sigma(x) = x + 1 \text{ iff } x \geq 2 \text{ is a prime;} \tag{5}$$

$$\sigma(x) = x + 2 \text{ is not solvable;} \tag{6}$$

$$\sigma(x) = x + 3 \text{ iff } x = 4. \tag{7}$$

$$\sigma(x) = x + 4 \text{ iff } x = 9. \tag{7'}$$

Proof. (5) is well-known and follows from $\sigma(x) \geq x + 1$, with equality only if x is a prime.
As $\sigma(x) = (x + 1) + 1$ and 1 and $x > 1$ are distinct divisors of x, clearly (6) is not possible.
Writing (7) as $\sigma(x) = x + 1 + 2$, it follows that $2 \| x$, and there are no other divisors; so $x = 2^k$; and it is immediate that $k = 2$; so $x = 4$.
Writing (7') as $\sigma(x) = x + 1 + 3$, it follows in a similar manner that $x = 9$. □

Theorem 1. *If Equation (1) is solvable, then we must have $n = k = p$, where p is a prime number.*

Proof. As $(k, k + 1) = 1$ we must have $n = \text{multiple of } k$, i.e., $n = k \cdot A$ ($A \geq 1$ integer). Thus (1) becomes

$$\sigma(k \cdot A) = (k + 1) \cdot A. \tag{8}$$

By applying inequality (4), one has $\sigma(k \cdot A) \geq A \cdot \sigma(k)$, with equality only for $A = 1$, so by (8) we get $\sigma(k) \leq k + 1$. By (5) we get that $k = p = \text{prime}$. As $A = 1$, we have $n = p \cdot 1 = p$, and the proof is complete. □

Remark 1. *Suppose that $k = q - 1$, where $q \geq 3$ is a prime. On the basis of Theorem 1, the equation*

$$\sigma(n) = \frac{q}{q - 1} \cdot n \tag{9}$$

can be solved only if $q - 1 = p$ is a prime. This is possible only if $q = 3$, $p = 2$, so the only solution of the Equation (9) is $n = 2$. This offers a complete solution of equation from [3].

Theorem 2. *If Equation (2) is solvable, then k must be a prime $(k = p)$ and $n = p^2$.*

Proof. Equation (2) with $n = k \cdot A$ ($A \geq 1$) can be rewritten as $\sigma(kA) = (k + 1) \cdot A + 1$. As by (4) one has $\sigma(kA) \geq A \cdot \sigma(k)$, we get that one must have

$$A \cdot [\sigma(k) - (k + 1)] \leq 1. \tag{10}$$

As $\sigma(k)-(k+1)\geq 0$ and $A\geq 1$, inequality (10) can be true only if:

i) $\sigma(k)-(k+1)=0$ and A arbitrary;

ii) $A=1$, $\sigma(k)-(k+1)=1$.

Remark that Case ii) is not possible by (6), so remains only Case i), when we get that k is prime, and A is arbitrary. Let $k=p$, and write $n=p^a\cdot N$ $(a\geq 1, N\geq 1)$ and $(p,N)=1$.

As $\sigma(n)=\sigma(p^a)\sigma(N)=\dfrac{p^{a+1}-1}{p-1}\cdot\sigma(N)$, Equation (2) can be rewritten as

$$p\cdot(p^{a+1}-1)\sigma(N)=(p^2-1)p^a\cdot N+p\cdot(p-1). \tag{11}$$

Let $\sigma(N)=N+T$, where $T\geq 0$ is an integer. Equation (11) can be rewritten as

$$N\cdot(p^a-p)+T\cdot(p^{a+2}-p)=p^2-p. \tag{12}$$

If $a=1$, (12) is possible only if $N=1$ and $T=0$; thus $n=1$, which is not possible as we have assumed $n>1$.

If $a\geq 2$, (12) is possible only if $N=1$, $T=0$ and $a=2$, so we get $n=p^2$, and the theorem is proved. □

Remark 2. *Particularly, when $k=2$, we get that only solution of* (2) *is $n=4$, proved in [4].*

Theorem 3. *Equation* (3) *is solvable only if $n=k=4$.*

Proof. As $n=k\cdot A$ $(A\geq 1)$, Equation (3) can be rewritten as

$$\sigma(kA)=(k+1)\cdot A+2. \tag{13}$$

As $\sigma(kA)\geq A\sigma(k)$, from (13) we get the inequality

$$A\cdot[\sigma(k)-(k+1)]\leq 2. \tag{14}$$

Logically, there are possible four distinct cases:

i) A arbitrary, $\sigma(k)-(k+1)=0$;

ii) $A=1, \sigma(k)-(k+1)=2$;

iii) $A=2, \sigma(k)-(k+1)=1$;

iv) $A=1, \sigma(k)-(k+1)=1$.

Clearly, Cases iii), iv) are not possible by (6). In Case ii) one has by (7) that $k=4$ and $A=1$; so $n=4$.

Let us consider now Case i). In this case $k=p=\text{prime}$. Similarly to Equation (11), we get:

$$N\cdot(p^a-p)+T\cdot(p^{a+2}-p)=2\cdot(p^2-p). \tag{15}$$

If $T\geq 1$, as $a\geq 1$ remark that $T\cdot(p^{a+2}-p)>2(p^2-p)$, as $p^{a+2}-p>2(p^2-p)$ can be written as $p\cdot(p^{a+1}-2p+1)>0$, which is true, as $p^{a+1}-2p=p(p^a-2)\geq 0$. Thus one must have $T=0$, in which case $n=1$ and (15) is not possible.

Therefore, only the Case ii) is possible, i.e., $n=p=4$, and this finishes the proof of the theorem. □

Theorem 4. *All solutions of Equation* (3') *are* $n = 9$ *and* $n = 2q$*, where* $q \geq 3$ *is an arbitrary prime.*

Proof. Proceeding similarly as in the proof of Theorem 3, one can deduce that when $k = p$ is a prime, and for the solutions $n = p^a \cdot N$ one must have an analoque of Equation (15):

$$N \cdot (p^a - p) + T \cdot (p^{a+2} - p) = 3 \cdot (p^2 - p), \tag{16}$$

where $\sigma(N) = N + T$. Now the situation is distinct from the case of (15), as now for $a = 1$, the equation $T \cdot (p^3 - p) = 3(p^2 - p)$ has the solution $T = 1, p = 2$. As $T = 1$ and $\sigma(N) = N + 1$, we get that N is a prime: $N = q$. In this case $n = 2q$, so the Equation (3') has infinitely many solutions, contrary to Equation (3).

Similarly to (14), one can have also the cases $A = 1$ and $\sigma(k) - (k + 1) = 2$, in which case $k = 4$ and $n = 4$, and we do not have a solution; or $A = 1$ and $\sigma(k) - (k + 1) = 3$, so on the basis of (7') we get $k = 9$ and $n = 9$, which is a solution. □

Finally, an extension of Theorems 3 and 4 is contained in the following Theorem 5.

Theorem 5. *The equation*

$$\sigma(n) = \frac{k+1}{k} \cdot n + r, \tag{17}$$

where r *is a given positive integer can have at most a finite number of solutions* n*, and/or an infinite number solutions of type* $n = q \cdot N$*, where* q *is an arbitrary prime, and* N *can take at most a finite number of values.*

Proof. As in the proof of the preceding theorems, for $n = kA$ one gets the inequality

$$A \cdot [\sigma(k) - (k + 1)] \leq r. \tag{18}$$

As r is finite, A and $\sigma(k) - (k + 1)$ can have at most a finite number of values. The equation $\sigma(k) - (k + 1) = m$, or $\sigma(k) = k + M$ $(M = m + 1)$ can have at most a finite number of solutions, as all divisors of k must be certain divisors of M, and there are at most a finite number of such values.

In the case when $\sigma(k) - (k+1) = 0$, i.e., $k = p = \text{prime}$, one gets $n = p^a \cdot N$ with $(p, N) = 1$, and writing $\sigma(N) = N + T$, the equation similar to (16) will be

$$N \cdot (p^a - p) + T \cdot (p^{a+2} - p) = r \cdot (p^2 - p). \tag{19}$$

When $a = 1$, the equation $T \cdot (p^3 - p) = r(p^2 - p)$ can be written as $T \cdot (p + 1) = r$, which can have a finite number of values of T, so by $\sigma(N) = N + T$, there will be a finite number of values of N. Thus $n = p \cdot N$ and the proof of the theorem is complete. □

References

[1] Sándor, J. (1989). On the composition of some arithmetic functions. *Studia Universitatis Babeş- Bolyai, Mathematica*, 34(1), 7–14.

[2] Sándor, J. (2001). On even perfect and superperfect numbers. *Notes on Number Theory and Discrete Mathematics*, 7(1), 4–5.

[3] Sándor, J. (2006). An extension of k-perfect numbers. *Research Group in Mathematical Inequalities and Applications*, 9(4), Article 4.

[4] Sándor, J. (2008). On k_p-perfect numbers. *Matlap*, 12(8), 297–298 (in Hungarian).

[5] Sándor, J., & Atanassov, K. T. (2021). *Arithmetic Functions*. Nova Science Publishers, New York.

[6] Sándor, J., Mitrinović, D. S., & Crstici, B. (2006). *Handbook of Number Theory I*. Springer, Dordrecht.