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# Some equalities and binomial sums about the generalized Fibonacci number $u_n$

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**Abstract:** In this study, we take the generalized Fibonacci sequence $\{u_n\}$ as $u_0 = 0, u_1 = 1$ and $u_n = ru_{n-1} + u_{n-2}$ for $n > 1$, where $r$ is a non-zero integer. Based on Halton's paper in [4], we derive three interrelated functions involving the terms of generalized Fibonacci sequence $\{u_n\}$. Using these three functions we introduce a simple approach to obtain a lot of identities, binomial sums and alternate binomial sums involving the terms of generalized Fibonacci sequence $\{u_n\}$.
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## 1 Introduction

For $n > 1$, the second order linear recurrence sequence $\{w_n(a, b; r, s)\}$ is defined by

$$w_n = rw_{n-1} - sw_{n-2},$$

where $w_0 = a, w_1 = b$. This sequence was introduced by Horadam in [5, 6] and it generalizes many sequences (see [11]). Fibonacci number sequence $\{F_n\} = \{w_n(0, 1; 1, -1)\}$, Lucas number sequence $\{L_n\} = \{w_n(2,1;1,-1)\}$, Pell number sequence $\{P_n\} = \{w_n(0,1;2,-1)\}$ and Pell–Lucas number sequence $\{Q_n\} = \{w_n(2,2;2,-1)\}$ are well-known examples of the sequence $\{w_n\}$. In this study, we take the generalized Fibonacci sequence $\{u_n\} = \{w_n(0,1;r,-1)\}$, where $r$ is a non-zero integer. Thus

$$u_n = ru_{n-1} + u_{n-2} \tag{1}$$

where $u_0 = 0, u_1 = 1$. It is cleared that $u_n = F_n$ (n-th Fibonacci number) and $u_n = P_n$ ($n$-th Pell number) for $r = 1$ and $r = 2$, respectively. Let $\alpha$ be a positive root of the quadratic equation $x^2 - rx - 1 = 0$ and $\beta$ its negative root. Then we have the Binet's formula

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

From the Binet's formula, one can see that

$$u_{-n} = (-1)^{n+1} u_n \tag{2}$$

for $n \geq 0$.

Numerous authors appear to have been fascinated by the many interesting summation identities involving the Fibonacci and generalized Fibonacci numbers. There are many types of identities involving sums of products of binomial coefficients and Fibonacci or Lucas numbers (see [1, 4, 10–12]). Many authors have been concerned with the generalized Fibonacci sequence $\{u_n\}$. They have been searched for the binomial sums, alternate binomial sums, weighted binomial sums of the terms of this sequence and the binomial sums of products of these terms using by matrix methods, generating function methods or in different ways (see [2, 3, 5–9]).

In this study, based on Halton's paper in [4], we derive three interrelated functions involving the terms of generalized Fibonacci sequence $\{u_n\}$ and introduce a simple approach to obtain a lot of identities, binomial sums and alternate binomial sums involving the terms of generalized Fibonacci sequence $\{u_n\}$ using these functions. Numerous new identities, binomial sums and alternate binomial sums as well as those found in the existing literature are included a single identity.

# 2 On three interrelated functions involving the generalized Fibonacci numbers $u_n$

In this section, we will define three interrelated functions involving the generalized Fibonacci numbers $u_n$ and give some properties of these functions.

Firstly, we define a function

$$S_1(m,n) = u_m u_n - u_{m+1}u_{n-1} - (-1)^{n-1}u_{m-n+1}. \tag{3}$$

For all integers $m$ and $n$, we get

$$S_1(m+1,n) = rS_1(m,n) + S_1(m-1,n) \tag{4}$$

using equalities (1) and (3). It is also clear from equalities (1), (2) and (3) that we can write:

$$S_1(0,n)=0, \tag{5}$$
$$S_1(1,n)=0. \tag{6}$$

If the equalities (5) and (6) are taken into account in equality (4), it is easily obtained that

$$S_1(m,n)=0 \tag{7}$$

for all integers $m$ and $n$.

Now we consider a function

$$S_2(t,m,n)=u_m u_n - u_{m+t}u_{n-t} - (-1)^{n-t}u_t u_{m-n+t}. \tag{8}$$

Then by equalities (1) and (8), for all integers $t$, $m$ and $n$

$$S_2(t+1,m,n)=S_2(t-1,m,n-2)+rS_2(t,m,n-1). \tag{9}$$

Since $u_0=0$ and $u_1=1$, we can easily obtain the following equations from the equalities (1), (3), (7) and (8):

$$S_2(0,m,n)=0,$$
$$S_2(1,m,n)=S_1(m,n)=0.$$

Thus, from the equality (9), we obtain that

$$S_2(t,m,n)=0 \tag{10}$$

for all integers $t$, $m$ and $n$.

Finally, we define a function

$$S_3(k,t,m,n)=u_m^k u_n - \sum_{i=0}^{k}\binom{k}{i}(-1)^{(m+1)i}u_{m+t}^{k-i}u_t^i u_{n-kt-mi}. \tag{11}$$

**Theorem 2.1.** *For all integers $t$, $m$, $n$ and $k\geq 0$, we have*

$$S_3(k+1,t,m,n)=u_m S_3(k,t,m,n). \tag{12}$$

*Proof.* From the definition of the function $S_3(k,t,m,n)$, we can write

$$S_3(k+1,t,m,n)=u_m^{k+1}u_n - \sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^{(m+1)i}u_{m+t}^{k+1-i}u_t^i u_{n-(k+1)t-mi}.$$

Since

$$\binom{k+1}{i}=\binom{k}{i}+\binom{k}{i-1}$$

and for $i<0$ or $0\leq k<i$

$$\binom{k}{i}=0,$$

we obtain that

$$\begin{aligned}S_3(k+1,t,m,n) &= u_m^{k+1}u_n - \sum_{i=0}^{k}\binom{k}{i}(-1)^{(m+1)i}u_{m+t}^{k+1-i}u_t^i u_{n-(k+1)t-mi}\\ &\quad - \sum_{i=0}^{k}\binom{k}{i}(-1)^{(m+1)(i+1)}u_{m+t}^{k-i}u_t^{i+1}u_{n-(k+1)t-m(i+1)}\\ &= u_m^{k+1}u_n - \sum_{i=0}^{k}\binom{k}{i}(-1)^{(m+1)i}u_{m+t}^{k-i}u_t^i(u_{m+t}u_{n-(k+1)t-mi}\\ &\quad + (-1)^{m+1}u_t u_{n-(k+1)t-m(i+1)}).\end{aligned}$$

Since the equality (10) is true for all integers $m$, $n$ and $t$, taking $n-tk-mi$ instead of $n$ in equality (8), we can write

$$\begin{aligned}u_m u_{n-tk-mi} &= u_{m+t}u_{n-t(k+1)-mi} + (-1)^{n-t(k+1)-mi}u_t u_{m(i+1)-n+t(k+1)}\\ &= u_{m+t}u_{n-t(k+1)-mi} + (-1)^{m+1}u_t u_{n-m(i+1)-t(k+1)}\end{aligned}$$

using by equality (2). Thus, we have

$$\begin{aligned}S_3(k+1,t,m,n) &= u_m^{k+1}u_n - u_m\sum_{i=0}^{k}\binom{k}{i}(-1)^{(m+1)i}u_{m+t}^{k-i}u_t^i u_{n-tk-mi}\\ &= u_m S_3(k,t,m,n)\end{aligned}$$

by equality (11). □

**Theorem 2.2.** *For all integers $t$, $m$, $n$ and $k \geq 0$, we have*

$$S_3(k,t,m,n) = 0.$$

*Proof.* Taking $k=0$ in equality (11), we get

$$S_3(0,t,m,n) = u_n - u_n = 0.$$

Similarly, if we take $k=1$ in equality (11), we get

$$\begin{aligned}S_3(1,t,m,n) &= u_m u_n - u_{m+t}u_{n-t} - (-1)^{m+1}u_t u_{n-t-m}\\ &= u_m u_n - u_{m+t}u_{n-t} - (-1)^{n-t}u_t u_{m-n+t}\\ &= S_2(t,m,n)\end{aligned}$$

by equalities (2) and (8). From the equalities (10), we get $S_3(1,t,m,n) = 0$. Thus, the desired result is obtained from the Theorem 2.1. □

# 3 Some equalities and binomial sums about the generalized Fibonacci numbers $u_n$

In this section, we will give some equalities, binomial sums and alternate binomial sums about the generalized Fibonacci number $u_n$ that can be derived from some special cases of the equality (11). Since sums and identities involving Fibonacci or generalized Fibonacci numbers are in a closed form, it is very interesting to investigate these types of identities and sums.

Now let us take $-m$ and $-n$ instead of $m$ and $n$ respectively in equality (11), we get

$$u_{-m}^k u_{-n} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(-m+1)i} u_{-m+t}^{k-i} u_t^i u_{-n-kt+mi}$$

by Theorem 2.2. Since $u_{-n} = (-1)^{n+1} u_n$ for $n \geq 0$, we have

$$u_m^k u_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(m-t)i} u_{m-t}^{k-i} u_t^i u_{n+kt-mi}.$$

This final sum can also be obtained by substituting $-t$ for $t$ in equality (11).

On the left of each equalities below, we write the function $S_3(k, t, m, n)$ as $(k, t, m, n)$ briefly and use the equality (2) to remove the negative subscripts in some cases.

$$(k, t, -m, -kt-n) : u_m^k u_{kt+n} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(m-t)i} u_{m-t}^{k-i} u_t^i u_{n+2kt-mi}$$

$$(k, t, m, -kt) : u_m^k u_{kt} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt+i} u_{m+t}^{k-i} u_t^i u_{2kt+mi}$$

$$(k, t, m, m) : u_m^{k+1} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{m-i+1+kt} u_{m+t}^{k-i} u_t^i u_{m(i-1)+kt}$$

$$(k, t, m, nt) : u_m^k u_{nt} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(m+1)i} u_{m+t}^{k-i} u_t^i u_{(n-k)t-mi}$$

$$(k, t, mt, n) : u_{mt}^k u_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(mt+1)i} u_{(m+1)t}^{k-i} u_t^i u_{n-(mi+k)t}$$

$$(k, t, m, 0) : 0 = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt+i+1} u_{m+t}^{k-i} u_t^i u_{mi+kt}$$

$$(k, t, m, 1) : u_m^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt+i} u_{m+t}^{k-i} u_t^i u_{mi+kt-1}$$

$$(k, t, mt, 0) : 0 = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt+i+1} u_{(m+1)t}^{k-i} u_t^i u_{(mi+k)t}$$

$$(k, t, 1, n) : u_n = \sum_{i=0}^{k} \binom{k}{i} u_{t+1}^{k-i} u_t^i u_{n-kt-i}$$

$$(k, t, 1, -n-kt) : u_{n+kt} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt-i} u_{t+1}^{k-i} u_t^i u_{n+2kt+i}$$

$$(k, t, 1, -kt) : u_{kt} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt-i} u_{t+1}^{k-i} u_t^i u_{2kt+i}$$

$$(k, t, 1, nt) : u_{nt} = \sum_{i=0}^{k} \binom{k}{i} u_{t+1}^{k-i} u_t^i u_{(n-k)t-i}$$

$$(k,t,2,n) : r^k u_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^i u_{t+2}^{k-i} u_t^i u_{n-kt-2i}$$

$$(k,t,2,-kt) : r^k u_{kt} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt-i} u_{t+2}^{k-i} u_t^i u_{2(kt+i)}$$

$$(k,1,m,n) : u_m^k u_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(m+1)i} u_{m+1}^{k-i} u_{n-k-mi}$$

$$(k,1,m,k) : u_m^k u_k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} u_{m+1}^{k-i} u_{mi}$$

$$(k,2,m,n) : u_m^k u_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^{(m+1)i} r^i u_{m+2}^{k-i} u_{n-2k-mi}$$

$$(k,t,1,0) : 0 = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt+i+1} u_{t+1}^{k-i} u_t^i u_{kt+i}$$

$$(k,t,2,0) : 0 = \sum_{i=0}^{k} \binom{k}{i} (-1)^{kt+i+1} u_{t+2}^{k-i} u_t^i u_{kt+2i}$$

$$(k,1,m,0) : 0 = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k+i+1} u_{m+1}^{k-i} u_{k+mi}$$

$$(k,1,m,1) : u_m^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k+i} u_{m+1}^{k-i} u_{k+mi-1}$$

$$(k,1,1,-n) : u_n = \sum_{i=0}^{k} \binom{k}{i} (-r)^{k-i} u_{n+k+i}$$

$$(k,1,1,-kn) : u_{kn} = \sum_{i=0}^{k} \binom{k}{i} (-r)^{k-i} u_{k(n+1)+i}$$

$$(k,1,1,-k) : u_k = \sum_{i=0}^{k} \binom{k}{i} (-r)^{k-i} u_{2k+i}$$

$$(k,2,-1,-n) : u_n = \sum_{i=0}^{k} \binom{k}{i} (-r)^i u_{n+2k-i}$$

$$(k,-1,2,-n) : r^k u_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} u_{n-k+2i}$$

$$(k,2,-1,-2k) : u_{2k} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i r^i u_{4k-i}$$

$$(k,-1,2,k) : r^k u_k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i u_{2(k-i)}$$

$$(k,1,1,0) : 0 = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k+i+1} r^{k-i} u_{k+i}$$

$$
\begin{aligned}
(k,2,-1,0) &: 0=\sum_{i=0}^{k}\binom{k}{i}(-1)^{i+1}r^{i}u_{2k-i}\\
(k,-1,2,0) &: 0=\sum_{i=0}^{k}\binom{k}{i}(-1)^{i}u_{k-2i}\\
(1,t,m,n) &: u_mu_n-u_{m+t}u_{n-t}=(-1)^{m+1}u_tu_{n-m-t}\\
(1,t,m,-n) &: u_mu_n=(-1)^{t}(u_{m+t}u_{n+t}-u_tu_{n+m+t})\\
(1,t,m,-m) &: u_m^2=(-1)^{t}(u_{m+t}^2-u_tu_{2m+t})\\
(1,t,m,n-t) &: u_mu_{n-t}=u_{m+t}u_{n-2t}+(-1)^{m+1}u_tu_{n-m-2t} \qquad (13)\\
(1,t-s,m+s,-m+s) &: u_{m+s}u_{m-s}=(-1)^{s-m-1}u_{m+t}u_{n-t}+u_{s-t}u_{s+t}\\
(1,1,m,n) &: u_{m+1}u_{n-1}-u_mu_n=(-1)^{m}u_{n-m-1}\\
(1,1,m,n-1) &: u_{m+1}u_{n-2}-u_mu_{n-1}=(-1)^{m}u_{n-m-2}\\
(1,2,m-1,n-1) &: u_{m-1}u_{n-1}=u_{m+1}u_{n-3}+(-1)^{m}ru_{n-m-2}\\
(1,1,m,m) &: u_m^2=u_{m+1}u_{m-1}+(-1)^{m+1}\\
(1,1,m,m-1) &: u_{m+1}u_{m-2}-u_mu_{m-1}=(-1)^{m+1}r\\
(1,2,m-1,m-1) &: u_{m+1}u_{m-3}-u_{m-1}^2=(-1)^{m}r^2\\
(1,1,m,-m) &: u_m^2+u_{m+1}^2=u_{2m+1}\\
(1,2,m,-m) &: u_{m+2}^2-u_m^2=ru_{2m+2}\\
(1,1,m+1,-m+1) &: u_{m+1}u_{m-1}+u_{m+2}u_m=u_{2m+1}\\
(1,1,-m,m+1) &: u_{2m}=u_m(u_{m+1}+u_{m-1}) \qquad (14)
\end{aligned}
$$

Now we will give an example how different sums can also be obtained by using the above identities. It is clear that many equalities can be obtained in this way.

If we substitute $t-m$ instead of $n-t$ in the equality (13) and use the equality (2), we can easily obtain the identity

$$
\begin{aligned}
u_mu_{t-m}&=u_{m+t}u_{-m}+(-1)^{m+1}u_tu_{-2m}\\
&=(-1)^{m+1}(u_{m+t}u_m-u_tu_{2m}).
\end{aligned}
$$

Dividing the obtained equality by $u_m$ and using the equality (2), we have

$$
u_t(u_{m+1}+u_{m-1})=u_{m+t}+(-1)^{m}u_{t-m}. \qquad (15)
$$

Let us add $-u_t-(-1)^m u_t$ to both sides of equality (15), substitute $tm+n$ for $t$ in the resulting identity, and sum from $t=1$ to $t=k$, then we have the following sum:

$$
\sum_{t=1}^{k}u_{tm+n}=\frac{u_{(k+1)m+n}-u_{m+n}-(-1)^{m}(u_{km+n}-u_n)}{u_{m+1}+u_{m-1}-1-(-1)^{m}}. \qquad (16)
$$

It can also be seen that this sum can be formulated by Binet's formula or recurrence relation (1) (see [13] and [14]). Taking $m=1$ in equality (16), we have

$$
\sum_{t=1}^{k}u_{t+n}=\frac{1}{r}(u_{k+n+1}+u_{k+n}-u_{n+1}-u_n).
$$

Now if we take $m = 2$, $n = 2s$ and $m = 2$, $n = 2s - 1$ in equality (16), we obtain

$$\sum_{t=1}^{k} u_{2(t+s)} = \frac{1}{r}(u_{2(k+s)+1} - u_{2s+1})$$

and

$$\sum_{t=1}^{k} u_{2(t+s)-1} = \frac{1}{r}(u_{2(k+s)} - u_{2s}),$$

respectively, using the recurrence relation (1).

Similarly, taking $m = 3$ and $n = 3s$ in equality (16), we get

$$\sum_{t=1}^{k} u_{3(t+s)} = \frac{1}{r(r^2+3)}(u_{3(k+s+1)} + u_{3(k+s)} - u_{3(s+1)} - u_{3s}).$$


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