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q-Fibonacci bicomplex and q-Lucas bicomplex numbers

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Abstract: In the paper, we define the q-Fibonacci bicomplex numbers and the q-Lucas bicomplex numbers, respectively. Then, we give some algebraic properties of the q-Fibonacci bicomplex numbers and the q-Lucas bicomplex numbers.
Keywords: Bicomplex number, q-integer (q-number), Fibonacci number, Bicomplex Fibonacci number, q-Fibonacci number.
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1 Introduction

W. R. Hamilton defined the four-dimensional real quaternions in 1843. The real quaternions constitute an extension of complex numbers into a four-dimensional space and can be considered as four-dimensional vectors, in the same way that complex numbers are considered as two-dimensional vectors. Hamilton [9] introduced the set of quaternions which can be represented as:

$$H = \{\, q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \,|\, q_s \in \mathbb{R}\,, s = 0,1,2,3\,\} \tag{1}$$

where

$$i^2 = j^2 = k^2 = -1\,,\ i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j\,. \tag{2}$$

Horadam [10, 11] defined complex Fibonacci and Lucas quaternions as follows:

$$QF_n = F_n + F_{n+1}\,i + F_{n+2}\,j + F_{n+3}\,k \tag{3}$$

and

$$QL_n = L_n + L_{n+1}\, i + L_{n+2}\, j + L_{n+3}\, k, \tag{4}$$

where $F_n = F_{n-1} + F_{n-2},\ n \geq 2$ and $L_n = L_{n-1} + L_{n-2},\ n \geq 2$ denote the n-th Fibonacci and Lucas numbers with the initial conditions $F_0 = 0,\ F_1 = 1,\ L_0 = 2$ and $L_1 = 1$, respectively. Also, the imaginary quaternion units $i\,,\ j\,,\ k$ have the following rules

$$i^2 = j^2 = k^2 = -1\,,\ i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j$$

There are several studies on quaternions as the Fibonacci and Lucas quaternions and their generalizations, for example, [3, 6–8, 12, 13, 16, 19, 23–25].

In recent years, fractal structures of bicomplex numbers have also been studied [20, 22]. The set of bicomplex numbers can be expressed by the basis $\{1\,, i\,, j\,, i\,j\,\}$ (Table 1) as,

$$q = q_1 + i\,q_2 + j\,q_3 + i\,j\,q_4 = (q_1 + i\,q_2) + (q_3 + i\,q_4)\,j, \tag{5}$$

where i, j and $i\,j$ satisfy the conditions

$$i^2 = -1,\ \ j^2 = -1,\ \ i\,j = j\,i. \tag{6}$$

Then, for any $q = q_1 + i\,q_2 + j\,q_3 + i\,j\,q_4$ and $p = p_1 + i\,p_2 + j\,p_3 + i\,j\,p_4$, the addition and subtraction and the multiplication of the bicomplex quaternions is defined by

$$q \pm p \ = (q_1 \pm p_1) + i\,(q_2 \pm p_2) + j\,(q_3 \pm p_3) + i\,j\,(q_4 \pm p_4). \tag{7}$$

$$\begin{aligned} q \times p \ &= (q_1p_1 - q_2p_2 - q_3p_3 + q_4p_4) + i\,(q_1p_2 + q_2p_1 - q_3p_4 - q_4p_3) \\ &\quad + j\,(q_1p_3 + q_3p_1 - q_2p_4 - q_4p_2) + ij\,(q_1p_4 + q_4p_1 + q_2p_3 + q_3p_2)\,, \end{aligned} \tag{8}$$

respectively. Here $q \times p = p \times q$.

Bicomplex numbers have three different conjugations as follows:

$$\begin{aligned} q_i{}^* &= q_1 - iq_2 + jq_3 - ijq_4 = (q_1 - iq_2) + j\,(q_3 - iq_4), \\ q_j{}^* &= q_1 + iq_2 - jq_3 - ijq_4 = (q_1 + iq_2) - j\,(q_3 + iq_4), \\ q_{ij}{}^* &= q_1 - iq_2 - jq_3 + ijq_4 = (q_1 - iq_2) - j\,(q_3 - iq_4). \end{aligned} \tag{9}$$

For more details about the bicomplex numbers, the readers can refer to [17, 18, 21, 22].

In 2015, the bicomplex Fibonacci and Lucas numbers defined by Nurkan and Güven [20], respectively, as follows:

$$\begin{aligned} \mathbb{BF}_n &= F_n + i\,F_{n+1} + j\,F_{n+2} + k\,F_{n+3}, \\ \mathbb{BL}_n &= L_n + i\,L_{n+1} + j\,L_{n+2} + k\,L_{n+3}, \end{aligned} \tag{10}$$

where F_n is the n-th Fibonacci number, L_n is the n-th Lucas number and $i,\ j,\ k$ are bicomplex units which satisfy the commutative multiplication rules:

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = 1,$$

$$i\,j = j\,i = k, \quad j\,k = k\,j = -i, \quad i\,k = k\,i = -j\,.$$

Starting from $n = 0$, the bicomplex Fibonacci and bicomplex Lucas numbers can be written respectively as:

$$BF_0 = 1i + 1j + 2k, \qquad BF_1 = 1 + 1i + 2j + 3k, \quad BF_2 = 1 + 2i + 3j + 5k, \ldots$$
$$BL_0 = 2 + 1i + 3j + 4k, \quad BL_1 = 1 + 3i + 4j + 7k, \quad BL_2 = 3 + 4i + 7j + 11k, \ldots$$

In 2018, the bicomplex Fibonacci numbers were defined by Aydın Torunbalcı [5] as follows:

$$\begin{aligned} \mathbb{BF}_n &= F_n + i\,F_{n+1} + j\,F_{n+2} + i\,j\,F_{n+3} \\ &= (F_n + i\,F_{n+1}) + (F_{n+2} + i\,F_{n+3})\,j, \end{aligned} \tag{11}$$

where i, j and $i\,j$ satisfy the conditions

$$i^2 = -1, \quad j^2 = -1, \quad j = j\,i, \quad (i\,j)^2 = 1. \tag{12}$$

The theory of the quantum q-calculus has been extensively studied in many branches of mathematics as well as in other areas in biology, physics, electrochemistry, economics, probability theory, and statistics [1,4] For $n \in \mathbb{N}_0$, the q-integer $[n]_q$ is defined as follows:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{13}$$

By (13), for all $m, n \in \mathbb{Z}$, it can be easily obtained $[m+n]_q = [m]_q + q^m\,[n]_q$.
For more details related to the quantum q-calculus, we refer to [2, 14].

In 2019, q-Fibonacci hybrid and q-Lucas hybrid numbers defined by Kızılateş [15] as follows

$$\mathbb{HF}_n(\alpha;q) = \alpha^{n-1}[n]_q + \alpha^{n1}[n+1]_q\,\mathbf{i} + \alpha^{n+1}[n+2]_q\,\boldsymbol{\varepsilon} + \alpha^{n+2}[n+3]_q\,\mathbf{h} \tag{14}$$

and

$$\mathbb{HL}_n(\alpha;q) = \alpha^n\frac{[2\,n]_q}{[n]_q} + \alpha^{n+1}\,\frac{[2\,n+2]_q}{[n+1]_q}\,\mathbf{i} + \alpha^{n+2}\,\frac{[2\,n+4]_q}{[n+2]_q}\,\boldsymbol{\varepsilon} + \alpha^{n+3}\frac{[2\,n+6]_q}{[n+3]_q}\,\mathbf{h}, \tag{15}$$

where $\mathbf{i}$, ε and $\mathbf{h}$ satisfy the conditions

$$\mathbf{i^2} = -1, \quad \boldsymbol{\varepsilon}^{\mathbf{2}} = 0, \quad \mathbf{h^2} = 1, \quad \mathbf{i}\,\mathbf{h} = \mathbf{h}\,\mathbf{i} = \boldsymbol{\varepsilon} + \mathbf{i}. \tag{16}$$

Also, Kızılateş derived several interesting properties of these numbers such as Binet-like formulas, exponential generating functions, summation formulas, Cassini-like identities, Catalan-like identities and d'Ocagne-like identities [15].

2 q-Fibonacci bicomplex and q-Lucas bicomplex numbers

In this section, we define q-Fibonacci bicomplex numbers and q-Lucas bicomplex numbers by using the basis $\{1,\ i,\ j,\ i\,j\}$, where i, j and $i\,j$ satisfy the conditions (6) as follows:

$$\begin{aligned} \mathbb{BF}_n(\alpha;q) &= \alpha^{n-1}[n]_q + \alpha^n[n+1]_q\,i + \alpha^{n+1}[n+2]_q\,j + \alpha^{n+2}[n+3]_q\,ij \\ &= \alpha^n\left(\tfrac{1-q^n}{\alpha-\alpha\,q}\right) + \alpha^{n+1}\left(\tfrac{1-q^{n+1}}{\alpha-\alpha\,q}\right) i + \alpha^{n+2}\left(\tfrac{1-q^{n+2}}{\alpha-\alpha\,q}\right) j + \alpha^{n+3}\left(\tfrac{1-q^{n+3}}{\alpha-\alpha\,q}\right)\,ij \\ &= \tfrac{\alpha^n}{\alpha-(\alpha\,q)}\left[\,1 + \alpha\,i + \alpha^2\,j + \alpha^3\,ij\,\right] - \tfrac{(\alpha\,q)^n}{\alpha-(\alpha\,q)}\,\left[\,1 + (\alpha\,q)\,i + (\alpha\,q)^2\,j + (\alpha\,q)^3\,ij\,\right] \end{aligned} \tag{17}$$

and

$$\begin{aligned}
\mathbb{BL}_n(\alpha;q) &= \alpha^n \tfrac{[2\,n]_q}{[n]_q} + \alpha^{n+1}\, \tfrac{[2\,n+2]_q}{[n+1]_q}\, i + \alpha^{n+2}\, \tfrac{[2\,n+4]_q}{[n+2]_q}\, j + \alpha^{n+3}\, \tfrac{[2\,n+6]_q}{[n+3]_q}\, i\, j. \\
&= \alpha^{2n} \,(\tfrac{1-q^{2n}}{\alpha^n-(\alpha\,q)^n}) + \alpha^{2n+2} \,(\tfrac{1-q^{2n+2}}{\alpha^{n+1}-(\alpha\,q)^{n+1}})\, i \\
&\quad + \alpha^{2n+4} \,(\tfrac{1-q^{2n+4}}{\alpha^{n+2}-(\alpha\,q)^{n+2}})\, j + \alpha^{2n+6} \,(\tfrac{1-q^{2n+6}}{\alpha^{n+3}-(\alpha\,q)^{n+3}})\, i\, j \\
&= (\tfrac{\alpha^{2n}-(\alpha\,q)^{2n}}{\alpha^n-(\alpha\,q)^n}) + (\tfrac{\alpha^{2n+2}-(\alpha\,q)^{2n+2}}{\alpha^{n+1}-(\alpha\,q)^{n+1}})\, i + (\tfrac{\alpha^{2n+4}-(\alpha\,q)^{2n+4}}{\alpha^{n+2}-(\alpha\,q)^{n+2}})\, j + (\tfrac{\alpha^{2n+6}-(\alpha\,q)^{2n+6}}{\alpha^{n+3}-(\alpha\,q)^{n+3}})\, i\, j \qquad (18) \\
&= (\alpha^n + (\alpha\,q)^n) + (\alpha^{n+1} + (\alpha\,q)^{n+1})\, i + (\alpha^{n+2} + (\alpha\,q)^{n+2})\, j \\
&\quad + (\alpha^{n+3} + (\alpha\,q)^{n+3})\, i\, j \\
&= \alpha^n\,(1 + \alpha\, i + \alpha^2\, j + \alpha^3\, i\, j\,) + (\alpha\,q)^n\,(1 + (\alpha\,q)\, i + (\alpha\,q)^2\, j + (\alpha\,q)^3\, i\, j\,).
\end{aligned}$$

For $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $(\alpha\,q) = \dfrac{-1}{\alpha}$ q-Fibonacci bicomplex number $\mathbb{BF}_n(\alpha;q)$ becomes the bicomplex Fibonacci numbers $\mathbb{BF}_n$ and q-Lucas bicomplex number $\mathbb{BL}_n(\alpha;q)$ becomes the bicomplex Lucas numbers $\mathbb{BL}_n$.

The addition, subtraction and multiplication by real scalars of two q-Fibonacci bicomplex numbers gives a q-Fibonacci bicomplex number.

Then, the addition, subtraction and multiplication by a scalar of q-Fibonacci bicomplex numbers are defined by

$$\begin{aligned}
\mathbb{BF}_n(\alpha;q) \pm \mathbb{BF}_m(\alpha;q) &= (\tfrac{\alpha^n-(\alpha\,q)^n}{\alpha-\alpha\,q} \pm \tfrac{\alpha^m-(\alpha\,q)^m}{\alpha-\alpha\,q}) + (\tfrac{\alpha^{n+1}-(\alpha\,q)^{n+1}}{\alpha-\alpha\,q} \pm \tfrac{\alpha^{m+1}-(\alpha\,q)^{m+1}}{\alpha-\alpha\,q})\, i \\
&\quad + (\tfrac{\alpha^{n+2}-(\alpha\,q)^{n+2}}{\alpha-\alpha\,q} \pm \tfrac{\alpha^{m+2}-(\alpha\,q)^{m+2}}{\alpha-\alpha\,q})\, j + (\tfrac{\alpha^{n+3}-(\alpha\,q)^{n+3}}{\alpha-\alpha\,q} \pm \tfrac{\alpha^{m+3}-(\alpha\,q)^{m+3}}{\alpha-\alpha\,q})\, i\, j \\
&= \tfrac{1}{\alpha-\alpha\,q}\{(\alpha^n \pm \alpha^m)(1 + \alpha\, i + \alpha^2\, j + \alpha^3\, i\, j) \\
&\quad -((\alpha\,q)^n \pm (\alpha\,q)^m\,)(1 + (\alpha\,q)\, i + (\alpha\,q)^2\, j + (\alpha\,q)^3\, i\, j)\},
\end{aligned} \qquad (19)$$

$$\begin{aligned}
\mathbb{BL}_n(\alpha;q) \pm \mathbb{BL}_m(\alpha;q) = \;& (\tfrac{\alpha^{2n}-(\alpha\,q)^{2n}}{\alpha^n-(\alpha\,q)^n} \pm \tfrac{\alpha^{2m}-(\alpha\,q)^{2m}}{\alpha^m-(\alpha\,q)^m}) + (\tfrac{\alpha^{2n+2}-(\alpha\,q)^{2n+2}}{\alpha^{n+1}-(\alpha\,q)^{n+1}} \pm \tfrac{\alpha^{2m+2}-(\alpha\,q)^{2m+2}}{\alpha^{m+1}-(\alpha\,q)^{m+1}})\, i \\
& + (\tfrac{\alpha^{2n+4}-(\alpha\,q)^{2n+4}}{\alpha^{n+2}-(\alpha\,q)^{n+2}} \pm \tfrac{\alpha^{2m+4}-(\alpha\,q)^{2m+4}}{\alpha^{m+2}-(\alpha\,q)^{m+2}})\, j \\
& + (\tfrac{\alpha^{2n+6}-(\alpha\,q)^{2n+6}}{\alpha^{n+3}-(\alpha\,q)^{n+3}} \pm \tfrac{\alpha^{2n+6}-(\alpha\,q)^{2n+6}}{\alpha^{n+3}-(\alpha\,q)^{n+3}})\, i\, j \\
= \;& (\alpha^n + (\alpha\,q)^n \pm \alpha^m + (\alpha\,q)^m) \\
& + (\alpha^{n+1} + (\alpha\,q)^{n+1} \pm \alpha^{m+1} + (\alpha\,q)^{m+1})\, i \\
& + (\alpha^{n+2} + (\alpha\,q)^{n+2} \pm \alpha^{m+2} + (\alpha\,q)^{m+2})\, j \\
& + (\alpha^{n+3} + (\alpha\,q)^{n+3} \pm \alpha^{m+3} + (\alpha\,q)^{m+3})\, i\, j \\
= \;& (\alpha^n \pm \alpha^m)\,(1 + \alpha\, i + \alpha^2\, j + \alpha^3\, i\, j\,) \\
& + ((\alpha\,q)^n \pm (\alpha\,q)^m)\,(1 + (\alpha\,q)\, i + (\alpha\,q)^2\, j + (\alpha\,q)^3\, i\, j\,)
\end{aligned} \qquad (20)$$

The multiplication of q-Fibonacci bicomplex and q-Lucas bicomplex number by the real scalar λ is defined as

$$\begin{aligned}\lambda\,\mathbb{BF}_n(\alpha;q) &= \tfrac{\lambda}{\alpha-\alpha\,q}\{\,\alpha^n(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j\,)\\ &\quad+(\alpha\,q)^n\,(\,1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j\,)\,\}\,,\end{aligned} \tag{21}$$

$$\begin{aligned}\lambda\,\mathbb{BL}_n(\alpha;q) &= \lambda\,\alpha^n(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j\,)\\ &\quad+\lambda\,(\alpha\,q)^n\,(\,1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j\,).\end{aligned} \tag{22}$$

The multiplication of two q-Fibonacci bicomplex numbers is defined by

$$\begin{aligned}\mathbb{BF}_n(\alpha;q)\times\,\mathbb{BF}_m(\alpha;q) &= (\alpha^{n-1}[n]_q+\alpha^n[n+1]_q\,i+\alpha^{n+1}[n+2]_q\,j+\alpha^{n+2}[n+3]_q\,i\,j)\\ &\quad\times(\alpha^{m-1}[m]_q+\alpha^m[m+1]_q\,i+\alpha^{m+1}[m+2]_q\,j+\alpha^{m+2}[m+3]_q\,i\,j)\\ &= \tfrac{\alpha^{n+m}}{(\alpha-\alpha\,q)^2}\,\{(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j)^2\\ &\quad-q^{n+m}(1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j\,)^2\\ &\quad-(q^m+q^n)(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j)(1+(\alpha\,q)\,i\\ &\qquad+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j\,)\,\}\\ &= \mathbb{BF}_m(\alpha;q)\times\,\mathbb{BF}_n(\alpha;q)\end{aligned} \tag{23}$$

The multiplication of two q-Lucas bicomplex numbers is defined by

$$\begin{aligned}\mathbb{BL}_n(\alpha;q)\times\mathbb{BL}_m(\alpha;q) &= (\alpha^n\tfrac{[2\,n]_q}{[n]_q}+\alpha^{n+1}\,\tfrac{[2\,n+2]_q}{[n+1]_q}\,i+\alpha^{n+2}\,\tfrac{[2\,n+4]_q}{[n+2]_q}\,j+\alpha^{n+3}\tfrac{[2\,n+6]_q}{[n+3]_q}\,i\,j)\\ &\quad\times(\alpha^m\tfrac{[2\,m]_q}{[m]_q}+\alpha^{m+1}\tfrac{[2\,m+2]_q}{[m+1]_q}\,i+\alpha^{m+2}\tfrac{[2\,m+4]_q}{[m+2]_q}\,j+\alpha^{m+3}\tfrac{[2\,m+6]_q}{[m+3]_q}\,i\,j)\\ &= \alpha^{n+m}\,\{(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j\,)^2\\ &\quad+q^{n+m}\,(1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j\,)^2\\ &\quad+(q^m+q^n)(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j)\,(1+(\alpha\,q)\,i+(\alpha\,q)^2\,j\\ &\quad+(\alpha\,q)^3\,i\,j\,)\,\}\\ &= \mathbb{BL}_m(\alpha;q)\times\,\mathbb{BL}_n(\alpha;q)\end{aligned} \tag{24}$$

The multiplication is done using bicomplex units (Table 1), and this product is commutative.

x	1	i	j	$i\,j$
1	1	i	j	$i\,j$
i	i	-1	$i\,j$	$-j$
j	j	$i\,j$	-1	$-i$
$i\,j$	$i\,j$	$-j$	$-i$	1

Table 1. Multiplication scheme of bicomplex units

Three kinds of conjugation can be defined for bicomplex numbers [20,22]. Therefore, conjugation of the q-Fibonacci bicomplex number is defined in three different ways as follows

$$(\mathbb{BF}_n(\alpha;q))^{*_1} = (\alpha^{n-1}[n]_q - i\,\alpha^n[n+1]_q + j\,\alpha^{n+1}[n+2]_q - i\,j\,\alpha^{n+2}[n+3]_q), \tag{25}$$

$$(\mathbb{BF}_n(\alpha;q))^{*_2} = (\alpha^{n-1}[n]_q + i\,\alpha^n[n+1]_q - j\,\alpha^{n+1}[n+2]_q - i\,j\,\alpha^{n+2}[n+3]_q), \tag{26}$$

$$(\mathbb{BF}_n(\alpha;q))^{*_3} = (\alpha^{n-1}[n]_q - i\,\alpha^n[n+1]_q - j\,\alpha^{n+1}[n+2]_q + i\,j\,\alpha^{n+2}[n+3]_q). \tag{27}$$

Therefore, the norm of the q-Fibonacci bicomplex number $\mathbb{BF}_n(\alpha;q)$ is defined in three different ways as follows

$$N_{\mathbb{BF}_n(\alpha;q)^{*_1}} = \|(\mathbb{BF}_n(\alpha;q)) \times (\mathbb{BF}_n(\alpha;q))^{*_1}\|^2, \tag{28}$$

$$N_{\mathbb{BF}_n(\alpha;q))^{*_2}} = \|(\mathbb{BF}_n(\alpha;q)) \times (\mathbb{BF}_n(\alpha;q))^{*_2}\|^2, \tag{29}$$

$$N_{\mathbb{BF}_n(\alpha;q))^{*_3}} = \|(\mathbb{BF}_n(\alpha;q)) \times (\mathbb{BF}_n(\alpha;q))^{*_3}\|^2. \tag{30}$$

Now, we give another relations related to q-Fibonacci bicomplex numbers. For $n \geq 1$,

$$\mathbb{BF}_{n+1}(\alpha;q) = \mathbb{BF}_n(\alpha;q) + \mathbb{BF}_{n-1}(\alpha;q), \tag{31}$$

$$\mathbb{BF}_{n+1}(\alpha;q) - \mathbb{BF}_{n-1}(\alpha;q) = \frac{1}{\sqrt{\Delta}}\,\mathbb{BL}_n(\alpha;q), \tag{32}$$

with the initial values

$$\mathbb{BF}_0(\alpha;q) = \frac{1}{\alpha - \alpha\,q}\{\,(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j\,) - (1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j\,)\},$$
$$\mathbb{BF}_1(\alpha;q) = \frac{1}{\alpha - \alpha\,q}\{\,\alpha\,(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j\,) - (\alpha\,q)\,(1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j\,)\},$$

where $\Delta = (\alpha - \alpha\,q)^2$.
For example, for equality (31):

$$\begin{aligned}
\mathbb{BF}_2(\alpha;q) &= \frac{1}{\alpha-(\alpha\,q)}[\alpha^2(1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j) \\
&\quad - (\alpha\,q)^2(1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j)\,] \\
\mathbb{BF}_1(\alpha;q)+\mathbb{BF}_0(\alpha;q) &= \frac{1}{\alpha-(\alpha\,q)}\{\,[\,(\alpha+1)+(\alpha^2+\alpha)\,i+(\alpha^3+\alpha^2)\,j+(\alpha^4+\alpha^3)\,i\,j\,] \\
&\quad - [\,((\alpha\,q)+1)+((\alpha\,q)^2+(\alpha\,q))\,i+((\alpha\,q)^3+(\alpha\,q)^2)\,j \\
&\quad + ((\alpha\,q)^4+(\alpha\,q)^3)\,i\,j\,]\},
\end{aligned}$$

where

$$\alpha+1=\alpha^2, \quad \alpha^2+\alpha=\alpha^3, \quad \alpha^3+\alpha^2=\alpha^4, \quad \alpha^4+\alpha^3=\alpha^5$$

and

$$(\alpha\,q)+1=(\alpha\,q)^2,\ (\alpha\,q)^2+(\alpha\,q)=(\alpha\,q)^3,$$
$$(\alpha\,q)^3+(\alpha\,q)^2=(\alpha\,q)^4,\ (\alpha\,q)^4+(\alpha\,q)^3=(\alpha\,q)^5.$$

For equality (32):

$$\begin{aligned}\mathbb{BF}_2(\alpha;q)-\mathbb{BF}_0(\alpha;q)=&\frac{1}{\alpha-(\alpha\,q)}[(\alpha^2-1)+(\alpha^3-\alpha)\,i+(\alpha^4-\alpha^2)\,j+(\alpha^5-\alpha^3)\,i\,j\,]\\&-[\,((\alpha\,q)^2-1)+((\alpha\,q)^3-(\alpha\,q))\,i+((\alpha\,q)^4-(\alpha\,q)^2)\,j\\&+((\alpha\,q)^5-(\alpha\,q)^3)\,i\,j\,]\}\\\frac{1}{\sqrt{\Delta}}\,\mathbb{BL}_1(\alpha;q)=&\frac{1}{\sqrt{\Delta}}\{\,[\,\alpha+\alpha^2\,i+\alpha^3\,j+\alpha^4\,i\,j\,]+[\,\alpha\,q+\alpha\,q^2\,i+\alpha\,q^3\,j+\alpha\,q^4\,i\,j\,]\end{aligned}$$

where

$$\alpha^2-1=\alpha,\quad \alpha^3-\alpha=\alpha^2,\quad \alpha^4-\alpha^2=\alpha^3,\quad \alpha^5-\alpha^3=\alpha^4$$

and

$$\begin{gathered}(\alpha\,q)^2-1=\alpha\,q,\quad (\alpha\,q)^3-\alpha\,q=(\alpha\,q)^2,\\(\alpha\,q)^4-(\alpha\,q)^2=(\alpha\,q)^3,\quad (\alpha\,q)^5-(\alpha\,q)^3=(\alpha\,q)^4.\end{gathered}$$

In this section, we give Binet-Like formulas, exponential generating functions and some other identities for the q-Fibonacci bicomplex numbers.

Theorem 1 (Binet's formula). *Let $\mathbb{BF}_n(\alpha;q)$ and $\mathbb{BL}_n(\alpha;q)$ be the q-Fibonacci bicomplex number and the q-Lucas bicomplex number. For $n\geq 1$, Binet's formula for these numbers respectively, is as follows:*

$$\mathbb{BF}_n(\alpha;q)=\frac{\alpha^n\,\widehat{\gamma}-(\alpha\,q)^n\,\widehat{\delta}}{\alpha-\alpha\,q},\tag{33}$$

and

$$\mathbb{BL}_n(\alpha;q)=\alpha^n\,\widehat{\gamma}+(\alpha\,q)^n\,\widehat{\delta}\tag{34}$$

where

$$\widehat{\gamma}=1+\alpha\,i+\alpha^2\,j+\alpha^3\,i\,j,\quad \alpha=\tfrac{1+\sqrt{5}}{2}$$

and

$$\widehat{\delta}=1+(\alpha\,q)\,i+(\alpha\,q)^2\,j+(\alpha\,q)^3\,i\,j,\quad \alpha\,q=\tfrac{-1}{\alpha}.$$

Proof. (33): Using (13) and (17), we find that

$$\begin{aligned}\mathbb{BF}_n(\alpha;q)=&\;\alpha^{n-1}[n]_q+\alpha^n[n+1]_q\,\mathbf{i}+\alpha^{n+1}[n+2]_q\,\mathbf{j}+\alpha^{n+2}[n+3]_q\,\mathbf{i}\,\mathbf{j}\\=&\;\alpha^n\,\tfrac{1-q^n}{\alpha-\alpha\,q}+\alpha^{n+1}\,\tfrac{1-q^{n+1}}{\alpha-\alpha\,q}\,\mathbf{i}+\alpha^{n+2}\,\tfrac{1-q^{n+2}}{\alpha-\alpha\,q}\,\mathbf{j}+\alpha^{n+3}\,\tfrac{1-q^{n+3}}{\alpha-\alpha\,q}\,\mathbf{i}\,\mathbf{j}\\=&\;\tfrac{\alpha^n\,[\,1+\alpha\,\mathbf{i}+\alpha^2\,\mathbf{j}+\alpha^3\,\mathbf{i}\,\mathbf{j}\,]-(\alpha\,q)^n\,[\,1+(\alpha\,q)\,\mathbf{i}+(\alpha\,q)^2\,\mathbf{j}+(\alpha\,q)^3\,\mathbf{i}\,\mathbf{j}\,]}{\alpha-(\alpha\,q)}\\=&\;\tfrac{\alpha^n\,\widehat{\gamma}-(\alpha\,q)^n\,\widehat{\delta}}{\alpha-\alpha\,q}.\end{aligned}$$

In a similar way, equality (34) can be derived as follows

$$\begin{aligned}
\mathbb{BL}_n(\alpha;q) = {} & \alpha^n \tfrac{[2\,n]_q}{[n]_q} + \alpha^{n+1}\tfrac{[2\,n+2]_q}{[n+1]_q}\, i + \alpha^{n+2}\tfrac{[2\,n+4]_q}{[n+2]_q}\, j + \alpha^{n+3}\tfrac{[2\,n+6]_q}{[n+3]_q}\, i\, j \\
= {} & \alpha^{2n}\,(\tfrac{1-q^{2n}}{\alpha^n-(\alpha\,q)^n}) + \alpha^{2n+2}\,(\tfrac{1-q^{2n+2}}{\alpha^{n+1}-(\alpha\,q)^{n+1}})\, i \\
& + \alpha^{2n+4}\,(\tfrac{1-q^{2n+4}}{\alpha^{n+2}-(\alpha\,q)^{n+2}})\, j + \alpha^{2n+6}\,(\tfrac{1-q^{2n+6}}{\alpha^{n+3}-(\alpha\,q)^{n+3}})\, i\, j \\
= {} & \alpha^n\,(1 + \alpha\, i + \alpha^2\, j + \alpha^3\, i\, j\,) \\
& + (\alpha\, q)^n\,(1 + (\alpha\, q)\, i + (\alpha\, q)^2\, j + (\alpha\, q)^3\, i\, j\,) \\
= {} & \alpha^n\,\widehat{\gamma} + (\alpha\, q)^n\,\widehat{\delta}
\end{aligned}$$

where $\widehat{\gamma} = 1 + \alpha\, i + \alpha^2\, j + \alpha^3\, i\, j,\ \widehat{\delta} = 1 + (\alpha\, q)\, i + (\alpha\, q)^2\, j + (\alpha\, q)^3\, i\, j$ and $\widehat{\gamma}\,\widehat{\delta} = \widehat{\delta}\,\widehat{\gamma}$.
Thus, the proof is completed. □

Theorem 2 (Generating function). *Let $\mathbb{BF}_n(\alpha;q)$ be the q-Fibonacci bicomplex number and $\mathbb{BL}_n(\alpha;q)$ be the q-Lucas bicomplex number. For the generating function for these numbers is as follows:*

$$g_{\mathbb{BF}_n(\alpha;q)}\,(t) = \sum_{n=0}^{\infty} \mathbb{BF}_n(\alpha;q)\, t^n = \frac{(1-t+\alpha\, t)\,\widehat{\gamma} - (1-t+(\alpha\, q)\, t)\,\widehat{\delta}}{(\alpha - \alpha\, q)\,(1-t-t^2)}, \tag{35}$$

$$g_{\mathbb{BL}_n(\alpha;q)}\,(t) = \sum_{n=0}^{\infty} \mathbb{BL}_n(\alpha;q)\, t^n = \frac{(1-t+\alpha\, t)\,\widehat{\gamma} + (1-t+(\alpha\, q)\, t)\,\widehat{\delta}}{1-t-t^2}. \tag{36}$$

Proof. (35): Using the definition of generating function, we obtain

$$g_{\mathbb{BF}_n(\alpha;q)}\,(t) = \mathbb{BF}_0(\alpha;q) + \mathbb{BF}_1(\alpha;q)\, t + \mathbb{BF}_2(\alpha;q)\, t^2 + \cdots + \mathbb{BF}_n(\alpha;q)\, t^n + \cdots . \tag{37}$$

Multiplying by $(1-t-t^2)$ both sides of (37) and using (31), we have

$$\begin{aligned}
(1-t-t^2)\, g_{\mathbb{BF}_n(\alpha;q)}\,(t) = {} & \mathbb{BF}_0(\alpha;q) + [\,\mathbb{BF}_1(\alpha;q) - \mathbb{BF}_0(\alpha;q)\,]\, t \\
& + [\mathbb{BF}_2(\alpha;q) - \mathbb{BF}_1(\alpha;q) - \mathbb{BF}_0(\alpha;q)\,]\, t^2 \\
& + [\mathbb{BF}_3(\alpha;q) - \mathbb{BF}_2(\alpha;q) - \mathbb{BF}_1(\alpha;q)\,]\, t^3 + \ldots \\
& + [\mathbb{BF}_{k+1}(\alpha;q) - \mathbb{BF}_k(\alpha;q) - \mathbb{BF}_{k-1}(\alpha;q)\,]\, t^{k+1} + \ldots \\
g_{\mathbb{BF}_n(\alpha;q)}\,(t) = {} & \frac{(1-t+\alpha\, t)\,\widehat{\gamma} - (1-t+(\alpha\, q)\, t)\,\widehat{\delta}}{(\alpha - \alpha\, q)\,(1-t-t^2)}.
\end{aligned}$$

where

$$\begin{gathered}
\mathbb{BF}_2(\alpha;q) - \mathbb{BF}_1(\alpha;q) - \mathbb{BF}_0(\alpha;q) = 0, \\
\vdots \\
\mathbb{BF}_{k+1}(\alpha;q) - \mathbb{BF}_k(\alpha;q) - \mathbb{BF}_{k-1}(\alpha;q) = 0.
\end{gathered}$$

In a similar way, (36) can be derived.
Thus, the proof is completed. □

Theorem 3 (Exponential generating function). *Let $\mathbb{BF}_n(\alpha;q)$ be the q-Fibonacci bicomplex number and $\mathbb{BL}_n(\alpha;q)$ be the q-Lucas bicomplex number. For the exponential generating functions for these numbers are as follows:*

$$g_{\mathbb{BF}_n(\alpha;q)}\,(\frac{t^n}{n!})=\sum_{n=0}^{\infty}\mathbb{BF}_n(\alpha;q)\,\frac{t^n}{n!}=\frac{\widehat{\gamma}\,e^{\alpha\,t}-\widehat{\delta}\,e^{(\alpha\,q)\,t}}{\alpha-\alpha\,q} \tag{38}$$

$$g_{\mathbb{BL}_n(\alpha;q)}\,(\frac{t^n}{n!})=\sum_{n=0}^{\infty}\mathbb{BL}_n(\alpha;q)\,\frac{t^n}{n!}=\widehat{\gamma}\,e^{\alpha\,t}+\widehat{\delta}\,e^{(\alpha\,q)\,t}\,. \tag{39}$$

Proof. (38): Using the definition of exponential generating function, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{BF}_n(\alpha;q)\,\tfrac{t^n}{n!}&=\sum_{n=0}^{\infty}(\tfrac{\alpha^n\,\widehat{\gamma}-(\alpha\,q)^n\,\widehat{\delta}}{\alpha-\alpha\,q})\,\tfrac{t^n}{n!}\\&=\tfrac{\widehat{\gamma}}{\alpha-\alpha\,q}\sum_{n=0}^{\infty}\tfrac{t^n}{n!}-\tfrac{\widehat{\delta}}{\alpha-\alpha\,q}\sum_{n=0}^{\infty}\tfrac{(\alpha\,q\,t)^n}{n!}\\&=\tfrac{\widehat{\gamma}\,e^{\alpha\,t}-\widehat{\delta}\,e^{(\alpha\,q)\,t}}{\alpha-\alpha\,q}.\end{aligned}$$

(39):

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{BL}_n(\alpha;q)\,\tfrac{t^n}{n!}&=\sum_{n=0}^{\infty}(\alpha^n\,\widehat{\gamma}+(\alpha\,q)^n\,\widehat{\delta})\,\tfrac{t^n}{n!}\\&=\widehat{\gamma}\sum_{n=0}^{\infty}\tfrac{(\alpha\,t)^n}{n!}+\widehat{\delta}\sum_{n=0}^{\infty}\tfrac{(\alpha\,q\,t)^n}{n!}\\&=\widehat{\gamma}\,e^{\alpha\,t}+\widehat{\delta}\,e^{(\alpha\,q)\,t}\,.\end{aligned}$$

Thus, the proof is completed. □

Theorem 4. *Let q-Fibonacci bicomplex number $(\mathbb{BF}_n(\alpha;q))$ and q-Lucas bicomplex number $(\mathbb{BL}_n(\alpha;q))$. In this case, for nonnegative integer numbers n and k, we can give the following relations:*

$$\sum_{i=0}^{n}\binom{n}{i}(-\alpha^2\,q)^{n-i}\,\mathbb{BF}_{2\,i+k}(\alpha;q)=\begin{cases}\Delta^{\frac{n}{2}}\,\mathbb{BF}_{n+k}(\alpha;q) & \textit{if } n \textit{ is even},\\ \Delta^{\frac{n-1}{2}}\,\mathbb{BL}_{n+k}(\alpha;q) & \textit{if } n \textit{ is odd},\end{cases} \tag{40}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-\alpha^2\,q)^{n-i}\,\mathbb{BL}_{2\,i+k}(\alpha;q)=\begin{cases}\Delta^{\frac{n}{2}}\,\mathbb{BL}_{n+k}(\alpha;q) & \textit{if } n \textit{ is even},\\ \Delta^{\frac{n+1}{2}}\,\mathbb{BF}_{n+k}(\alpha;q) & \textit{if } n \textit{ is odd},\end{cases} \tag{41}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\,(-\alpha^2\,q)^{n-i}\,\mathbb{BF}_{2\,i+k}(\alpha;q)=(-\alpha[2]_q)^n\,\mathbb{BF}_{n+k}(\alpha;q)\,, \tag{42}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\,(-\alpha^2\,q)^{n-i}\,\mathbb{BL}_{2\,i+k}(\alpha;q)=(-\alpha[2]_q)^n\,\mathbb{BL}_{n+k}(\alpha;q)\,. \tag{43}$$

where $\Delta=(\alpha-\alpha\,q)^2$ and $(-\alpha[2]_q)^n=(-\alpha(1+q))^n$.

Proof. (40): If n is even. Using the binomial coefficients and (33), we have

$$\begin{aligned}\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^2\,q)^{n-i}\,\mathbb{BF}_{2\,i+k}(\alpha;q)=&\sum_{i=0}^{n}\tbinom{n}{i}(-\alpha^2\,q)^{n-i}\,\tfrac{\alpha^{2\,i+k}\,\widehat{\gamma}-(\alpha\,q)^{2\,i+k}\,\widehat{\delta}}{\alpha-\alpha\,q}\\=&\tfrac{(\alpha^2-\alpha^2\,q)^n\,\alpha^k\,\widehat{\gamma}-(\alpha^2\,q^2-\alpha^2\,q)^n\,(\alpha\,q)^k\,\widehat{\delta}}{\alpha-\alpha\,q}\\=&\tfrac{(\alpha\,\sqrt{\Delta})^n\,\alpha^k\,\widehat{\gamma}-(-\alpha\,q\,\sqrt{\Delta})^n\,(\alpha\,q)^k\,\widehat{\delta}}{\alpha-\alpha\,q}\\=&\sqrt{\Delta}^n\,(\tfrac{\alpha^{n+k}\,\widehat{\gamma}-(\alpha\,q)^{n+k}\,\widehat{\delta}}{\alpha-\alpha\,q})\\=&\Delta^{\frac{n}{2}}\,\mathbb{BF}_{n+k}(\alpha;q)\,.\end{aligned}$$

If n is odd. Using the binomial coefficients and (33), we have

$$\begin{aligned}
\sum_{i=0}^{n}\binom{n}{i}(-\alpha^2 q)^{n-i}\,\mathbb{BF}_{2i+k}(\alpha;q) &= \sum_{i=0}^{n}\binom{n}{i}(-\alpha^2 q)^{n-i}\,\frac{\alpha^{2i+k}\,\widehat{\gamma}-(\alpha q)^{2i+k}\,\widehat{\delta}}{\alpha-\alpha q}\\
&= \frac{(\alpha^2-\alpha^2 q)^n\,\alpha^k\,\widehat{\gamma}-(\alpha^2 q^2-\alpha^2 q)^n\,(\alpha q)^k\,\widehat{\delta}}{\alpha-\alpha q}\\
&= \frac{(\alpha\sqrt{\Delta})^n\,\alpha^k\,\widehat{\gamma}+(\alpha q\sqrt{\Delta})^n\,(\alpha q)^k\,\widehat{\delta}}{\alpha-\alpha q}\\
&= \sqrt{\Delta}^{n-1}\,(\alpha^{n+k}\,\widehat{\gamma}+(\alpha q)^{n+k}\,\widehat{\delta})\\
&= \Delta^{\frac{n-1}{2}}\,\mathbb{BL}_{n+k}(\alpha;q)\,.
\end{aligned}$$

In a similar way, (41), (42) and (43) can be derived.
Thus, the proof is completed. □

Theorem 5 (Honsberger identity). *For $n, m \geq 0$ the Honsberger identity for the q-Fibonacci bicomplex numbers $\mathbb{BF}_n(\alpha;q)$ and $\mathbb{BF}_m(\alpha;q)$ and the q-Lucas bicomplex numbers $\mathbb{BL}_n(\alpha;q)$ and $\mathbb{BL}_m(\alpha;q)$, respectively, are as follows:*

$$\begin{aligned}
&\mathbb{BF}_n(\alpha;q)\,\mathbb{BF}_m(\alpha;q)+\mathbb{BF}_{n+1}(\alpha;q)\,\mathbb{BF}_{m+1}(\alpha;q)\\
&=\frac{\alpha^{n+m}}{(\alpha-\alpha q)^2}\left\{(1+\alpha^2)\,\widehat{\gamma}^2-(1+\alpha(\alpha q))\,(q^n+q^m)\,\widehat{\gamma}\,\widehat{\delta}+(1+(\alpha q)^2\,q^{n+m}\,\widehat{\delta}^2\right\},
\end{aligned}\tag{44}$$

and

$$\begin{aligned}
&\mathbb{BL}_n(\alpha;q)\,\mathbb{BL}_m(\alpha;q)+\mathbb{BL}_{n+1}(\alpha;q)\,\mathbb{BL}_{m+1}(\alpha;q)\\
&=\alpha^{n+m}\left\{(1+\alpha^2)\,\widehat{\gamma}^2+(1+\alpha(\alpha q))\,(q^n+q^m)\,\widehat{\gamma}\,\widehat{\delta}+(1+(\alpha q)^2\,q^{n+m}\,\widehat{\delta}^2\right\}.
\end{aligned}\tag{45}$$

Proof. (44): By using (17) and (33) we get

$$\begin{aligned}
&\mathbb{BF}_n(\alpha;q)\,\mathbb{BF}_m(\alpha;q)+\mathbb{BF}_{n+1}(\alpha;q)\,\mathbb{BF}_{m+1}(\alpha;q)\\
&=\left(\frac{\alpha^n\,\widehat{\gamma}-(\alpha q)^n\,\widehat{\delta}}{\alpha-\alpha q}\right)\left(\frac{\alpha^m\,\widehat{\gamma}-(\alpha q)^m\,\widehat{\delta}}{\alpha-\alpha q}\right)+\left(\frac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha q}\right)\left(\frac{\alpha^{m+1}\,\widehat{\gamma}-(\alpha q)^{m+1}\,\widehat{\delta}}{\alpha-\alpha q}\right)\\
&=\frac{\alpha^{n+m}}{(\alpha-\alpha q)^2}\left\{(\widehat{\gamma}-q^n\,\widehat{\delta})(\widehat{\gamma}-q^m\,\widehat{\delta})\right\}+\frac{\alpha^{n+m+2}}{(\alpha-\alpha q)^2}\left\{\widehat{\gamma}-q^{n+1}\,\widehat{\delta})(\widehat{\gamma}-q^{m+1}\,\widehat{\delta})\right\}\\
&=\frac{\alpha^{n+m}}{(\alpha-\alpha q)^2}\left\{(1+\alpha^2)\,\widehat{\gamma}^2-(1+\alpha(\alpha q))\,(q^n+q^m)\,\widehat{\gamma}\,\widehat{\delta}+(1+(\alpha q)^2)\,q^{n+m}\,\widehat{\delta}^2\right\}.
\end{aligned}$$

For example, for $n=2$ and $m=1$,

$$\begin{aligned}
&\mathbb{BF}_2(\alpha;q)\,\mathbb{BF}_1(\alpha;q)+\mathbb{BF}_3(\alpha;q)\,\mathbb{BF}_2(\alpha;q)\\
&=\frac{\widehat{\gamma}^2(\alpha^3+\alpha^5)-\widehat{\gamma}\,\widehat{\delta}(\alpha^2(\alpha q)+\alpha(\alpha q)^2+\alpha^3(\alpha q)^2+\alpha^2(\alpha q)^3)+\widehat{\delta}^2((\alpha q)^3+(\alpha q)^5)}{(\alpha-\alpha q)^2}\\
&=\frac{\alpha^3}{(\alpha-\alpha q)^2}\left\{(1+\alpha^2)\,\widehat{\gamma}^2-(1+\alpha(\alpha q))\,(q^2+q)\,\widehat{\gamma}\,\widehat{\delta}+(1+(\alpha q)^2\,q^3\,\widehat{\delta}^2\right\}
\end{aligned}$$

(45): By using (18) and (34) we get,

$$\begin{aligned}
&\mathbb{BL}_n(\alpha;q)\,\mathbb{BL}_m(\alpha;q)+\mathbb{BL}_{n+1}(\alpha;q)\,\mathbb{BL}_{m+1}(\alpha;q)\\
&=(\alpha^n\,\widehat{\gamma}+(\alpha q)^n\,\widehat{\delta})\,(\alpha^m\,\widehat{\gamma}+(\alpha q)^m\,\widehat{\delta})+(\alpha^{n+1}\,\widehat{\gamma}+(\alpha q)^{n+1}\,\widehat{\delta})\,(\alpha^{m+1}\,\widehat{\gamma}+(\alpha q)^{m+1}\,\widehat{\delta})\\
&=\alpha^{n+m}\left\{(1+\alpha^2)\,\widehat{\gamma}^2+(q^n+q^m)\,(1+\alpha(\alpha q))\,\widehat{\gamma}\,\widehat{\delta}+q^{n+m}\,(1+(\alpha q)^2)\,\widehat{\delta}^2\right\}.
\end{aligned}$$

For example, for $n=2$ and $m=1$,

$$\begin{aligned}
&\mathbb{BL}_2(\alpha;q)\,\mathbb{BL}_1(\alpha;q)+\mathbb{BL}_3(\alpha;q)\,\mathbb{BL}_2(\alpha;q)\\
&=(\alpha^3+\alpha^5)\widehat{\gamma}^2+\widehat{\gamma}\,\widehat{\delta}(\alpha^2(\alpha\,q)+\alpha(\alpha\,q)^2+\alpha^3(\alpha\,q)^2+\alpha^2(\alpha\,q)^3)+((\alpha\,q)^3+(\alpha\,q)^5)\widehat{\delta}^2\\
&=\alpha^3\{\,(1+\alpha^2)\,\widehat{\gamma}^2+(q^2+q)\,(1+\alpha(\alpha\,q))\,\widehat{\gamma}\,\widehat{\delta}+q^3\,(1+(\alpha\,q)^2)\,\widehat{\delta}^2\},
\end{aligned}$$

where $\widehat{\gamma}\,\widehat{\delta}=\widehat{\delta}\,\widehat{\gamma}$.
Thus, the proof is completed. □

Theorem 6 (d'Ocagne's identity). *For $n,m\geq 0$, the d'Ocagne's identity for the q-Fibonacci bicomplex numbers $\mathbb{BF}_n(\alpha;q)$ and the q-Lucas bicomplex numbers $\mathbb{BL}_n(\alpha;q)$, respectively, are as follows:*

$$\mathbb{BF}_n(\alpha;q)\,\mathbb{BF}_{m+1}(\alpha;q)-\mathbb{BF}_{n+1}(\alpha;q)\,\mathbb{BF}_m(\alpha;q)=\tfrac{\alpha^{n+m-1}(q^m-q^n)\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)}. \tag{46}$$

$$\mathbb{BL}_n(\alpha;q)\,\mathbb{BL}_{m+1}(\alpha;q)-\mathbb{BL}_{n+1}(\alpha;q)\,\mathbb{BL}_m(\alpha;q)=\alpha^{n+m+1}\{\,(1-q)\,(q^n-q^m)\,\widehat{\gamma}\,\widehat{\delta}\}\,. \tag{47}$$

Proof. (46): By using (17) and (33) we get,

$$\begin{aligned}
&\mathbb{BF}_n(\alpha;q)\,\mathbb{BF}_{m+1}(\alpha;q)-\mathbb{BF}_{n+1}(\alpha;q)\,\mathbb{BF}_m(\alpha;q)\\
&=(\tfrac{\alpha^n\,\widehat{\gamma}-(\alpha\,q)^n\,\widehat{\delta}}{\alpha-\alpha\,q})\,(\tfrac{\alpha^{m+1}\,\widehat{\gamma}-(\alpha\,q)^{m+1}\,\widehat{\delta}}{\alpha-\alpha\,q})-(\tfrac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha\,q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha\,q})\,(\tfrac{\alpha^m\,\widehat{\gamma}-(\alpha\,q)^m\,\widehat{\delta}}{\alpha-\alpha\,q})\\
&=\tfrac{\alpha^{n+m+1}}{(\alpha-\alpha\,q)^2}\,\{(1-q)\,(q^m-q^n)\,\widehat{\gamma}\,\widehat{\delta}\,\}\\
&=\tfrac{\alpha^{n+m-1}(q^m-q^n)\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)}\,.
\end{aligned}$$

For example, for $m=2$ and $n=1$,

$$\begin{aligned}
&\mathbb{BF}_2(\alpha;q)\,\mathbb{BF}_2(\alpha;q)-\mathbb{BF}_3(\alpha;q)\,\mathbb{BF}_1(\alpha;q)\\
&=\tfrac{\widehat{\gamma}\,\widehat{\delta}\,(-\alpha^2(\alpha\,q)^2-\alpha^2(\alpha\,q)^2+\alpha^3(\alpha\,q)+\alpha(\alpha\,q)^3)}{(\alpha-\alpha\,q)^2}\\
&=\tfrac{\alpha^4}{(\alpha-\alpha\,q)^2}\,\{(1-q)\,(q-q^2)\,\widehat{\gamma}\,\widehat{\delta}\,\}\\
&=\tfrac{\alpha^2(q-q^2)\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)}.
\end{aligned}$$

(47): By using (18) and (34) we get,

$$\begin{aligned}
&\mathbb{BL}_n(\alpha;q)\,\mathbb{BL}_{m+1}(\alpha;q)-\mathbb{BL}_{n+1}(\alpha;q)\,\mathbb{BL}_m(\alpha;q)\\
&=(\alpha^n\,\widehat{\gamma}+(\alpha\,q)^n\,\widehat{\delta})\,(\alpha^{m+1}\,\widehat{\gamma}+(\alpha\,q)^{m+1}\,\widehat{\delta})\\
&\qquad+(\alpha^{n+1}\,\widehat{\gamma}+(\alpha\,q)^{n+1}\,\widehat{\delta})\,(\alpha^m\,\widehat{\gamma}+(\alpha\,q)^m\,\widehat{\delta})\\
&=\alpha^{n+m+1}\,\{\,(q^{m+1}-q^m)\widehat{\gamma}\,\widehat{\delta}+(q^n-q^{n+1})\,\widehat{\delta}\,\widehat{\gamma}\,\}\\
&=\alpha^{n+m+1}\,\{(1-q)\,(q^n-q^m)\,\widehat{\gamma}\,\widehat{\delta}\,\}\,.
\end{aligned}$$

For example, for $m = 2$ and $n = 1$,

$$\begin{aligned}&\mathbb{BL}_2(\alpha;q)\,\mathbb{BL}_2(\alpha;q) - \mathbb{BL}_3(\alpha;q)\,\mathbb{BL}_1(\alpha;q)\\ &= \alpha^4\,\{\,(q^3-q^2)\widehat{\gamma}\,\widehat{\delta} + (q-q^2)\,\widehat{\delta}\,\widehat{\gamma}\,\}\\ &= \alpha^4\,\{(1-q)\,(q^2-q)\,\widehat{\gamma}\,\widehat{\delta}\,\}\,.\end{aligned}$$

Here, $\widehat{\gamma}\,\widehat{\delta} = \widehat{\delta}\,\widehat{\gamma}$ is used.

Thus, the proof is completed. □

Theorem 7 (Cassini's identity). *For $n \geq 1$, Cassini's identity for the q-Fibonacci bicomplex numbers $\mathbb{BF}_n(\alpha;q)$ and the q-Lucas bicomplex numbers $\mathbb{BL}_n(\alpha;q)$ respectively, are as follows:*

$$\mathbb{BF}_{n+1}(\alpha;q)\,\mathbb{BF}_{n-1}(\alpha;q) - \mathbb{BF}_n^2(\alpha;q) = \frac{\alpha^{2n-2}\,q^n\,(1-q^{-1})\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)}. \tag{48}$$

$$\mathbb{BL}_{n+1}(\alpha;q)\,\mathbb{BL}_{n-1}(\alpha;q) - \mathbb{BL}_n{}^2(\alpha;q) = \alpha^{2n}\,q^{n-1}\,(1-q)^2)\,\widehat{\gamma}\,\widehat{\delta}\,. \tag{49}$$

Proof. (48): By using (17) and (33) we get

$$\begin{aligned}\mathbb{BF}_{n+1}(\alpha;q)\,\mathbb{BF}_{n-1}(\alpha;q) - \mathbb{BF}_n^2(\alpha;q) &= (\tfrac{\alpha^{n+1}\,\widehat{\gamma}-(\alpha\,q)^{n+1}\,\widehat{\delta}}{\alpha-\alpha\,q})\,(\tfrac{\alpha^{n-1}\,\widehat{\gamma}-(\alpha\,q)^{n-1}\,\widehat{\delta}}{\alpha-\alpha\,q}) - (\tfrac{\alpha^{n}\,\widehat{\gamma}-(\alpha\,q)^{n}\,\widehat{\delta}}{(\alpha-\alpha\,q)})^2\\ &= \tfrac{\alpha^{2n}\,q^n\,(1-q)(1-q^{-1})\,\widehat{\gamma}\,\widehat{\delta}}{(\alpha-\alpha\,q)^2}\\ &= \tfrac{\alpha^{2n-2}\,q^n\,(1-q^{-1})\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)}\,.\end{aligned}$$

For example, for $n = 2$,

$$\begin{aligned}\mathbb{BF}_3(\alpha;q)\,\mathbb{BF}_1(\alpha;q) - \mathbb{BF}_2^2(\alpha;q) &= \tfrac{\widehat{\gamma}\,\widehat{\delta}\,(\alpha^2(\alpha\,q)^2+\alpha^2(\alpha\,q)^2-\alpha^3(\alpha\,q)-\alpha(\alpha\,q)^3)}{(\alpha-\alpha\,q)^2}\\ &= \tfrac{\alpha^4\,q^2\,(1-q)(1-q^{-1})\,\widehat{\gamma}\,\widehat{\delta}}{(\alpha-\alpha\,q)^2}\\ &= \tfrac{\alpha^2\,q^2\,(1-q^{-1})\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)}\,.\end{aligned}$$

(49): By using (18) and (34) we get

$$\begin{aligned}\mathbb{BL}_{n+1}(\alpha;q)\,\mathbb{BL}_{n-1}(\alpha;q) - \mathbb{BL}_n{}^2(\alpha;q) &= (\alpha^{n+1}\,\widehat{\gamma} + (\alpha\,q)^{n+1}\,\widehat{\delta})\,(\alpha^{n-1}\,\widehat{\gamma} + (\alpha\,q)^{n-1}\,\widehat{\delta})\\ &\quad -(\alpha^n\,\widehat{\gamma} + (\alpha\,q)^n\,\widehat{\delta}\,)^2\\ &= \alpha^{2n}\,q^n\,(q+q^{-1}-2)\,\widehat{\gamma}\,\widehat{\delta}\\ &= \alpha^{2n}\,q^{n-1}\,(1-q)^2\,\widehat{\gamma}\,\widehat{\delta}\,.\end{aligned}$$

For example: For $n = 2$,

$$\begin{aligned}\mathbb{BL}_3(\alpha;q)\,\mathbb{BL}_1(\alpha;q) - \mathbb{BL}_2^2(\alpha;q) &= [\,\alpha^3\,(\alpha\,q) + \alpha(\alpha\,q)^3 - \alpha^2(\alpha\,q)^2 - \alpha^2(\alpha\,q)^2\,]\,\widehat{\gamma}\,\widehat{\delta}\\ &= \alpha^4\,(q+q^3-q^2-q^2)\,\widehat{\gamma}\,\widehat{\delta}\\ &= \alpha^4\,q^2\,(q+q^{-1}-2)\,\widehat{\gamma}\,\widehat{\delta}\end{aligned}$$

Here, $\widehat{\gamma}\,\widehat{\delta} = \widehat{\delta}\,\widehat{\gamma}$ is used.

Thus, the proof is completed. □

Theorem 8 (Catalan's identity). *For $n \geq r$, Catalan's identity for the q-Fibonacci bicomplex numbers $\mathbb{BF}_n(\alpha;q)$ and the q-Lucas bicomplex numbers $\mathbb{BL}_n(\alpha;q)$ respectively, are as follows:*

$$\mathbb{BF}_{n+r}(\alpha;q)\,\mathbb{BF}_{n-r}(\alpha;q)-\mathbb{BF}^2_n(\alpha;q)=\frac{\alpha^{2n-2}\,q^n\,(1-q^r)(1-q^{-r})\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)^2}\,. \tag{50}$$

$$\mathbb{BL}_{n+r}(\alpha;q)\,\mathbb{BL}_{n-r}(\alpha;q)-\mathbb{BL}_n{}^2(\alpha;q)=\alpha^{2n}\,q^{n-r}\,(1-q^r)^2\,\widehat{\gamma}\,\widehat{\delta}\,. \tag{51}$$

Proof. (50): By using (17) and (33) we get

$$\begin{aligned}
\mathbb{BF}_{n+r}(\alpha;q)\,\mathbb{BF}_{n-r}(\alpha;q)-\mathbb{BF}^2_n(\alpha;q) &= \left(\tfrac{\alpha^{n+r}\,\widehat{\gamma}-(\alpha\,q)^{n+r}\,\widehat{\delta}}{\alpha-\alpha\,q}\right)\left(\tfrac{\alpha^{n-r}\,\widehat{\gamma}-(\alpha\,q)^{n-r}\,\widehat{\delta}}{\alpha-\alpha\,q}\right)-\left(\tfrac{\alpha^{n}\,\widehat{\gamma}-(\alpha\,q)^{n}\,\widehat{\delta}}{(\alpha-\alpha\,q)}\right)^2\\
&= \tfrac{-\alpha^{2n}\,q^{n-r}\,\widehat{\gamma}\,\widehat{\delta}-\alpha^{2n}\,q^{n+r}\,\widehat{\gamma}\,\widehat{\delta}+2\,\alpha^{2n}\,q^{n}\,\widehat{\gamma}\,\widehat{\delta}}{(\alpha-\alpha\,q)^2}\\
&= -\tfrac{\alpha^{2n}\,q^{n}\,\widehat{\gamma}\,\widehat{\delta}\,[\,(1-q^{-r})+(1-q^{r})\,]}{(\alpha-\alpha\,q)^2}\\
&= \tfrac{\alpha^{2n-2}\,q^{n}\,(1-q^{-r})(1-q^{r})\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)^2}\,.
\end{aligned}$$

For example, for $n=2$,

$$\begin{aligned}
\mathbb{BF}_{2+r}(\alpha;q)\,\mathbb{BF}_{2-r}(\alpha;q)-\mathbb{BF}^2_2(\alpha;q) &= \tfrac{-\alpha^{2+r}(\alpha\,q)^{2-r}-\alpha^{2-r}(\alpha\,q)^{2+r}+2\,\alpha^{2}(\alpha\,q)^{2}\,\widehat{\gamma}\,\widehat{\delta}}{(\alpha-\alpha\,q)^2}\\
&= \tfrac{\alpha^{4}\,(-q^{2-r}-q^{2+r}+2\,q^{2})\,\widehat{\gamma}\,\widehat{\delta}}{(\alpha-\alpha\,q)^2}\\
&= \tfrac{\alpha^{2}\,q^{2}\,(1-q^{-r})(1-q^{r})\,\widehat{\gamma}\,\widehat{\delta}}{(1-q)^2}
\end{aligned}$$

(51): By using (18) and (34) we get

$$\begin{aligned}
\mathbb{BL}_{n+r}(\alpha;q)\,\mathbb{BL}_{n-r}(\alpha;q)-\mathbb{BL}_n{}^2(\alpha;q) &= (\alpha^{n+r}\,\widehat{\gamma}+(\alpha\,q)^{n+r}\,\widehat{\delta})\,(\alpha^{n-r}\,\widehat{\gamma}+(\alpha\,q)^{n-r}\,\widehat{\delta})\\
&\quad -(\alpha^{n}\,\widehat{\gamma}+(\alpha\,q)^{n}\,\widehat{\delta})^2\\
&= \alpha^{2n}\,q^{n}\,(q^{-r}+q^{r}-2)\,\widehat{\gamma}\,\widehat{\delta}\\
&= \alpha^{2n}\,q^{n-r}\,(1-q^{r})^2\,\widehat{\gamma}\,\widehat{\delta}\,.
\end{aligned}$$

For example, for $n=2$,

$$\begin{aligned}
\mathbb{BL}_{2+r}(\alpha;q)\,\mathbb{BL}_{2-r}(\alpha;q)-\mathbb{BF}^2_2(\alpha;q) &= [\alpha^{2+r}\,(\alpha\,q)^{2-r}+\alpha^{2-r}\,(\alpha\,q)^{2+r}-2\,\alpha^{2}\,(\alpha\,q)^{2}\,]\,\widehat{\gamma}\,\widehat{\delta}\\
&= \alpha^{4}\,q^{2-r}\,(1-q^{r})^2\,\widehat{\gamma}\,\widehat{\delta}\,.
\end{aligned}$$

Here, $\widehat{\gamma}\,\widehat{\delta}=\widehat{\delta}\,\widehat{\gamma}$ is used.
Thus, the proof is completed. □

3 Conclusion

In this paper, algebraic and analytic properties of the q-Fibonacci bicomplex numbers and the q-Lucas bicomplex numbers are investigated. Thanks to the q-calculus, many mathematical concepts are generalized. In this study, we generalized the bicomplex Fibonacci numbers using the q-calculus.

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