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Note on some sequences having periods that divide $(p^p - 1) / (p - 1)$

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Abstract: In this paper, we use the properties of the classical umbral calculus to determine sequences related to the Bell numbers and having periods divide $(p^p - 1) / (p - 1)$.
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1 Introduction

The periodicity of the Bell numbers has been studied by several authors. In [17], William showed that the number $\mathcal{N}_p := (p^p - 1) / (p - 1)$ is the period of the Bell numbers. In [12], Radoux conjectured that $\mathcal{N}_p$ is the minimum period of the Bell numbers, which have been verified by some authors for most primes below 180. In [4, 5], the authors gave some properties of the period $\mathcal{N}_p$. In this paper, for a given prime p, we use the properties of the classical umbral calculus to determine sequences related to the Bell numbers and having periods divide $\mathcal{N}_p$. Before starting, we recall some definitions and properties of several versions on the enumeration of the set-partitions. Indeed, the n-th Bell number $\mathcal{B}_n$ counts the number of all partitions of the set $[n] := \{1, \dots, n\}$, and the n-th r-Bell number $\mathcal{B}_{n;r}$ counts the number of all partitions of $[n + r]$ such that the first r elements are in distinct subsets, see [9]. More generally, the $(r_1, \dots, r_q)$-Bell number $\mathcal{B}_{n,r_1,\ldots,r_q}$ counts the number of all partitions of $[n]$ such that the elements of each of the q sets

$$\begin{aligned}[R_1] &:= \{1,\ldots,r_1\}\,,\\ [R_2] &:= \{r_1+1,\ldots,r_1+r_2\}\,,\\ &\vdots\\ [R_q] &:= \{r_1+\cdots+r_{q-1}+1,\ldots,r_1+\cdots+r_q\}\,,\end{aligned}$$

are in distinct subsets, see [8]. Furthermore, the number of partitions of $[n]$ without singletons and the number of partitions of $[n+1]$ with the large singleton $k+1$ are denoted, respectively, by $\mathcal{V}_n$ and $\mathcal{V}_{n;k}$, see [16]. Let $\mathbf{B}$ be the Bell umbra introduced by Rota et al. [13–15], given by $\mathbf{B}^n = \mathcal{B}_n$. The above numbers can be represented by $\mathbf{B}$, as follows [2, 16]:

$$\begin{aligned}\mathcal{B}_{n;r} &= (\mathbf{B}+r)^n\,,\\ \mathcal{B}_{n;r_1,\ldots,r_q} &= \mathbf{B}^n\,(\mathbf{B})_{r_1}\cdots(\mathbf{B})_{r_q}\,,\\ \mathcal{V}_n &= (\mathbf{B}-1)^n\,,\\ \mathcal{V}_{n;k} &= \mathbf{B}^k\,(\mathbf{B}-1)^{n-k}\,,\end{aligned}$$

where $(x)_n$ is the falling factorial defined by

$$(x)_n = x\,(x-1)\cdots(x-n+1)\,,\text{if } n\geq 1 \text{ and } (x)_0 = 1.$$

Also, for any polynomial f, and any non-negative integer n, we have, [7]

$$(\mathbf{B})_n\, f(\mathbf{B}) = f(\mathbf{B}+n)\,. \tag{1.1}$$

In particular, for $f(x) = 1$, we obtain

$$(\mathbf{B})_n = 1. \tag{1.2}$$

For more information on the umbral calculus and its applications one can see [1–3, 7, 13, 15]. For any non-negatives integers $n, s\geq 1$, and any prime p, it is known from [6] that we have

$$\mathcal{B}_{n+p^s} \equiv s\mathcal{B}_n + \mathcal{B}_{n+1} \pmod{p}\,. \tag{1.3}$$

We also have, [10–12]

$$\mathcal{B}_{n+\mathcal{N}_p} \equiv \mathcal{B}_n \pmod{p}\,,$$

where

$$\mathcal{N}_s = 1+p+\cdots+p^{s-1} = \frac{p^s-1}{p-1},\ s = 1,2,\ldots.$$

In the remainder of this paper, we use the identity

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k\,, \tag{1.4}$$

where $\left\{ {n \atop k} \right\}$ is the (n,k)-th Stirling number of the second kind, which counts the number of partitions of the set $[n]$ into k non-empty subsets. Also, for any numbers a and b we denote by $a\equiv b$ to mean $a\equiv b \pmod{p}$.

2 Period of some sequences linked to Bell numbers

The key of the main results is given by the following theorem.

Theorem 2.1. *Let f be a polynomial in $\mathbb{Z}[x]$. Then for any prime p, and any non-negative integers $n, s \geq 0$, there holds*

$$\mathbf{B}^{\mathcal{N}_{s+1}-1} f(\mathbf{B}) \equiv (\mathbf{B}+s)_s f(\mathbf{B}). \tag{2.1}$$

Proof. It suffices to take $f(x) = x^n$. We proceed by induction on s. It is obvious that the congruence (2.1) is true for $s = 0$ and for $s = 1$, by (1.3) we have

$$\mathbf{B}^{n+\mathcal{N}_1-1} = \mathbf{B}^n \equiv (\mathbf{B}+1)_0 \mathbf{B}^n,$$

and

$$\mathbf{B}^{n+\mathcal{N}_2-1} = \mathbf{B}^{n+p} \equiv \mathbf{B}^n + \mathbf{B}^{n+1} = (\mathbf{B}+1)_1 \mathbf{B}^n.$$

Assume that $\mathbf{B}^{n+\mathcal{N}_s-1} \equiv (\mathbf{B}+s-1)_{s-1} \mathbf{B}^n$, $s \geq 1$. Then

$$\mathbf{B}^{n+\mathcal{N}_{s+1}-1} = \mathbf{B}^{p^s}\mathbf{B}^{n+\mathcal{N}_s-1} \equiv \mathbf{B}^{p^s}(\mathbf{B}+s-1)_{s-1} \mathbf{B}^n = (\mathbf{B}+s-1)_{s-1} \mathbf{B}^{n+p^s}.$$

By (1.3) we have $\mathbf{B}^{n+p^s} \equiv s\mathbf{B}^n + \mathbf{B}^{n+1}$. Then

$$\begin{aligned}\mathbf{B}^{n+\mathcal{N}_{s+1}-1} &\equiv (\mathbf{B}+s-1)_{s-1} \mathbf{B}^{n+p^s}\\ &\equiv (\mathbf{B}+s-1)_{s-1} \left(s\mathbf{B}^n + \mathbf{B}^{n+1}\right)\\ &= (\mathbf{B}+s-1)_{s-1} (\mathbf{B}+s) \mathbf{B}^n\\ &= (\mathbf{B}+s)_s \mathbf{B}^n.\end{aligned}$$

This completes the inductive step. □

Proposition 2.2. *Let f be a polynomial in $\mathbb{Z}[X]$. Then for any prime number p and any integer r such that $0 \leq r \leq p-1$, we have*

$$(\mathbf{B}-1)_r \mathbf{B}^{\mathcal{N}_{p-r}} f(\mathbf{B}) \equiv f(\mathbf{B}). \tag{2.2}$$

In particular for $r = 0$, we get

$$\mathbf{B}^{\mathcal{N}_p} f(\mathbf{B}) \equiv f(\mathbf{B}). \tag{2.3}$$

Proof. By Theorem 2.1, the identity $(x)_{n+m} = (x)_n (x-n)_m$ and the congruence

$$\mathbf{B}^p - \mathbf{B} \equiv (\mathbf{B})_p = 1,$$

we obtain

$$\begin{aligned}(\mathbf{B}-1)_r \mathbf{B}^{\mathcal{N}_{p-r}} f(\mathbf{B}) &= (\mathbf{B}(\mathbf{B}-1)_r)\left(\mathbf{B}^{\mathcal{N}_{p-r}-1} f(\mathbf{B})\right)\\ &\equiv (\mathbf{B})_{r+1} (\mathbf{B}+p-r-1)_{p-r-1} f(\mathbf{B})\\ &\equiv (\mathbf{B})_{r+1} (\mathbf{B}-r-1)_{p-r-1} f(\mathbf{B})\\ &= (\mathbf{B})_p f(\mathbf{B})\\ &= f(\mathbf{B}).\end{aligned}$$

□

Corollary 2.1. *For any prime number p and any non-negative integer n, we have*

$$\mathcal{B}_{n+\mathcal{N}_p} \equiv \mathcal{B}_n, \tag{2.4}$$

$$\mathcal{B}_{n+\mathcal{N}_p;r} \equiv \mathcal{B}_{n;r}, \tag{2.5}$$

$$\mathcal{B}_{n+\mathcal{N}_p;r_1,\ldots,r_q} \equiv \mathcal{B}_{n;r_1,\ldots,r_q}, \tag{2.6}$$

$$\mathcal{V}_{n+\mathcal{N}_p;k+\mathcal{N}_p} \equiv \mathcal{V}_{n;k}. \tag{2.7}$$

Proof. We use the identity $(\mathbf{B})_n f(\mathbf{B}) = f(\mathbf{B}+n)$ and apply the congruence (2.3) on the polynomials $x^n,\ (x)_r x^n,\ x^n (x)_{r_1} \cdots (x)_{r_q}$ and $x^k (x-1)^{n-k}$. □

Remark 2.3. *Using the identities* (1.1) *and* (1.4)*, the congruence* (2.3) *can be written as*

$$\sum_{j=1}^{\mathcal{N}_p} \left\{ \begin{matrix} \mathcal{N}_p \\ j \end{matrix} \right\} f(\mathbf{B}+j) \equiv f(\mathbf{B}). \tag{2.8}$$

Example 1. *By application of the congruence* (2.8) *on the polynomials $(x+r)^n,\ (x-1)^n$ and $x^n (x)_{r_1} \cdots (x)_{r_q}$, we get*

$$\sum_{j=1}^{\mathcal{N}_p} \left\{ \begin{matrix} \mathcal{N}_p \\ j \end{matrix} \right\} \mathcal{B}_{n;r+j} \equiv \mathcal{B}_{n;r},$$

$$\sum_{j=1}^{\mathcal{N}_p} \left\{ \begin{matrix} \mathcal{N}_p \\ j \end{matrix} \right\} \mathcal{B}_{n;j-1} \equiv \mathcal{V}_n,$$

$$\sum_{j=1}^{\mathcal{N}_p} \left\{ \begin{matrix} \mathcal{N}_p \\ j \end{matrix} \right\} \mathcal{B}_{n;r_1,\ldots,r_q,j} \equiv \mathcal{B}_{n;r_1,\ldots,r_q}.$$

Corollary 2.2. *For any non-negative integers n, r and any prime number $p > r$, we have*

$$\mathcal{B}_{n+\mathcal{N}_p-\mathcal{N}_{p-r}} \equiv \mathcal{B}_{n-1,r+1}. \tag{2.9}$$

In particular, for $r = 1$ or $r = 2$, we obtain

$$\mathcal{B}_{n+p^{p-1}} \equiv \mathcal{B}_{n-1,2},\ \mathcal{B}_{n+p^{p-2}+p^{p-1}} \equiv \mathcal{B}_{n-1,3}, \tag{2.10}$$

and by replacing n by $n+1+\mathcal{N}_{p-r}$, we get

$$\mathcal{B}_{n+\mathcal{N}_{p-r},r+1} \equiv \mathcal{B}_{n+1}.$$

Proof. If we take $f(x) = x^{n+\mathcal{N}_p-\mathcal{N}_{p-r}}$ in (2.2), then

$$\mathbf{B}^{n+\mathcal{N}_p} (\mathbf{B}-1)_r \equiv \mathbf{B}^{n+\mathcal{N}_p-\mathcal{N}_{p-r}}.$$

On the other hand, we have

$$\mathbf{B}^{n+\mathcal{N}_p} (\mathbf{B}-1)_r \equiv (\mathbf{B}-1)_r \mathbf{B}^n = (\mathbf{B})_{r+1} \mathbf{B}^{n-1} = (\mathbf{B}+r+1)^{n-1},$$

which gives $\mathbf{B}^{n+\mathcal{N}_p-\mathcal{N}_{p-r}} \equiv (\mathbf{B}+r+1)^{n-1}$, i.e., $\mathcal{B}_{n+\mathcal{N}_p-\mathcal{N}_{p-r}} \equiv \mathcal{B}_{n-1,r+1}$. □

Remark 2.4. *The congruences* (2.10) *can also be obtained by taking $s = p-1$ or $p-2$ in the congruence*

$$\mathcal{B}_{n+p^s} \equiv s\mathcal{B}_n + \mathcal{B}_{n+1},$$

and can be written as

$$\mathcal{B}_{n+p^{p-1}} \equiv \mathcal{B}_{n+1} - \mathcal{B}_n,$$

$$\mathcal{B}_{n+p^{p-2}+p^{p-1}} \equiv \mathcal{B}_{n+1+p^{p-2}} - \mathcal{B}_{n+p^{p-2}} \equiv \mathcal{B}_{n+2} - 3\mathcal{B}_{n+1} + 2\mathcal{B}_n.$$

Corollary 2.3. *Let $n, r_1, \ldots, r_m$ be non-negative integers, and let $p > \max(r_1, \ldots, r_m)$ be a prime number. Then for any polynomial f in $\mathbb{Z}[X]$, there holds*

$$(\mathbf{B}-1)_{r_1} \cdots (\mathbf{B}-1)_{r_m} \mathbf{B}^{\mathcal{N}_{p-r_1}+\cdots+\mathcal{N}_{p-r_m}} f(\mathbf{B}) \equiv f(\mathbf{B}). \tag{2.11}$$

Proof. By (2.2), we have

$$(\mathbf{B}-1)_r \mathbf{B}^{\mathcal{N}_{p-r}} f(\mathbf{B}) \equiv f(\mathbf{B}), \quad 0 \le r \le p-1. \tag{2.12}$$

This congruence proves the following

$$\begin{aligned} & (\mathbf{B}-1)_{r_1} \cdots (\mathbf{B}-1)_{r_m} \mathbf{B}^{\mathcal{N}_{p-r_1}+\cdots+\mathcal{N}_{p-r_m}} f(\mathbf{B}) \\ = \; & (\mathbf{B}-1)_{r_1} \cdots (\mathbf{B}-1)_{r_{m-1}} \mathbf{B}^{\mathcal{N}_{p-r_1}+\cdots+\mathcal{N}_{p-r_{m-1}}} \left[(\mathbf{B}-1)_{r_m} \mathbf{B}^{\mathcal{N}_{p-r_m}} f(\mathbf{B})\right] \\ \equiv \; & (\mathbf{B}-1)_{r_1} \cdots (\mathbf{B}-1)_{r_{m-1}} \mathbf{B}^{\mathcal{N}_{p-r_1}+\cdots+\mathcal{N}_{p-r_{m-1}}} f(\mathbf{B}) \\ \equiv \; & \cdots \\ \equiv \; & (\mathbf{B}-1)_{r_1} \mathbf{B}^{\mathcal{N}_{p-r_1}} f(\mathbf{B}) \\ \equiv \; & f(\mathbf{B}). \end{aligned}$$

□

Example 2. *For $f(x) = x^{n+m}$ in* (2.11)*, we get*

$$\mathbf{B}^n (\mathbf{B})_{r_1+1} \cdots (\mathbf{B})_{r_m+1} \mathbf{B}^{\mathcal{N}_{p-r_1}+\cdots+\mathcal{N}_{p-r_m}} \equiv \mathbf{B}^{n+m},$$

or, equivalently,

$$\mathcal{B}_{n+\mathcal{N}_{p-r_1}+\cdots+\mathcal{N}_{p-r_m};r_1+1,\ldots,r_m+1} \equiv \mathcal{B}_{n+m}.$$

For

$$r_1 = \cdots = r_m = p-1, \;\; r_1 = \cdots = r_m = 0 \;\text{ or }\; m = 1,$$

we obtain the congruences

$$\mathcal{B}_{n+m;p,\ldots,p} \equiv \mathcal{B}_{n+m}, \quad \mathcal{B}_{n+m\mathcal{N}_p;1,\ldots,1} \equiv \mathcal{B}_{n+m}, \quad \mathcal{B}_{n+\mathcal{N}_{p-r};r+1} \equiv \mathcal{B}_{n+1}.$$

3 Conclusion

Knowing that $\mathcal{N}_p$ is the period of the Bell numbers, we managed to generate sequences related to Bell numbers having the same period. Among these sequences, let us quote the r-Bell numbers, the $(r_1, \ldots, r_q)$-Bell numbers, and a sequence of numbers related to the partitions on a finite set. More generally, according to congruences (2.2) and (2.3), one can generate several sequences having the same period $\mathcal{N}_p$. So, some questions about the minimum period may arise: when does such a sequence have the same minimum period as that of Bell numbers? In particular, for what values of r, the numbers $\mathcal{B}_n$ and $\mathcal{B}_{n,r}$ have the same minimum period?

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