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On the equation $f(n^2 - Dnm + m^2) = f^2(n) - Df(n)f(m) + f^2(m)$

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Abstract: We give all solutions $f : \mathbb{N} \to \mathbb{C}$ of the functional equation

$$f(n^2 - Dnm + m^2) = f^2(n) - Df(n)f(m) + f^2(m),$$

where $D \in \{1, 2\}$.
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1 Introduction

Let, as usual, $\mathcal{P}, \mathbb{N}, \mathbb{C}$ be the sets of primes, positive integers and complex numbers, respectively. A function $f : \mathbb{N} \to \mathbb{C}$ is multiplicative if

$$f(nm) = f(n)f(m) \text{ for every } n, m \in \mathbb{N}, (n, m) = 1.$$

Let $\mathcal{M}$ be the set of complex-valued multiplicative functions.

A characterization of the identity function was studied by C. Spiro [7], J.-M. De Koninck, I. Kátai and B. M. Phong [1], B. M. Phong [4, 5] and by others.

In 1992, C. Spiro [7] proved that if $f \in \mathcal{M}$ satisfies

$$f(p+q) = f(p) + f(q) \quad (\forall p, q \in \mathcal{P}) \text{ and } f(p_0) \neq 0 \quad \text{for some} \quad p_0 \in \mathcal{P},$$

then $f(n)$ is the identity function.

In 1997, J.-M. De Koninck, I. Kátai and B. M. Phong [1] proved that if a function $f \in \mathcal{M}$ satisfies the condition

$$f(p+m^2) = f(p) + f(m^2) \text{ for every } p \in \mathcal{P}, m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$.

Recently Poo-Sung Park [3] proved the following results:

Theorem A. *If a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfies*

$$f(n^2+nm+m^2) = f^2(n) + f(n)f(m) + f^2(m) \text{ for every } n, m \in \mathbb{N},$$

then f is the identity function.

Theorem B. *A multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfies*

$$f(n^2-nm+m^2) = f^2(n) - f(n)f(m) + f^2(m)$$

if and only if f is one of the following:

1. *the identity function $f(n) = n$;*
2. *the constant function $f(n) = 1$;*
3. *function f_p defined by:*

$$f_p(n) = \begin{cases} 0, & \text{if } p \mid n \\ 1, & \text{if } p \nmid n \end{cases}$$

for some prime $p \equiv 2 \pmod 3$.

For some generalizations of Theorem A we refer the works of B. M. M. Khanh [2], B. M. Phong and R. B. Szeidl [6]. They prove that if $D \in \{1, 2, 3\}$ and an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the conditions $f(1) = 1$ and

$$f(n^2+Dnm+m^2) = f^2(n) + Df(n)f(m) + f^2(m) \text{ for every } n, m \in \mathbb{N},$$

then f is the identity function.

In this note, we improve Theorem B as follows:

Theorem 1. *An arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfies*

$$f(n^2-nm+m^2) = f^2(n) - f(n)f(m) + f^2(m) \text{ for every } n, m \in \mathbb{N}$$

if and only if f is one of the following:

(a) $f(n) = 0$ *for every* $n \in \mathbb{N}$,

(b) $f(1)=0$ *and* $f(n)=1$ *for every* $n\in\mathbb{N}, n\geq 2$,

(c) $f(n)=\Theta_M(n)=\begin{cases} 0, & \text{if} \quad M \mid n \\ 1, & \text{if} \quad M \nmid n, \end{cases}$ *for every* $n\in\mathbb{N}$, *where*

  (c1) *either* $M=2$,

  (c2) *or* $M=q_1\cdots q_s\geq 5$ *is a square-free number,* $q_i\equiv 2 \pmod 3$ $(i=1,\ldots,s)$,

(d) $f(n)=1$ *for every* $n\in\mathbb{N}$,

(e) $f(n)=n$ *for every* $n\in\mathbb{N}$.

Theorem 2. *An arithmetical function* $f:\mathbb{N}\to\mathbb{C}$ *satisfies*

$$f(n^2-2nm+m^2)=f^2(n)-2f(n)f(m)+f^2(m) \text{ for every } n,m\in\mathbb{N}$$

if and only if f *is one of the following:*

(A) $f(n)=0$ *for every* $n\in\mathbb{N}$,

(B) $f(n)=\chi_2(n)$ *for every* $n\in\mathbb{N}$,

(C) $f(n)=n$ *for every* $n\in\mathbb{N}$,

where $\chi_2(n)$ *is a Dirichlet character* $\pmod 2$.

We infer from Theorem 1 and Theorem 2 the following results.

Corollary 1. ***(Poo-Sung Park, Theorem A).*** *A function* $f\in\mathcal{M}$ *satisfies*

$$f(n^2-nm+m^2)=f^2(n)-f(n)f(m)+f^2(m) \text{ for every } n,m\in\mathbb{N}$$

if and only if $f\in\{\mathbb{U},\chi_q,I\}$, *where* $\mathbb{U}(n)=1,\ I(n)=n$ *for every* $n\in\mathbb{N}$ *and* χ_q *is the Dirichlet principal character* $\pmod q$, $q\in\mathcal{P}, q\equiv 2 \pmod 3$.

Corollary 2. *A multiplicative function* $f:\mathbb{N}\to\mathbb{C}$ *satisfies*

$$f(n^2-2nm+m^2)=f^2(n)-2f(n)f(m)+f^2(m) \text{ for every } n,m\in\mathbb{N}$$

if and only if $f\in\{\chi_2,I\}$, *where* $I(n)=n$ *for every* $n\in\mathbb{N}$ *and* χ_2 *is the Dirichlet character* $\pmod 2$.

2 Lemmas

Assume that $f:\mathbb{N}\to\mathbb{C}$ satisfies

$$f(n^2-nm+m^2)=f^2(n)-f(n)f(m)+f^2(m) \text{ for every } n,m\in\mathbb{N}. \tag{1}$$

First we prove the following lemma.

Lemma 2.1. *Assume that* $f:\mathbb{N}\to\mathbb{C}$ *satisfies (1). Then*

$$f(k^2)=f^2(k) \tag{2}$$

and

$$\Big(f(n)-f(m)\Big)\Big(f(n+m)-f(n)-f(m)\Big)=0. \tag{3}$$

hold for every $k,n,m\in\mathbb{N}$.

Proof. The relation (2) is obvious, because by taking $n = m = k$ into (1), we have

$$f(k^2) = f(k^2 - k \cdot k + k^2) = f^2(k) - f(k) \cdot f(k) + f^2(k) = f^2(k).$$

In order to prove (3), we start with the relation

$$(n+m)^2 - (n+m)m + m^2 = (n+m)^2 - (n+m)n + n^2,$$

consequently it follows from (1) that

$$f^2(n+m) - f(n+m)f(m) + f^2(m) = f^2(n+m) - f(n+m)f(n) + f^2(n),$$

and so

$$\Big(f(n) - f(m)\Big)\Big(f(n+m) - f(n) - f(m)\Big) = 0$$

holds for every $n, m \in \mathbb{N}$. Thus, (3) and Lemma 2.1 are true. □

Lemma 2.2. *Assume that $f : \mathbb{N} \to \mathbb{C}$ satisfies (1). Then*

$$\Big(f(1), f(2)\Big) \in \Big\{(0,0), (0,1), (1,0), (1,1), (1,2)\Big\}. \tag{4}$$

Proof. It is obvious from (2) that

$$f(1) = f^2(1), \text{ i.e., } f(1) \in \{0, 1\}.$$

First, we infer from (1) and from $3 = 2^2 - 2 \cdot 1 + 1^2$, $3^2 - 3 \cdot 2 + 2^2 = 3^2 - 3 \cdot 1 + 1^2$ that

$$f(3) = f^2(2) - f(2) \cdot f(1) + f^2(1)$$

and

$$\begin{aligned} 0 &= f^2(3) - f(3) \cdot f(2) + f^2(2) - (f^2(3) - f(3) \cdot f(1) + f^2(1)) \\ &= -f(3)\Big(f(2) - f(1)\Big) + \Big(f(2) - f(1)\Big)\Big(f(2) + f(1)\Big) \\ &= -\Big(f(2) - f(1)\Big)\Big(f(3) - f(2) - f(1)\Big) \\ &= -\Big(f(2) - f(1)\Big)\Big(f^2(2) - f(2)f(1) + f^2(1) - f(2) - f(1)\Big). \end{aligned} \tag{5}$$

By using (5), it is obvious that

$$\text{if } f(1) = 0, \text{ then } f(2)\Big(f^2(2) - f(2)\Big) = f^2(2)\Big(f(2) - 1\Big) = 0$$

and

$$\text{if } f(1) = 1, \text{ then } \Big(f(2) - 1\Big)\Big(f^2(2) - 2f(2)\Big) = f(2)\Big(f(2) - 1\Big)\Big(f(2) - 2\Big) = 0$$

The last two relations prove (4).

Lemma 2.2 is proved. □

3 Proof of Theorem 1

It is clear to check that the functions defined in (a), (b), (d) and (e) satisfy the functional equation (1). Let us consider the case (c). If $M = 2$, then $f(n) = \Theta_2(n) = \chi_2(n)$ is the Dirichlet character $(\bmod\ 2)$ and it is trivial that

$$\chi_2(n^2 - nm + m^2) = \chi_2^2(n) - \chi_2(n)\chi_2(m) + \chi_2^2(m) \text{ for every } n, m \in \mathbb{N}.$$

Now we consider the case $(c2)$. Let $M = q_1 \cdots q_s \geq 5$ be a square-free number and $q_i \equiv 2 \pmod 3$ for every $i \in \{1, \dots, s\}$. Let $f(n) = \Theta_M(n)$ be a function defined in (c). Then we infer from the facts $q_1 \equiv \cdots \equiv q_s \equiv 2 \pmod 3$ that:

$$\begin{aligned}
\Theta_M(n^2 - nm + m^2) = 0 &\iff n^2 - nm + m^2 \equiv 0 \pmod M \\
&\iff n^2 - nm + m^2 \equiv 0 \pmod{q_i} \text{ for every } i \in \{1, \dots, s\} \\
&\iff q_i | n \text{ and } q_i | m \text{ for every } i \in \{1, \dots, s\} \\
&\iff M | n \text{ and } M | m \\
&\iff \Theta_M^2(n) - \Theta_M(n)\Theta_M(m) + \Theta_M^2(m) = 0,
\end{aligned}$$

consequently

$$\Theta_M(n^2 - nm + m^2) = \Theta_M^2(n) - \Theta_M(n)\Theta_M(m) + \Theta_M^2(m)$$

for every $n, m \in \mathbb{N}$.

In the above proof we have used that $\Theta_M^2(n) - \Theta_M(n)\Theta_M(m) + \Theta_M^2(m) \in \{0, 1\}$ for every $n, m \in \mathbb{N}$.

Now let us prove the "only if" part.

As we seen in the Lemma 2.2 there are five possibilities according to

$$\Big(f(1), f(2)\Big) \in \Big\{(0,0), (0,1), (1,0), (1,1), (1,2)\Big\}.$$

(a) Assume that $f(1) = 0$ and $f(2) = 0$. We will prove that

$$f(n) = 0 \text{ for every } n \in \mathbb{N}. \tag{6}$$

It follows from (1) and (2) that

$$f(3) = f^2(2) - f(2)f(1) + f^2(1) = 0 \text{ and } f(4) = f^2(2) = 0.$$

If we assume that $f(1) = \cdots = f(N-1) = 0$ and $f(N) \neq 0$, then $N \geq 5$. Now we apply (3) to get

$$(f(1) - f(m))(f(m+1) - f(m) - f(1)) = -f(m)(f(m+1) - f(m)) = 0,$$

and so

$$f(m+1) = f(m) \quad \text{if} \quad f(m) \neq 0.$$

This with the fact $f(N) \neq 0$ implies that

$$f(n) = f(N) \text{ for every } n \in \mathbb{N}, n \geq N.$$

Since $N \geq 5$, we have

$$(N-1)^2 - (N-1) \cdot 1 + 1^2 = N^2 - 3N + 3 \geq 5N - 3N + 3 = 2N + 3 > N,$$

which implies

$$f\Big((N-1)^2 - (N-1) \cdot 1 + 1^2\Big) = f(N) \neq 0.$$

This is impossible, because by (1) we obtain that

$$f\Big((N-1)^2 - (N-1) \cdot 1 + 1^2\Big) = f^2(N-1) - f(N-1) \cdot f(1) + f^2(1) = 0.$$

Thus, we have proved that if $f(1) = f(2) = 0$ then (6) is true.

The proof of (a) is finished.

(b) Assume that $f(1)=0$ and $f(2)=1$. We will prove that $f(n)=1$ for every $n\geq 2$.

Indeed, we infer from (3) that

$$\Big(f(n)-f(1)\Big)\Big(f(n+1)-f(n)-f(1)\Big)=f(n)\Big(f(n+1)-f(n)\Big)=0,$$

consequently

$$f(n+1)=f(n)\quad\text{if}\quad f(n)\neq 0.$$

Since $f(2)=1$, the last relation shows that $f(3)=1, f(4)=1,\dots,$ and $f(n)=1$ for every $n\geq 2$.
The proof of (b) is finished.

$(c1)$ Assume that $f(1)=1$ and $f(2)=0$. By applying (3), we have

$$\Big(f(n)-f(2)\Big)\Big(f(n+2)-f(n)-f(2)\Big)=f(n)\Big(f(n+2)-f(n)\Big)=0,$$

consequently

$$f(n+2)=f(n)\quad\text{if}\quad f(n)\neq 0.$$

Thus, from the fact $f(1)=1$, we have

$$f(2k+1)=1\ \text{ for every }\ k\in\mathbb{N}. \tag{7}$$

Now we prove that

$$f^2(2k)=f(2k)\ \text{ for every }\ k\in\mathbb{N} \tag{8}$$

and

$$f(2k)\Big(f(2k+2)-1\Big)=0\ \text{ for every }\ k\in\mathbb{N}. \tag{9}$$

It follows from (3) and (7) that

$$\Big(f(2k)-1\Big)\Big(-f(2k)\Big)=\Big(f(2k)-f(1)\Big)\Big(f(2k+1)-f(2k)-f(1)\Big)=0,$$

which proves (8).

On the other hand, it follows from (3), (8) that

$$\begin{aligned} f(2k)f(2k+2)-f(2k) &= f(2k)f(2k+2)-f^2(2k)\\ &= f(2k)\big(f(2k+2)-f(2k)\big)\\ &= \big(f(2k)-f(2)\big)\big(f(2k+2)-f(2k)-f(2)\big)=0, \end{aligned}$$

which proves (9).
Finally, we will prove that

$$f(2k)=0\ \text{ for every }\ k\in\mathbb{N}. \tag{10}$$

Assume that there is a $M\in\mathbb{N}$ such that $f(2n)=0$ for every $n<M$, and $f(2M)\neq 0$. Then (8) implies that $f(2M)=1$, consequently it follows from (9) that

$$f(2n)=1\ \text{ for every }\ n\geq M. \tag{11}$$

Since $f(2)=0$ and $f(4)=f^2(2)=0$, we have $M\geq 3$.

Applying (3) for $n = 2M - 2$ and $m = 2$, we have $f(n) = f(2M-2) = 0$, $f(m) = f(2) = 0$ and so

$$f\Big((2M-2)^2 - (2M-2)\cdot 2 + 2^2\Big) = f^2(2M-2) - f(2M-2)\cdot f(2) + f^2(2) = 0.$$

But it follows from $M \geq 3$ that

$$(2M-2)^2 - (2M-2)\cdot 2 + 2^2 = 4M^2 - 12M + 12 > 2M,$$

which with (11) implies $f\Big((2M-2)^2 - (2M-2)\cdot 2 + 2^2\Big) = 1$. This is impossible.
Thus we have proved (10), which with (7) implies that $f(n) = \chi_2(n)$.
The proof of $(c1)$ for $M = 2$ is finished.

$(c2)$ Now assume that $f(1) = 1$, $f(2) = 1$ and $f(n) \neq 1$ for some $n \in \mathbb{N}$.
In this case we have

$$f(3) = f^2(2) - f(2)f(1) + f^2(1) = 1 \text{ and } f(4) = f^2(2) = 1.$$

It follows from our assumption that there is some number $M \in \mathbb{N}$, $M \geq 5$ such that

$$f(M) \neq 1 \text{ and } f(n) = 1 \text{ for every } n \in \{1, \dots, M-1\}. \tag{12}$$

First we prove that

$$f(M) = 0. \tag{13}$$

We infer from (3) that

$$\Big(f(M) - f(n)\Big)\Big(f(M+n) - f(M) - f(n)\Big) = \Big(f(M) - 1\Big)\Big(f(M+n) - f(M) - 1\Big) = 0$$

holds for every $n \in \{1, \dots, M-1\}$, consequently

$$f(M+n) = f(M) + 1 \text{ for every } n \in \{1, \dots, M-1\}.$$

This with $n = 1$ and $n = 2$ implies that

$$f(M+1) = f(M) + 1 \text{ and } f(M+2) = f(M) + 1.$$

On the other hand, we infer from (3) that

$$\begin{aligned} -f(M) &= \Big(f(M) + 1 - 1\Big)\Big(f(M) + 1 - (f(M)+1) - 1\Big) \\ &= \Big(f(M+1) - f(1)\Big)\Big(f(M+2) - f(M+1) - f(1)\Big) \\ &= 0. \end{aligned}$$

Thus, the proof of (13) is finished.
Let

$$I := \{n \in \mathbb{N} \mid f(n) = 1\}.$$

It follows from (12) that

$$\{1, \dots, M-1\} \subseteq I,$$

and so we infer from (3), (12) and (13) that if $n \in I$, then

$$f(n+M) - 1 = f(n+M) - f(n) - f(M) = \Big(f(n) - f(M)\Big)\Big(f(n+M) - f(n) - f(M)\Big) = 0.$$

Therefore, we have proved that

$$f(n+M)=1 \ \text{ for every } \ n\in I, \tag{14}$$

which, using the fact $\{1,\dots,M-1\}\subseteq I$ shows that

$$\{1,\dots,M-1,M+1,\dots,2M-1\}\subseteq I.$$

In the same way, we infer from (14) that

$$\mathbb{N}\setminus\{M,2M,\dots\}\subseteq I. \tag{15}$$

Now we will prove that

$$f(Mt)=0 \ \text{ for every } \ t\in\mathbb{N}. \tag{16}$$

Assume that (16) does not hold. Then there is a $t_0\in\mathbb{N}, t_0\geq 2$ such that

$$f(M)=\cdots=f((t_0-1)M)=0 \ \text{ and } \ f(t_0M)\neq 0.$$

By applying (3) and (15) with $n=1, m=t_0M$, we have

$$\Big(1-f(t_0M)\Big)\Big(-f(t_0M)\Big)=\Big(f(1)-f(t_0M)\Big)\Big(f(t_0M+1)-f(1)-f(t_0M)\Big)=0,$$

consequently

$$f(t_0M)\in\{0,1\}.$$

If $f(t_0M)\neq 0$, then $f(t_0M)=1, t_0M\in I$. This fact with (14) implies that

$$f(tM)=1 \ \text{ for every } \ t\in\mathbb{N}, t\geq t_0. \tag{17}$$

Now we apply (1) for $n=m=(t_0-1)M$, we obtain that

$$f\Big(\big((t_0-1)M\big)^2\Big)=f^2\Big((t_0-1)M\Big)=0.$$

This contradicts to (17), because

$$(t_0-1)^2M=5(t_0-1)^2\geq t_0,$$

which with (17) implies

$$f\Big(\big((t_0-1)M\big)^2\Big)=1.$$

Thus, the proof of (16) is finished, which proves that

$$f(n)=\Theta_M(n)=\begin{cases}0, & \text{if } \ M\mid n\\ 1, & \text{if } \ M\nmid n.\end{cases} \tag{18}$$

Now we prove that M is a square-free number and every prime divisor of M is congruent to $2 \pmod 3$. Let $M=P^2Q$, where $Q=q_1\cdots q_s$ is a square-free number. Then

$$M|(PQ)^2-PQM+M^2=M(Q-PQ+M),$$

and so we obtain from (18) that

$$0=f\Big((PQ)^2-PQM+M^2\Big)=f^2(PQ)-f(PQ)f(M)+f^2(M)=f^2(PQ),$$

which implies

$$f(PQ)=0 \ \text{ and } \ M=P^2Q\leq PQ.$$

This implies that $P = 1$ and $M = Q = q_1 \cdots q_s$. It is clear to show that $(M, 3) = 1$. Assume by contradiction that $M = 3R$, then $M \mid 3R^2 = R^2 - R \cdot (2R) + (2R)^2$, and so we have from (18)

$$0 = f(3R^2) = f^2(R) - f(R)f(2R) + f^2(2R) = 1^2 - 1^2 + 1^2 = 1,$$

which is impossible, because $M \nmid R,\ M \nmid 2R$ and $f(R) = f(2R) = 1$.

Now we prove that $q \equiv 2 \pmod 3$ for every prime $q \mid M$. Assume by contradiction that $q|M$ and $q \equiv 1 \pmod 3$. Then

$$x^2 - xy + y^2 \equiv 0 \pmod q \iff (2x - y)^2 \equiv -3y^2 \pmod q.$$

Since $q \equiv 1 \pmod 3$, we have $\left(\frac{-3}{q}\right) = \left(\frac{q}{3}\right) = 1$, consequently there are $n, m \in \mathbb{N}$ such that $n^2 - nm + m^2 \equiv 0 \pmod M$, $(nm, q) = 1$. Hence $M \nmid n, m$ and so $f(n) = f(m) = 1$, infer from (18) that

$$0 = f\Big(n^2 - nm + m^2\Big) = f^2(n) - f(n)f(m) + f^2(m) = 1 - 1 \cdot 1 + 1^2 = 1,$$

which is impossible.
Thus, we proved the assertions $(c2)$ and (c) of Theorem 1.

(d) Assume that $f(1) = 1$, $f(2) = 1$ and $f(n) = 1$ for every $n \in \mathbb{N}, n \geq 3$.
Then $f(n) = 1$ for every $n \in \mathbb{N}$, and so the assertion (d) is proved.

(e) Assume that $f(1) = 1$ and $f(2) = 2$.
Assume that $f(n) = n$ for every $n < N$, where $N \geq 3$. Then we infer from (3) that

$$\Big(N - 2\Big)\Big(f(N) - N\Big) = \Big(f(N-1) - f(1)\Big)\Big(f((N-1)+1) - f(N-1) - f(1)\Big) = 0,$$

which proves $f(N) = N$.
The case (e) is true, and so Theorem 1 is proved. □

4 Proof of Theorem 2

It is clear to check that the functions defined in (A), (B) and (C) satisfy the functional equation (1). Now we prove the "only if" part.

Assume that $f : \mathbb{N} \to \mathbb{C}$ satisfies

$$f(n^2 - 2nm + m^2) = f^2(n) - 2f(n)f(m) + f^2(m) \text{ for every } n, m \in \mathbb{N}. \tag{19}$$

First we note that

$$f(k^2) = \Big(f(n+k) - f(n)\Big)^2 \text{ for every } n, k \in \mathbb{N}. \tag{20}$$

This follows direct from (19) if $m = n + k$.

In the following we shall use the notation $[N, x, y, u, v] \in \mathcal{S}$ if

$$N = x^2 - 2xy + y^2 = u^2 - 2uv + v^2.$$

It is obvious that if $[N, x, y, u, v] \in \mathcal{S}$, then we infer from (19) that

$$E(N) := f^2(x) - 2f(x)f(y) + f^2(y) - \Big(f^2(u) - 2f(u)f(v) + f^2(v)\Big) = 0.$$

One can check that $[1, 3, 2, 2, 1] \in \mathcal{S}$, consequently

$$\begin{aligned} E(1) &= f^2(3) - 2f(3)f(2) + f^2(2) - \Big(f^2(2) - 2f(2)f(1) + f^2(1)\Big) \\ &= -\Big(f(3) - f(1)\Big)\Big(- f(3) + 2f(2) - f(1)\Big) = 0. \end{aligned}$$

Thus, we have two cases:

(A) $f(3) = f(1)$

(B) $f(3) \neq f(1)$ and $f(3) = 2f(2) - f(1)$.

Let us consider the case (A), i.e. $f(3) = f(1)$. In this case, we obtain from (20) that

$$f(4) = f(2^2) = \Big(f(1 + 2) - f(1)\Big)^2 = 0$$

and so the fact $[4, 4, 2, 3, 1] \in \mathcal{S}$ implies

$$0 = E(4) = f^2(4) - 2f(4)f(2) + f^2(2) - \Big(f^2(3) - 2f(3)f(1) + f^2(1)\Big) = f^2(2).$$

Since $f(3) = f(1)$ and $f(2) = f(4) = 0$, we may assume that $f(n) = f(1)\chi_2(n)$ holds for every $n \leq 4$. Then we infer from (20) that

$$0 = f(2^2) = \Big(f(n + 1) - f(n - 1)\Big)^2,$$

which gives

$$f(n + 1) = f(n - 1) = f(1)\chi_2(n - 1) = f(n - 1) = f(1)\chi_2(n + 1).$$

Thus, we have proved that $f(n) = f(1)\chi_2(n)$ holds for every $n \in \mathbb{N}$. Therefore we infer from (19) that $f(1) \in \{0, 1\}$. Thus we have two solutions: either $f(n) = 0$, or $f(n) = \chi_2(n)$ for every $n \in \mathbb{N}$.

Now we consider the case (B). Assume that $f(3) \neq f(1)$ and $f(3) = 2f(2) - f(1)$. Then $f(2) - f(1) \neq 0$. We infer from the fact $[4, 5, 3, 3, 1] \in \mathcal{S}$ that

$$\begin{aligned} E(4) &= f^2(5) - 2f(5)f(3) + f^2(3) - \Big(f^2(3) - 2f(3)f(1) + f^2(1)\Big) \\ &= \Big(f(5) - f(1)\Big)\Big(f(5) - 2f(3) + f(1)\Big) \\ &= \Big(f(5) - f(1)\Big)\Big(f(5) - 4f(2) + 3f(1)\Big) = 0. \end{aligned}$$

First we prove that $f(5) - f(1) \neq 0$, consequently

$$f(5) = 4f(2) - 3f(1). \tag{21}$$

Assume that $f(5) = f(1)$. Then we infer from the facts $[1, 5, 4, 4, 3] \in \mathcal{S}$ and

$$f(4) = (f(3) - f(1))^2 = 4(f(2) - f(1))^2 = 4f(1)$$

that

$$\begin{aligned} E(1) &= f^2(5) - 2f(5)f(4) + f^2(4) - \Big(f^2(4) - 2f(4)f(3) + f^2(3)\Big) \\ &= f^2(1) - 8f^2(1) + 8f(1)(2f(2) - f(1)) - (2f(2) - f(1))^2 \\ &= 4\Big(f(2) - f(1)\Big)\Big(4f(1) - f(2)\Big) = 0. \end{aligned}$$

Since $f(2) - f(1) \neq 0$, the last relation implies that $f(2) = 4f(1)$, and so $f(3) = 2f(2) - f(1) = 7f(1)$. Since $f(3) = 7f(1) \neq f(1)$, we have $f(1) \neq 0$. But one can check that $[4, 5, 3, 4, 2] \in \mathcal{S}$, consequently

$$\begin{aligned} E(4) &= f^2(5) - 2f(5)f(3) + f^2(3) - \Big(f^2(4) - 2f(4)f(2) + f^2(2)\Big) \\ &= f^2(1) - 14f(1)f(1) + 49f^2(1) - \Big(16f^2(1) - 32f(1)f(1) + 16f^2(1)\Big) \\ &= 36f^2(1) = 0. \end{aligned}$$

This is impossible, because $f(1) \neq 0$. Therefore, we have proved that (21) is true.

Now assume that (21) is true, i.e., $f(5) = 4f(2) - 3f(1)$. Then we infer from the facts $f(3) = 2f(2) - f(1)$, $f(4) = 4f(1)$ and $[1, 5, 4, 4, 3] \in \mathcal{S}$ that

$$\begin{aligned} E(1) &= f^2(5) - 2f(5)f(4) + f^2(4) - \Big(f^2(4) - 2f(4)f(3) + f^2(3)\Big) \\ &= \Big(4f(2) - 3f(1)\Big)^2 - 8f(1)\Big(4f(2) - 3f(1)\Big) + \Big(4f(1)\Big)^2 \\ &\quad - \Big(4f(1)\Big)^2 + 8f(1)\Big(2f(2) - f(1)\Big) - \Big(2f(2) - f(1)\Big)^2 \\ &= 12\Big(f(2) - f(1)\Big)\Big(f(2) - 2f(1))\Big) = 0. \end{aligned}$$

Since $f(2) - f(1) \neq 0$, the last relation shows that $f(2) = 2f(1)$.

On other hand, we infer from (20) that $f(1) = f(1^2) = (f(2) - f(1))^2 = f^2(1)$, which with the fact $f(2) - f(1) = 2f(1) - f(1) = f(1) \neq 0$ implies that $f(1) = 1$. Consequently

$$f(1) = 1, f(2) = 2f(1) = 2, f(3) = 2F(2) - f(1) = 3f(1) = 3 \text{ and } f(4) = 4f(1) = 4.$$

Assume that $f(n) = n$ for every $n \leq N$, where $N \geq 4$. The we obtain from (20) that

$$1 = f(1^2) = \Big(f(N) - f(N-1)\Big)^2 = \Big(f(N) - (N-1)\Big)^2 = f^2(N) - 2(N-1)f(N) + (N-1)^2$$

and

$$\begin{aligned} 4 = f(4) = f(2^2) &= \Big(f(N) - f(N-2)\Big)^2 = \Big(f(N) - (N-2)\Big)^2 \\ &= f^2(N) - 2(N-2)f(N) + (N-2)^2. \end{aligned}$$

These imply that

$$-2(N-2)f(N)+(N-2)^2-(-2(N-1)f(N)+(N-1)^2)=4-1=3$$

and so $2f(N) = 2N$. Thus we have proved that $f(N) = N$, consequently $f(n) = n$ for every $n \in \mathbb{N}$.

Theorem 2 is proved. □

5 Proof of the corollaries

Since $f \in \mathcal{M}$, we have $f(1) = 1$. It is obvious that if $f(M) = f(q_1 \cdots q_s) = f(q_1) \cdots f(q_s) = 0$ and M is minimal, then $M = q \in \mathcal{P}$. Therefore, Corollary 1 and Corollary 2 are true. □

References

[1] De Koninck, J.-M., Kátai, I., & Phong, B. M. (1997). A new characteristic of the identity function. *Journal of Number Theory*, 63(2), 325–338.

[2] Khanh, B. M. M. (2021). Characterization of the identity function with an equation function. *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös Nominatae. Sectio Computatorica*, 52, 195–216.

[3] Park, P.-S. (in press). Multiplicative functions commutable with binary quadratic forms $x^2 \pm xy + y^2$. *Bulletin of the Korean Mathematical Society*.

[4] Phong, B. M. (2006). A characterization of the identity function with the equation of Hosszú type. *Publicationes Mathematicae Debrecen*, 69(1–2), 219–226.

[5] Phong, B. M. (2016). A characterization of identity with function equation II. *Acta Mathematica Hungarica*, 148(2), 450–465.

[6] Phong, B. M., & Szeidl, R. B. (2021). On the equation $f(n^2 + nm + m^2) = f(n)^2 + f(n)f(m)+f(m)^2$. *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös Nominatae. Sectio Computatorica*, 52, 255–278.

[7] Spiro, C. (1992). Additive uniqueness set for arithmetic functions. *Journal of Number Theory*, 42, 232–246.