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A generalization of multiple zeta values. Part 2: Multiple sums

Roudy El Haddad

Engineering Department, La Sagesse University
Beirut, Lebanon
e-mail: roudy1581999@live.com

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Abstract: Multiple zeta values have become of great interest due to their numerous applications in mathematics and physics. In this article, we present a generalization, which we will refer to as *multiple sums*, where the reciprocals are replaced with arbitrary sequences. We develop formulae to help with manipulating such sums. We develop variation formulae that express the variation of multiple sums in terms of lower order multiple sums. Additionally, we derive a set of partition identities that we use to prove a reduction theorem that expresses multiple sums as a combination of simple sums. We present a variety of applications including applications concerning polynomials and MZVs such as generating functions and expressions for $\zeta(\{2p\}_m)$ and $\zeta^{\star}(\{2p\}_m)$. Finally, we establish the connection between multiple sums and a type of sums called recurrent sums. By exploiting this connection, we provide additional partition identities for odd and even partitions.
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1 Introduction and notation

The harmonic series is a divergent series as was proven independently by Nicole Oresme [30], Pietro Mengoli [27], Johann Bernoulli [6], and Jacob Bernoulli [4, 5]. However, elevating the terms to a power $s > 1$, we obtain a convergent series. Sums of the form:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

where s is a real number, were first studied by Euler. The case for $s=2$ is the celebrated Basel problem. Its fame came from the originality of the way Euler proved that $\zeta(2)=\frac{\pi^2}{6}$ [13, 14, 17]. More rigorous proofs were later developed [12]. By the method used to find $\zeta(2)$, Euler was able to obtain a general formula for this zeta function for positive even values of s. In 1859, a generalized form of $\zeta(s)$ where s is a complex variable was introduced by Riemann in his article "On the Number of Primes Less Than a Given Magnitude" [34]. In recent years, mathematicians like Zagier [38], Hoffman [24], and Granville [20] have introduced a generalized form of the zeta function which they call the multiple harmonic series (MHS) or multiple zeta values (MZV). However, interest in these sums dates back to when Euler studied the case of length 2 [16].

Since the beginning of the 1990s, such series/sums have been heavily studied by mathematicians such as Hoffman and Zagier. Interest in such sums grew from their tremendous importance in Number Theory and numerous applications [37]. For example, the multiple harmonic series is directly related to the Riemann zeta function $\zeta(s)$ [20,24]. Interest in these series extends beyond mathematics. In fact, they appear in many fields of physics. The number $\zeta(\overline{6},\overline{2})$ appeared in the quantum field theory literature in 1986 [9]. They are of fundamental importance for the connection of knot theory with quantum field theory [10, 25]. Multiple harmonic sums became even more important when higher order calculations in quantum electrodynamics (QED) and quantum chromodynamics (QCD) started needing them [7, 8].

A multiple harmonic series (MHS) or multiple zeta values (MZV) is defined as:

$$\zeta(s_1,s_2,\ldots,s_k)=\sum_{1\le N_1<N_2<\cdots<N_k}\frac{1}{N_1^{s_1}N_2^{s_2}\cdots N_k^{s_k}}.$$

Its partial sum is written as:

$$\zeta_n(s_1,s_2,\ldots,s_k)=\sum_{1\le N_1<N_2<\cdots<N_k\le n}\frac{1}{N_1^{s_1}N_2^{s_2}\cdots N_k^{s_k}}.$$

These sums are a particular case of what we called multiple sums as they are of the form $\sum_{1\le N_1<\cdots<N_m\le n}a_{(m);N_m}\cdots a_{(1);N_1}$ with $a_{(i);N_i}=\frac{1}{N_i^{s_i}}$ for all i. Note, however, that while this particular case has been heavily studied, the general case has received much less attention. Meanwhile, there are thousands of formulae for multiple harmonic sums/series in the literature, not much can be found for general multiple sums/series. In this article, we derive some essential formulae for dealing with general multiple sums.

The aim of this paper is to develop methods/formulae that allow us to better the way we work with multiple sums. We will develop formulae to calculate the variation of such sums as well as formulae to express multiple sums in terms of simple sums. Partition identities needed to prove these formulae as well as partition identities that can be derived from these formulae will be presented. These identities include sums over partitions involving Bernoulli numbers and the zeta function as well as a new definition of binomial coefficients as a sum over partitions. Likewise, by complementing this work with that done in Part 1 of this study [22], we will extend the partition identities of this article to sums over odd and even partitions. As a matter of fact, multiple sums are intimately related to recurrent sums (presented in [22]) as we will show in this paper.

Additionally, note that like the MHS, the general multiple sum structure is not recent, it goes back to the 17-th century. In 1646, Viète proved that a polynomial can be represented as a product of factors (Viète's theorem) as well as he developed the relations linking the coefficients of a polynomial to its roots for positive roots (Viète's formula) [36]. Viète's formula was later proven to hold for any roots or coefficients by A. Girard [19]. The type of sums represented by Viète's formula is a general multiple sum. Hence, according to Viète's formula, multiple sums are fundamental for linking the roots and coefficients of a polynomial.

The importance of this article is based on how it improves our ability to study sums of this form. The theorems presented in this paper can be used to develop new theorems involving multiple sums or to improve upon previously obtained results as we will illustrate in this article. Applications include generalizing the Faulhaber formula for the sum of powers to a formula for the multiple sum of powers. They also include simplifying the relation between roots and coefficients of a polynomial as well as linking the roots of a polynomial to those of its derivatives. And probably, most importantly, determining generating functions for MZVs as well as several formulae for computing $\zeta(\{2p\}_m)$ and $\zeta^\star(\{2p\}_m)$ in terms of p and m. The partition identities are also a major part of the importance of this paper. This article includes partition identities not only for partitions in general but also for odd and even partitions. These identities could be key in deriving new theorems involving odd or even partitions.

Now let us define a notation for multiple sums which we will use in the remainder of this paper: For any $m, q, n \in \mathbb{N}$ where $n \geq q+m-1$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n]$, let $P_{m,q,n}(a_{(1);N_1}, \ldots, a_{(m);N_m})$ represent the general multiple sum of order m for the sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ with lower and upper bounds respectively q and n. For simplicity, however, we will denote it simply as $P_{m,q,n}$.

$$\begin{aligned} P_{m,q,n} &= \sum_{q \leq N_1 < \cdots < N_m \leq n} a_{(m);N_m} \cdots a_{(2);N_2} a_{(1);N_1} \\ &= \sum_{N_m=q+m-1}^{n} \cdots \sum_{N_2=q+1}^{N_3-1} \sum_{N_1=q}^{N_2-1} a_{(m);N_m} \cdots a_{(2);N_2} a_{(1);N_1} \\ &= \sum_{N_m=q+m-1}^{n} a_{(m);N_m} \cdots \sum_{N_2=q+1}^{N_3-1} a_{(2);N_2} \sum_{N_1=q}^{N_2-1} a_{(1);N_1}. \end{aligned} \tag{1}$$

The most common case of a multiple sum is that where all sequences are the same,

$$\begin{aligned} P_{m,q,n}(a_{N_1}, \ldots, a_{N_m}) &= \sum_{q \leq N_1 < \cdots < N_m \leq n} a_{N_m} \cdots a_{N_2} a_{N_1} \\ &= \sum_{N_m=q+m-1}^{n} \cdots \sum_{N_2=q+1}^{N_3-1} \sum_{N_1=q}^{N_2-1} a_{N_m} \cdots a_{N_2} a_{N_1} \\ &= \sum_{N_m=q+m-1}^{n} a_{N_m} \cdots \sum_{N_2=q+1}^{N_3-1} a_{N_2} \sum_{N_1=q}^{N_2-1} a_{N_1}. \end{aligned} \tag{2}$$

For simplicity, we will denote it as $\hat{P}_{m,q,n}$. We could also denote it as $P_{m,q,n}(a_N)$.

Remark 1. Knowing that adding zeros to a sum does not change the sum and noticing that for $N_2 = q$, we get $q \leq N_1 \leq N_2 - 1 = q - 1$ which would lead to an empty sum for this value. Hence, we can start N_2 at q. Similarly, for $N_3 = q, q+1$ or $\cdots$ or $N_m = q, \ldots, q+m-2$, all would lead to zeros. Hence, we could start all the variables at q.

$$P_{m,q,n} = \sum_{N_m=q}^{n} \cdots \sum_{N_2=q}^{N_3-1} \sum_{N_1=q}^{N_2-1} a_{(m);N_m} \cdots a_{(2);N_2} a_{(1);N_1}.$$

Remark 2. If $m > n-q+1$ (or $n < q+m-1$), the multiple sum can still be considered defined and will be zero ($P_{m,q,n} = 0$).

Remark 3. A multiple sum of order 0 is always equal to 1 ($P_{0,i,j} = 1, \forall i, j \in \mathbb{N}$). It is not equivalent to an empty sum (which is equal to 0).

In Section 2, formulas for the calculation of variation of these sums in terms of lower order multiple sums will be presented. Then, in Section 3, we will present a reduction formula that allows the representation of a multiple sum as a combination of simple sums. In Section 4, the relations developed will be applied to Viète's formula in order to simplify the relation linking the coefficients of a polynomial to its roots. Additionally, some theorems related to polynomials will be developed. A generalization of the binomial theorem will also be developed. In Section 5, the reduction theorem will be used to calculate certain special sums such as the multiple harmonic sum and the multiple power sum. In Section 6, we investigate the relation between recurrent sums and multiple sums then, using these links, we derive some odd and even partition identities.

2 Variation formulas

In this section, we will develop formulas to express the variation of a multiple sum of order m ($P_{m,q,n+1} - P_{m,q,n}$) in terms of lower order multiple sums. Equivalently, these formulas can be used to express $P_{m,q,n+1}$ in terms of $P_{m,q,n}$ and lower order multiple sums.

Remark 4. In this section, for concisness, we will omit the proofs as they are simple and repeat the procedure of the proofs of Section 2 of Part 1 of this study [22].

2.1 Simple expression

We begin by presenting the simplest case of the variation formula in Lemma 2.1. This basic form is needed in order to prove the general form.

Lemma 2.1. *For any $m, q, n \in \mathbb{N}$ where $n \geq q+m-1$, we have that*

$$P_{m,q,n+1} = P_{m,q,n} + a_{(m);n+1} P_{m-1,q,n}.$$

Based upon Lemma 2.1, a more generalized version of the variation formula can be developed which allows the representation of $P_{m,q,n+1}$ in terms of $P_{m,q,n}$ and lower order multiple sums (from order 0 to $(m-1)$).

Theorem 2.2. *For any $m, q, n \in \mathbb{N}$ where $n \geq q + m - 1$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n+1]$, we have that*

$$\sum_{q \leq N_1 < \cdots < N_m \leq n+1} a_{(m);N_m} \cdots a_{(1);N_1}$$
$$= \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1-j} \right) \left(\sum_{q \leq N_1 < \cdots < N_k \leq n-m+k} a_{(k);N_k} \cdots a_{(1);N_1} \right).$$

Using the notation from Eq. (1)*, this theorem can be written as*

$$P_{m,q,n+1} = \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1-j} \right) P_{k,q,n-m+k}.$$

Corollary 2.2.1. *If all sequences are the same, Theorem 2.2 becomes*

$$\sum_{q \leq N_1 < \cdots < N_m \leq n+1} a_{N_m} \cdots a_{N_1} = \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{n+1-j} \right) \left(\sum_{q \leq N_1 < \cdots < N_k \leq n-m+k} a_{N_k} \cdots a_{N_1} \right).$$

Using the notation from Eq. (2)*, this theorem can be written as*

$$\hat{P}_{m,q,n+1} = \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{n+1-j} \right) \hat{P}_{k,q,n-m+k}.$$

Example 2.1. For $m = 2$, we have

$$\sum_{q \leq N_1 < N_2 \leq n+1} b_{N_2} a_{N_1} - \sum_{q \leq N_1 < N_2 \leq n} b_{N_2} a_{N_1} = (b_{n+1}) \sum_{q \leq N_1 \leq n-1} a_{N_1} + (b_{n+1})(a_n).$$

2.2 Simple recurrent expression

Theorem 2.2 can be rewritten in a recursive way as illustrated by the following theorem.

Theorem 2.3. *For any $m, q, n \in \mathbb{N}$ where $n \geq q + m - 1$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n+1]$, we have that*

$$\sum_{q \leq N_1 < \cdots < N_m \leq n+1} a_{(m);N_m} \cdots a_{(1);N_1} - \sum_{q \leq N_1 < \cdots < N_m \leq n} a_{(m);N_m} \cdots a_{(1);N_1}$$
$$= a_{(m);n+1} \Bigg\{ a_{(m-1);n} \Bigg[\cdots a_{(2);n-m+3} \Bigg(a_{(1);n-m+2}(1) + \sum_{q \leq N_1 \leq n-m+1} a_{(1);N_1} \Bigg)$$
$$+ \sum_{q \leq N_1 < N_2 \leq n-m+2} a_{(2);N_2} a_{(1);N_1} \Bigg] + \sum_{q \leq N_1 < \cdots < N_{m-1} \leq n-1} a_{(m-1);N_{m-1}} \cdots a_{(1);N_1} \Bigg\}.$$

Using the notation from Eq. (1)*, this theorem can be written as*

$$P_{m,q,n+1} - P_{m,q,n}$$
$$= a_{(m);n+1} \left\{ a_{(m-1);n} \left[\cdots a_{(2);n-m+3} \left(a_{(1);n-m+2} \left(P_{0,q,n-m+1} \right) + P_{1,q,n-m+1} \right) + P_{2,q,n-m+2} \right] + P_{m-1,q,n-1} \right\},$$

where $P_{0,q,n-m+1} = 1$.

Corollary 2.3.1. *If all sequences are the same, Theorem 2.3 becomes*

$$\sum_{q\le N_1<\cdots<N_m\le n+1} a_{N_m}\cdots a_{N_1} - \sum_{q\le N_1<\cdots<N_m\le n} a_{N_m}\cdots a_{N_1}$$
$$= a_{n+1}\left\{a_n\left[\cdots a_{n-m+3}\left(a_{n-m+2}(1)+\sum_{q\le N_1\le n-m+1} a_{N_1}\right)+\sum_{q\le N_1<N_2\le n-m+2} a_{N_2}a_{N_1}\right]\right.$$
$$\left.+\sum_{q\le N_1<\cdots<N_{m-1}\le n-1} a_{N_{m-1}}\cdots a_{N_1}\right\}.$$

Using the notation from Eq. (2)*, this theorem can be written as*

$$\hat{P}_{m,q,n+1} = a_{n+1}\left\{a_n\left[\cdots a_{n-m+3}\left(a_{n-m+2}\left(\hat{P}_{0,q,n-m+1}\right)+\hat{P}_{1,q,n-m+1}\right)+\hat{P}_{2,q,n-m+2}\right]+\hat{P}_{m-1,q,n-1}\right\}+\hat{P}_{m,q,n},$$

where $\hat{P}_{0,q,n-m+1} = 1$.

Example 2.2. For $m = 2$, we have

$$\sum_{q\le N_1<N_2\le n+1} b_{N_2}a_{N_1} - \sum_{q\le N_1<N_2\le n} b_{N_2}a_{N_1} = (b_{n+1})\left\{\sum_{q\le N_1\le n-1} a_{N_1} + a_n(1)\right\}.$$

2.3 General expression

In order to represent the variation of a multiple sum of order m ($P_{m,q,n+1} - P_{m,q,n}$) only in terms of multiple sums of order going from p to $(m-1)$, a more general form of Theorem 2.2 can be developed.

Theorem 2.4. *For any* $m, q, n \in \mathbb{N}$ *where* $n \ge q+m-1$*, for any* $p \in [0, m]$*, and for any set of sequences* $a_{(1);N_1}, \ldots, a_{(m);N_m}$ *defined in the interval* $[q, n+1]$*, we have that*

$$\sum_{q\le N_1<\cdots<N_m\le n+1} a_{(m);N_m}\cdots a_{(1);N_1} = \sum_{k=p+1}^{m}\left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1-j}\right)\left(\sum_{q\le N_1<\cdots<N_k\le n-m+k} a_{(k);N_k}\cdots a_{(1);N_1}\right)$$
$$+\left(\prod_{j=0}^{m-p-1} a_{(m-j);n+1-j}\right)\left(\sum_{q\le N_1<\cdots<N_p\le n-m+p+1} a_{(p);N_p}\cdots a_{(1);N_1}\right).$$

Using the notation from Eq. (1)*, this theorem can be written as*

$$P_{m,q,n+1} = \sum_{k=p+1}^{m}\left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1-j}\right)P_{k,q,n-m+k} + \left(\prod_{j=0}^{m-p-1} a_{(m-j);n+1-j}\right)P_{p,q,n-m+p+1}.$$

Corollary 2.4.1. *If all sequences are the same, Theorem 2.4 simplifies to the following,*

$$\sum_{q\le N_1<\cdots<N_m\le n+1} a_{N_m}\cdots a_{N_1} = \sum_{k=p+1}^{m}\left(\prod_{j=0}^{m-k-1} a_{n+1-j}\right)\left(\sum_{q\le N_1<\cdots<N_k\le n-m+k} a_{N_k}\cdots a_{N_1}\right)$$
$$+\left(\prod_{j=0}^{m-p-1} a_{n+1-j}\right)\left(\sum_{q\le N_1<\cdots<N_p\le n-m+p+1} a_{N_p}\cdots a_{N_1}\right).$$

Using the notation from Eq. (2)*, this theorem can be written as*

$$\hat{P}_{m,q,n+1} = \sum_{k=p+1}^{m} \left(\prod_{j=0}^{m-k-1} a_{n+1-j} \right) \hat{P}_{k,q,n-m+k} + \left(\prod_{j=0}^{m-p-1} a_{n+1-j} \right) \hat{P}_{p,q,n-m+p+1}.$$

Example 2.3. For $p = 2$ and if the sequences are the same:

$$\sum_{q \leq N_1 < \cdots < N_m \leq n+1} a_{N_m} \cdots a_{N_1} = \sum_{k=3}^{m} \left(\prod_{j=0}^{m-k-1} a_{n+1-j} \right) \left(\sum_{q \leq N_1 < \cdots < N_k \leq n-m+k} a_{N_k} \cdots a_{N_1} \right) + \left(\prod_{j=0}^{m-3} a_{n+1-j} \right) \left(\sum_{q \leq N_1 < N_2 \leq n-m+3} a_{N_2} a_{N_1} \right).$$

Example 2.4. For $p = m - 2$ and if the sequences are the same:

$$\sum_{q \leq N_1 < \cdots < N_m \leq n+1} a_{N_m} \cdots a_{N_1} - \sum_{q \leq N_1 < \cdots < N_m \leq n} a_{N_m} \cdots a_{N_1}$$
$$= (a_{n+1}) \left(\sum_{q \leq N_1 < \cdots < N_{m-1} \leq n-1} a_{N_{m-1}} \cdots a_{N_1} \right) + (a_{n+1} a_n) \left(\sum_{q \leq N_1 < \cdots < N_{m-2} \leq n-1} a_{N_{m-2}} \cdots a_{N_1} \right).$$

2.4 General recurrent expression

The general expression of the variation formula (illustrated by Theorem 2.4) can be expressed in a recursive way as illustrated by the following theorem.

Theorem 2.5. *For any* $m, q, n \in \mathbb{N}$ *where* $n \geq q + m - 1$*, for any* $p \in [0, m]$*, and for any set of sequences* $a_{(1);N_1}, \ldots, a_{(m);N_m}$ *defined in the interval* $[q, n+1]$*, we have that*

$$P_{m,q,n+1} = a_{(m);n+1} \left\{ a_{(m-1);n} \left[\cdots a_{(p+2);n-m+p+3} \left(a_{(p+1);n-m+p+2} \left(P_{p,q,n-m+p+1} \right) + P_{p+1,q,n-m+p+1} \right) + P_{p+2,q,n-m+p+2} \right] + P_{m-1,q,n-1} \right\} + P_{m,q,n}.$$

Corollary 2.5.1. *If all sequences are the same, Theorem 2.5 simplifies to the following form,*

$$\hat{P}_{m,q,n+1} - \hat{P}_{m,q,n} = a_{n+1} \left\{ a_n \left[\cdots a_{n-m+p+3} \left(a_{n-m+p+2} \left(\hat{P}_{p,q,n-m+p+1} \right) + \hat{P}_{p+1,q,n-m+p+1} \right) + \hat{P}_{p+2,q,n-m+p+2} \right] + \hat{P}_{m-1,q,n-1} \right\}.$$

Example 2.5. For $p = m - 2$ and if the sequences are the same:

$$\sum_{q \leq N_1 < \cdots < N_m \leq n+1} a_{N_m} \cdots a_{N_1} - \sum_{q \leq N_1 < \cdots < N_m \leq n} a_{N_m} \cdots a_{N_1}$$
$$= a_{n+1} \left\{ \sum_{q \leq N_1 < \cdots < N_{m-1} \leq n-1} a_{N_{m-1}} \cdots a_{N_1} + a_n \left[\sum_{q \leq N_1 < \cdots < N_{m-2} \leq n-1} a_{N_{m-2}} \cdots a_{N_1} \right] \right\}.$$

3 Reduction formulas

The objective of this section is to introduce formulas which can be used to reduce multiple sums from their original form containing multiple summations to a form containing only simple sums. This will involve the use of partitions of an integer.

3.1 A brief introduction to partitions

In this paper, partitions are involved in the reduction formula for a multiple sum. For this reason, a brief introduction to partitions as well as to Stirling numbers is needed. For concisness, we omit this introduction here. See Section 4.1 of Part 1 [22] for the necessary introduction and notation.

Remark 5. A more in-depth explanation of partitions can be found in [1]. For further details on the partition function see [11, 15, 23, 32, 33]. Additional ways of representing partitions can be found in [31].

Before we can proceed to the next section, there is still an identity that needs to be presented. The unsigned Stirling numbers of the first kind, denoted $|S(m,r)|$ or $\left[{m \atop r}\right]$, can be expressed in terms of the rising factorial $x^{\overline{m}}$ [26] :

$$x^{\overline{m}} = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} x^k \qquad \text{or} \qquad \begin{bmatrix} m \\ r \end{bmatrix} = [x^r]\left(x^{\overline{m}}\right). \tag{3}$$

Remark 6. For $m \geq 0$, $\left[{m \atop m}\right] = 1$. For $m \geq 1$, $\left[{m \atop 0}\right] = 0$.

From this definition, the famous finite alternating sum of the unsigned Stirling numbers of the first kind can be directly deduced by substituting x by (-1) to get:

- If $m = 0$ or $m = 1$

$$\sum_{k=0}^{0} (-1)^k \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 = 0!\,(-1)^0. \tag{4}$$

$$\sum_{k=0}^{1} (-1)^k \begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 - 1 = 1!\,(-1)^1. \tag{5}$$

- If $m \geq 2$

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} = (-1)(-1+1)\cdots(-1+m-1) = 0. \tag{6}$$

Hence,

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} = \begin{cases} (-1)^m m! & \text{for } 0 \leq m \leq 1, \\ 0 & \text{for } m \geq 2. \end{cases} \tag{7}$$

3.2 Reduction Theorem and partition identities

In order to prove the main theorem of this section (Theorem 3.3, which we will call the reduction theorem), we have to prove a set of lemmas. There are 2 sets of lemmas needed: The first set of lemmas is also needed to prove the reduction theorem for Recurrent sums and, hence, was already proven in Part 1 of this study [22]. The second set of lemmas is specific to the type of sums studied in this paper and will be proven in this section.

We begin by proving the following identity involving an alternating sum over partitions.

Lemma 3.1. *Let m be a non-negative integer,*

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\pi(m)}(-1)^{\sum y_{k,i}}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!}=\begin{cases}(-1)^m & \text{for } 0\le m\le 1,\\ 0 & \text{for } m\ge 2.\end{cases}$$

Proof. Left to the reader. Can be proven using the same approach as Lemma 4.2 of Part 1 [22]. It requires the use of Lemma 4.1 from [22] as well as Eq. (7). □

A more general form of Lemma 3.1 is illustrated in the following lemma.

Lemma 3.2. *Let $(y_{k,1},\dots,y_{k,m})=\{(y_{1,1},\dots,y_{1,m}),(y_{2,1},\dots,y_{2,m}),\cdots\}$ be the set of all partitions of m. Let $(v_1,\dots,v_m)$ be a partition of $r\le m$.*

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ y_{k,i}\ge v_i}}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\begin{cases}(-1)^{m-r}\prod_{i=1}^{m}\frac{(-1)^{v_i}}{i^{v_i}(v_i)!} & \text{for } 0\le m-r\le 1,\\ 0 & \text{for } m-r\ge 2.\end{cases}$$

Remark 7. Knowing that the largest element of a partition of r is at most r, we can rewrite it as

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ y_{k,i}\ge v_i}}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\begin{cases}(-1)^{m-r}\prod_{i=1}^{r}\frac{(-1)^{v_i}}{i^{v_i}(v_i)!} & \text{for } 0\le m-r\le 1,\\ 0 & \text{for } m-r\ge 2.\end{cases}$$

Proof. Left to the reader. Can be proven using the same approach as in Lemma 4.4 of Part 1 [22]. It requires the use of Lemma 3.1. □

Remark 8. If $r>m$, then $\sum i.Y_{k,i}=m-r<0$ which makes Lemma 3.1 invalid. Hence, this lemma is invalid for $r>m$.

Now that all the required lemmas have been proven, we show the following theorem which allows the representation of a multiple sum in terms of simple sums.

Theorem 3.3 (Reduction Theorem)**.** *Let m be a non-negative integer, $p(m)$ be the number of partitions of m, k be the index of the k-th partition of m $(1\le k\le p(m))$, i be an integer between 1 and m, and $y_{k,i}$ be the multiplicity of i in the k-th partition of m. The reduction theorem for multiple sums is stated as follows:*

$$\sum_{q\le N_1<\cdots<N_m\le n}a_{N_m}\cdots a_{N_1}=(-1)^m\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

Remark 9. The theorem can also be written as

$$\sum_{q\le N_1<\cdots<N_m\le n}a_{N_m}\cdots a_{N_1}=\sum_{\pi(m)}(-1)^{m-\sum y_{k,i}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

Proof. Base case (for $n = q, \forall m \in \mathbb{N}$):

$$(-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=q}^{q} (a_N)^i \right)^{y_{k,i}} = (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} (a_q)^{i.y_{k,i}}$$
$$= (-a_q)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}}.$$

By applying Lemma 3.1, we get

$$(-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=q}^{q} (a_N)^i \right)^{y_{k,i}} = \begin{cases} (-a_q)^0(-1)^0 = 1 & \text{for } m = 0, \\ (-a_q)^1(-1)^1 = a_q & \text{for } m = 1, \\ (-a_q)^m(0) = 0 & \text{for } m \geq 2. \end{cases}$$

Likewise,

$$\sum_{q \leq N_1 < \cdots < N_m \leq q} a_{N_m} \cdots a_{N_1} = \begin{cases} 1 & \text{for } m = 0, \\ \sum_{q \leq N_1 \leq q} a_{N_1} = a_q & \text{for } m = 1, \\ \sum_{q \leq N_1 < \cdots < N_m \leq q} a_{N_m} \cdots a_{N_1} = 0 & \text{for } m \geq 2. \end{cases}$$

Induction hypothesis (for $n, \forall m \in \mathbb{N}$):

$$\sum_{q \leq N_1 < \cdots < N_m \leq n} a_{N_m} \cdots a_{N_1} = (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=q}^{n} (a_N)^i \right)^{y_{k,i}}.$$

Induction step: To be concise, we denote by I the right-hand side term of the equality to be proven, i.e.,

$$I = (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=q}^{n+1} (a_N)^i \right)^{y_{k,i}}$$
$$= (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \left(\sum_{N=q}^{n} (a_N)^i + (a_{n+1})^i \right)^{y_{k,i}}.$$

By applying the binomial theorem, we get that

$$I = (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \sum_{v=0}^{y_{k,i}} \binom{y_{k,i}}{v} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v} \left((a_{n+1})^i\right)^{y_{k,i}-v}$$
$$= (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \sum_{v=0}^{y_{k,i}} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v} (a_{n+1})^{i.y_{k,i}-i.v}.$$

Let $A_{v,i,k} = \frac{(-1)^{y_{k,i}}}{(y_{k,i})! i^{y_{k,i}}} \binom{y_{k,i}}{v} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v} (a_{n+1})^{i.y_{k,i}-i.v}$. By expanding then regrouping, it can be seen that

$$\prod_{i=1}^{m} \sum_{v=0}^{y_{k,i}} A_{v,i,k} = \sum_{v_m=0}^{y_{k,m}} \cdots \sum_{v_1=0}^{y_{k,1}} \prod_{i=1}^{m} A_{v_i,i,k} = \sum_{0 \leq v_i \leq y_{k,i}} \prod_{i=1}^{m} A_{v_i,i,k}.$$

This is because, for any given k, by expanding the product of sums (the left-hand side term), we will get a sum of products of the form $A_{v_1,1}A_{v_2,2}\cdots A_{v_m,m}$ ($\prod_{i=1}^{m} A_{v_i,i}$) for all combinations of $v_1, v_2, \ldots, v_m$ such that $0 \le v_1 \le y_{k,1}, \ldots, 0 \le v_m \le y_{k,m}$, which is equivalent to the right-hand side term. This then can be written more compactly by expressing the repeated sum over the v_i's with one sum that combines all the conditions. The set of conditions $0 \le v_1 \le y_{k,1}, \ldots, 0 \le v_m \le y_{k,m}$ can be expressed as the condition $0 \le v_i \le y_{k,i}$ for $i \in [1, m]$. Hence,

$$I = (-1)^m \sum_{\pi(m)} \sum_{0 \le v_i \le y_{k,i}} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i\right)^{v_i} (a_{n+1})^{i.y_{k,i} - i.v_i}.$$

Similarly, let j represent $\sum i.v_i$. Hence, we can add the trivial condition that is $j = \sum i.v_i$ to the sum over v_i. This condition is equivalent to the condition $\Pi(j)$ which refers to the sum being over all partitions Π of j. Additionally,

- $\sum i.v_i = j$ is minimal when $v_1 = 0, \ldots, v_m = 0$. Hence, $j_{min} = 0$.
- $\sum i.v_i = j$ is maximal when $v_1 = y_{k,1}, \ldots, v_m = y_{k,m}$. Hence, $j_{max} = \sum i.y_{k,i} = m$.

Therefore, we have that $0 \le j \le m$ or equivalently that j can go from 0 to m. Hence, knowing that adding a true statement to a condition does not change the condition, we can add this additional condition to get

$$I = (-1)^m \sum_{\pi(m)} \sum_{\substack{j=0 \\ \Pi(j) \\ 0 \le v_i \le y_{k,i}}}^{m} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i\right)^{v_i} (a_{n+1})^{i.y_{k,i} - i.v_i}.$$

Knowing that $\binom{y_{k,i}}{v_i} = 0$ if $v_i > y_{k,i}$, hence, the terms produced for $v_i > y_{k,i}$ would be zero. Thus, we can remove the condition $0 \le v_i \le y_{k,i}$ because terms that do not satisfy this condition will be zeros and, therefore, would not change the value of the sum.

$$I = (-1)^m \sum_{\pi(m)} \sum_{\substack{j=0 \\ \Pi(j)}}^{m} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i\right)^{v_i} (a_{n+1})^{i.y_{k,i} - i.v_i}.$$

We expand the expression then, from all values of k (from every partition $(y_{k,1}, \ldots, y_{k,m})$ of m), we regroup together the terms having a combination of exponents $(v_1, \ldots, v_m)$ that forms a partition of the same integer j and we do so $\forall j \in [0, m]$. Hence, performing this manipulation allows us to interchange the sum over $\pi(m)$ (over $\sum i.y_{k,i} = m$) with the sum over j. Thus, the expression becomes as follows,

$$\begin{aligned}
I &= (-1)^m \sum_{\substack{j=0 \\ \Pi(j)}}^{m} \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i\right)^{v_i} (a_{n+1})^{i.y_{k,i} - i.v_i} \\
&= (-1)^m \sum_{\substack{j=0 \\ \Pi(j)}}^{m} \sum_{\pi(m)} (a_{n+1})^{\sum i.y_{k,i} - \sum i.v_i} \left[\prod_{i=1}^{m} \left(\sum_{N=q}^{n} (a_N)^i\right)^{v_i}\right] \left[\prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i}\right] \\
&= (-1)^m \sum_{\substack{j=0 \\ \Pi(j)}}^{m} (a_{n+1})^{m-j} \left[\prod_{i=1}^{m} \left(\sum_{N=q}^{n} (a_N)^i\right)^{v_i}\right] \left(\sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i}\right).
\end{aligned}$$

Applying Lemma 3.2, we have

$$\sum_{\pi(m)} \prod_{i\,=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} = \begin{cases} (-1)^{m-j} \prod_{i=1}^{m} \frac{(-1)^{v_i}}{i^{v_i}(v_i)!} & \text{for } 0 \leq m-j \leq 1, \\ 0 & \text{for } m-j \geq 2, \end{cases}$$
$$= \begin{cases} (-1)^{m-j} \prod_{i=1}^{m} \frac{(-1)^{v_i}}{i^{v_i}(v_i)!} & \text{for } m-1 \leq j \leq m, \\ 0 & \text{for } 0 \leq j \leq m-2. \end{cases}$$

Hence,

$$I = (-1)^m \sum_{\substack{j=m-1 \\ \Pi(j)}}^{m} (a_{n+1})^{m-j} \left[\prod_{i=1}^{m} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right] (-1)^{m-j} \left(\prod_{i=1}^{m} \frac{(-1)^{v_i}}{i^{v_i}(v_i)!} \right)$$
$$= \sum_{\substack{j=m-1 \\ \Pi(j)}}^{m} (a_{n+1})^{m-j} (-1)^j \left(\prod_{i=1}^{m} \frac{(-1)^{v_i}}{i^{v_i}(v_i)!} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right).$$

Knowing that for any given value of j there are multiple combinations of $v_1, \ldots, v_m$ that satisfy $\sum i.v_i = j$. Hence, every value of j corresponds to a sum of the sum's argument for all partitions of j (for all combinations of $v_1, \ldots, v_m$ satisfying $\sum i.v_i = j$). Therefore, we can split the outer sum with two conditions into two sums each with one of the conditions as follows,

$$I = \sum_{j=m-1}^{m} (a_{n+1})^{m-j} (-1)^j \sum_{\Pi(j)} \left(\prod_{i=1}^{m} \frac{(-1)^{v_i}}{i^{v_i}(v_i)!} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right).$$

Knowing that the largest element of a partition of j is at most j, we can rewrite the product as going from 1 to j instead of 1 to m. Now, by using the induction hypothesis, the expression becomes

$$I = \sum_{j=m-1}^{m} (a_{n+1})^{m-j} \left(\sum_{q \leq N_1 < \cdots < N_j \leq n} a_{N_j} \cdots a_{N_1} \right)$$
$$= \left(\sum_{q \leq N_1 < \cdots < N_m \leq n} a_{N_m} \cdots a_{N_1} \right) + (a_{n+1}) \left(\sum_{q \leq N_1 < \cdots < N_{m-1} \leq n} a_{N_{m-1}} \cdots a_{N_1} \right).$$

Using Lemma 2.1, we get the case for $(n+1)$. The theorem is proven by induction. □

Example 3.1. *For $m = 2$, we have that*

$$\sum_{1 \leq N_1 < N_2 \leq n} a_{N_2} a_{N_1} = \frac{1}{2} \left(\sum_{N=1}^{n} a_N \right)^2 - \frac{1}{2} \left(\sum_{N=1}^{n} (a_N)^2 \right).$$

Example 3.2. *For $m = 3$, we have that*

$$\sum_{1\, \leq N_1 < N_2 < N_3 \leq n} a_{N_3} a_{N_2} a_{N_1} = \frac{1}{6} \left(\sum_{N=1}^{n} a_N \right)^3 - \frac{1}{2} \left(\sum_{N=1}^{n} a_N \right) \left(\sum_{N=1}^{n} (a_N)^2 \right) + \frac{1}{3} \left(\sum_{N=1}^{n} (a_N)^3 \right).$$

Corollary 3.3.1. *Let $m, n \in \mathbb{N}$, we have that*

$$(-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{n}{i}\right)^{y_{k,i}} = \binom{n}{m}.$$

Proof. From paper [21], we have the following relation,

$$\sum_{N_m=0}^{n-m} \cdots \sum_{N_1=0}^{N_2} 1 = \binom{n}{m}.$$

We shift the variables in the following way,

$$\sum_{1\le N_1<\cdots<N_m\le n} 1 = \sum_{N_m=m}^{n} \sum_{N_{m-1}=m-1}^{N_m-1} \cdots \sum_{N_1=1}^{N_2-1} 1 = \sum_{N_m=0}^{n-m} \sum_{N_{m-1}=m-1}^{N_m+m-1} \cdots \sum_{N_1=1}^{N_2-1} 1$$
$$= \sum_{N_m=0}^{n-m} \sum_{N_{m-1}=0}^{N_m} \sum_{N_{m-2}=m-2}^{N_{m-1}+m-2} \cdots \sum_{N_1=1}^{N_2-1} 1.$$

After completing the shifting for all variables, we get

$$\sum_{1\le N_1<\cdots<N_m\le n} 1 = \sum_{N_m=0}^{n-m} \cdots \sum_{N_1=0}^{N_2} 1 = \binom{n}{m}.$$

By applying Theorem 3.3, we get the corollary. □

Example 3.3. For $n = 1$, we get Lemma 3.1,

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} = (-1)^m \binom{1}{m} = \begin{cases} (-1)^m & \text{for } 0 \le m \le 1, \\ 0 & \text{for } m \ge 2. \end{cases}$$

Example 3.4. For $n = m$,

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-m)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} = (-1)^m \binom{m}{m} = (-1)^m.$$

Corollary 3.3.2. *For any $q, n \in \mathbb{N}$ where $n \ge q$, we have that*

$$(-1)^{n-q+1} \sum_{\pi(n-q+1)} \prod_{i=1}^{n-q+1} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=q}^{n} (a_N)^i\right)^{y_{k,i}} = \prod_{j=q}^{n} a_j.$$

Proof.

$$\sum_{q\le N_1<\cdots<N_{n-q+1}\le n} a_{N_{n-q+1}} \cdots a_{N_1} = \sum_{q=N_1, N_2=q+1, \ldots, N_{n-q+1}=n} a_{N_{n-q+1}} \cdots a_{N_1} = a_n \cdots a_q.$$

By using Theorem 3.3, we get the corollary. □

3.3 General Reduction Theorem

See Section 4.3 of Part 1 [22] for the necessary concepts and notation. Using the notation introduced, we can formulate a generalization of Theorem 3.3 where all sequences are distinct.

Theorem 3.4. *Let $m, n, q \in \mathbb{N}$ such that $n \geq q + m - 1$. Let $a_{(1);N}, \ldots, a_{(m);N}$ be m sequences defined in the interval $[q, n]$. we have that*

$$\sum_{\sigma \in S_m} \left(\sum_{q \leq N_1 < \cdots < N_m \leq n} a_{(\sigma(m));N_m} \cdots a_{(\sigma(1));N_1} \right)$$
$$= \sum_{P \in \Omega} (-1)^{m - \sum y_{k,i}} \prod_{i=1}^{m} [(i-1)!\,]^{y_{k,i}} \left[\prod_{g=1}^{y_{k,i}} \left(\sum_{N=q}^{n} \prod_{h \in P_{i,g}} a_{(h);N} \right) \right].$$

Remark 10. The theorem can also be written as

$$\sum_{\sigma \in S_m} \left(\sum_{q \leq N_1 < \cdots < N_m \leq n} a_{(\sigma(m));N_m} \cdots a_{(\sigma(1));N_1} \right)$$
$$= \sum_{\pi(m)} (-1)^{m - \sum y_{k,i}} \sum_{\Omega_k} \prod_{i=1}^{m} [(i-1)!\,]^{y_{k,i}} \left[\prod_{g=1}^{y_{k,i}} \left(\sum_{N=q}^{n} \prod_{h \in P_{i,g}} a_{(h);N} \right) \right]$$
$$= (-1)^m |S_m| \sum_{\pi(m)} \frac{1}{|\Omega_k|} \sum_{\Omega_k} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{y_{k,i}!\, i^{y_{k,i}}} \left[\prod_{g=1}^{y_{k,i}} \left(\sum_{N=q}^{n} \prod_{h \in P_{i,g}} a_{(h);N} \right) \right].$$

The first form is obtained by regrouping together, from the set of all partitions of the set $1, \ldots, m$, those who are associated with a given partition of m. The second expression is obtained by noting that

$$\frac{(-1)^m |S_m|}{|\Omega_k|} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{y_{k,i}!\, i^{y_{k,i}}} = (-1)^{m - \sum y_{k,i}} \prod_{i=1}^{m} [(i-1)!\,]^{y_{k,i}}.$$

These forms are shown as they can be more easily used to show that this theorem reduces to Theorem 3.3 if all sequences are the same.

Proof. The same terms appear in both sides of the theorem, hence, to prove the theorem, it suffices to prove that they appear with the same multiplicity on both sides.
We assume, without lost of generality, that all sequences are distinct. The left-hand side term can be written as follows

$$\sum_{\sigma \in S_m} \left(\sum_{q \leq N_1 < \cdots < N_m \leq n} a_{(\sigma(m));N_m} \cdots a_{(\sigma(1));N_1} \right) = \sum_{\sigma \in S_m} \left(\sum_{q \leq N_1 < \cdots < N_m \leq n} a_{(m);N_{\sigma(m)}} \cdots a_{(1);N_{\sigma(1)}} \right).$$

Consider the symmetric group S_m acting on $N = (N_1, \ldots, N_m)$. N has an isotropy group $S_m(N)$ and an associated partition ρ of the set $\{1, \ldots, m\}$. ρ is the set of equivalence classes of the relation given by $a \sim b$ if and only if $N_a = N_b$. $S_m(N) = \{\sigma \in S_m \mid \sigma(i) \sim i\}$. Hence, a term

$$a_{(m);N_m} \cdots a_{(1);N_1} \tag{8}$$

appears, in the left-hand side term, once if all the N_j's are distinct and none otherwise.

A term (8) appears

$$\sum_{P\succeq\rho}(-1)^{m-\sum y_{k,i}}\prod_{i=1}^{m}[(i-1)!\,]^{y_{k,i}} \tag{9}$$

times in the right hand side. Note that $\sum y_{k,i}$ is equal to the number of sets in P.

To prove the theorem, one has to show that

$$\sum_{P\succeq\rho}(-1)^{m-\sum y_{k,i}}\prod_{i=1}^{m}[(i-1)!\,]^{y_{k,i}}=\begin{cases}1, & \text{if } |\rho|=m,\\ 0, & \text{otherwise.}\end{cases} \tag{10}$$

We notice that the sign of $(-1)^{m-\sum y_{k,i}}\prod_{i=1}^{m}[(i-1)!\,]^{y_{k,i}}$ is positive if the permutations of cycle type P are even and negative if they are odd. Therefore, (9) is the signed sum of the number of even and odd permutations in the isotropy group $S_m(N)$. Let us also note that an isotropy group has the same number of even and odd permutations unless the associated partition ρ is $\{\{1\},\ldots,\{m\}\}$ ($|\rho|=m$). Hence, (9) is zero unless $|\rho|=m$. This concludes our proof of the theorem. □

Example 3.5. For $m=2$, Theorem 3.4 gives the following,

$$\sum_{q\le N_1<N_2\le n}a_{N_2}b_{N_1}+\sum_{q\le N_1<N_2\le n}b_{N_2}a_{N_1}=\left(\sum_{N=q}^{n}a_N\right)\left(\sum_{N=q}^{n}b_N\right)-\left(\sum_{N=q}^{n}a_Nb_N\right).$$

Example 3.6. For $m=3$, Theorem 3.4 gives the following,

$$\begin{aligned}&\sum_{\sigma\in S_3}\left(\sum_{q\le N_1<N_2<N_3\le n}a_{(\sigma(3));N_3}a_{(\sigma(2));N_2}a_{(\sigma(1));N_1}\right)\\ &=\left(\sum_{N=q}^{n}a_{(1);N}\right)\left(\sum_{N=q}^{n}a_{(2);N}\right)\left(\sum_{N=q}^{n}a_{(3);N}\right)\\ &\quad-\left(\sum_{N=q}^{n}a_{(1);N}\right)\left(\sum_{N=q}^{n}a_{(2);N}a_{(3);N}\right)-\left(\sum_{N=q}^{n}a_{(2);N}\right)\left(\sum_{N=q}^{n}a_{(1);N}a_{(3);N}\right)\\ &\quad-\left(\sum_{N=q}^{n}a_{(3);N}\right)\left(\sum_{N=q}^{n}a_{(1);N}a_{(2);N}\right)+2\left(\sum_{N=q}^{n}a_{(1);N}a_{(2);N}a_{(3);N}\right).\end{aligned}$$

Remark 11. The author thinks that a multiple sum with a specific ordering of the sequences cannot be isolated in Theorem 3.4 in the general case. However, given a specific set of sequences and using their properties, we should potentially be able to isolate the multiple sum with the desired order.

4 Applications to polynomials

Multiple sums have a variety of applications. However, the most famous one is it's usage in Viète's formula to relate the coefficients of a polynomial to its roots. This corresponds to the particular case where the sequence a_N represents the roots r_N of the polynomial. In this section, some applications of this type of sums to polynomials will be presented.

4.1 Relation between the roots and the coefficients of a polynomial

Let $P(x)$ be a polynomial of degree n,

$$P(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{11}$$

where a_i is the coefficient of x^i.

Viète's theorem allows us to rewrite this polynomial as a product of its factors,

$$P(x) = a_n \prod_{i=1}^{n} (x - r_i) = a_n (x - r_1) \cdots (x - r_n) \tag{12}$$

where r_i is the i-th root of the polynomial.

The relation between the roots and the coefficients of a polynomial of degree n (Viète's formula) is structured as a multiple sum of the roots of the polynomial,

$$\frac{a_{n-m}}{a_n} = (-1)^m \sum_{1 \leq N_1 < \cdots < N_m \leq n} r_{N_m} \cdots r_{N_1}. \tag{13}$$

This relation linking the roots and coefficients of a polynomial can be simplified by applying the reduction theorem to Viète's formula. By applying Theorem 3.3 to Viète's formula, we get the following theorem.

Theorem 4.1. *Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial and let $r_1, \ldots, r_n$ be its roots. The coefficients of this polynomial can be linked to its roots by the following relation,*

$$\frac{a_{n-m}}{a_n} = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=1}^{n} (r_N)^i \right)^{y_{k,i}}.$$

Some relations relating the coefficients of a polynomial to its roots for different values of m have been calculated using Theorem 4.1, and are as follows:

$$\frac{a_n}{a_n} = 1, \ \frac{a_{n-1}}{a_n} = -\sum_{N=1}^{n} r_N, \ \frac{a_{n-2}}{a_n} = \frac{1}{2}\left(\sum_{N=1}^{n} r_N\right)^2 - \frac{1}{2}\left(\sum_{N=1}^{n} (r_N)^2\right),$$

$$\frac{a_{n-3}}{a_n} = -\frac{1}{6}\left(\sum_{N=1}^{n} r_N\right)^3 + \frac{1}{2}\left(\sum_{N=1}^{n} r_N\right)\left(\sum_{N=1}^{n} (r_N)^2\right) - \frac{1}{3}\left(\sum_{N=1}^{n} (r_N)^3\right), \ \frac{a_0}{a_n} = (-1)^n \prod_{i=1}^{n} r_i.$$

4.2 Relation between the roots of a polynomial and those of its derivatives

With the formulas developed, we can go beyond just linking the coefficients and roots of a polynomial. In this section, we will develop a formula linking the roots of a polynomial to the roots of its derivatives as illustrated by the following theorem.

Theorem 4.2. *Let $f(x) = \sum_{i=0}^{n} a_i x^i = a_n(x-r_1)\cdots(x-r_n)$ be a polynomial of order n and let*

$$f^{(k)}(x) = \sum_{i=0}^{n-k} a_{(k);i}x^i = \sum_{j=k}^{n} \frac{j!}{(j-k)!}a_j x^{j-k} = a_{(k);n-k}(x-r_{(k);1})\cdots(x-r_{(k);n-k})$$

be its derivative of order k. The roots of $f(x)$ can be linked to the roots of $f^{(k)}(x)$ as follows,

$$\frac{(n-m)!}{n!}\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=1}^{n}(r_N)^i\right)^{y_{k,i}} = \frac{(n-m-k)!}{(n-k)!}\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=1}^{n-k}(r_{(k);N})^i\right)^{y_{k,i}}.$$

Proof. By comparing the coefficients of $f(x)$ and $f^{(k)}(x)$, we get

$$\frac{a_{(k);n-k-m}}{a_{(k);n-k}} = \frac{\frac{(n-m)!}{(n-m-k)!}a_{n-m}}{\frac{n!}{(n-k)!}a_n}\frac{\frac{(n-m)!}{n!}}{\frac{(n-m-k)!}{(n-k)!}}\frac{a_{n-m}}{a_n}.$$

By applying Theorem 4.1, we get this theorem. □

A special case of this theorem which is of special interest is the following.

Corollary 4.2.1. *Let $f(x) = \sum_{i=0}^{n} a_i x^i = a_n(x-r_1)\cdots(x-r_n)$ be a polynomial and let $f^{(k)}(x) = \sum_{i=0}^{n-k} a_{(k);i}x^i = a_{(k);n-k}(x-r_{(k);1})\cdots(x-r_{(k);n-k})$ be its derivative of order k. Let $\overline{x}$ be the average root value for $f(x)$ and $\overline{x_{(k)}}$ be the average root value for $f^{(k)}(x)$. we have that*

$$\overline{x} = \frac{r_1+\cdots+r_n}{n} = \frac{r_{(k);1}+\cdots+r_{(k);n-k}}{n-k} = \overline{x_{(k)}}.$$

4.3 Generating functions and the sum of multiple sums

In this section, we prove a formula for the sum of multiple sums. This identity is then used to prove a generalization of the binomial theorem as well as MZV identities.

Theorem 4.3. *Let $z \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$. We have the following generating function*

$$\sum_{m=0}^{n} z^m \sum_{1\le N_1<\cdots<N_m\le n} r_{N_m}\cdots r_{N_1} = z^n\prod_{N=1}^{n}\left(r_N+\frac{1}{z}\right) = \prod_{N=1}^{n}(r_N z+1).$$

$$\sum_{m=0}^{n}(-1)^m z^m\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_i}}{y_i!}\left(\frac{1}{i}\sum_{N=1}^{n}(r_N)^i\right)^{y_i} = z^n\prod_{N=1}^{n}\left(r_N+\frac{1}{z}\right) = \prod_{N=1}^{n}(r_N z+1).$$

$$\sum_{m=0}^{\infty} z^m \sum_{1\le N_1<\cdots<N_m} r_{N_m}\cdots r_{N_1} = \prod_{N=1}^{\infty}(r_N z+1).$$

Remark 12. The theorem holds when taking the limit for $z \to 0$. Also, knowing that the multiple sum is equal to 0 if $m > n$, we can extend the bounds of the first two equations for m going from 0 till ∞.

Proof. Let $f(x) = \sum_{m=0}^{n} a_m x^m = a_n(x-r_1)\cdots(x-r_n)$ be a polynomial of order n. Then

$$\frac{f(x)}{a_n} = \sum_{m=0}^{n} x^m\frac{a_m}{a_n} = \sum_{m=0}^{n} x^{n-m}\frac{a_{n-m}}{a_n} = (x-r_1)\cdots(x-r_n) = (-1)^n(r_1-x)\cdots(r_n-x).$$

Let $z = -1/x$. Applying Viète's formula, we get the first identity. Similarly, applying Theorem 4.1 instead leads to the second identity. Letting $n \to \infty$, we get the third equation. □

Corollary 4.3.1. *For $z = -1$, we get the alternating sum of multiple sums.*

$$\sum_{m=0}^{n} (-1)^m \sum_{1 \le N_1 < \cdots < N_m \le n} r_{N_m} \cdots r_{N_1} = \prod_{N=1}^{n} (1 - r_N).$$

Corollary 4.3.2. *For $z = 1$, we get the sum of multiple sums.*

$$\sum_{m=0}^{n} \sum_{1 \le N_1 < \cdots < N_m \le n} r_{N_m} \cdots r_{N_1} = \prod_{N=1}^{n} (r_N + 1).$$

Theorem 4.4. *For any $n \in \mathbb{N}^*$, we have that*

$$\begin{aligned}(a_1 + b_1) \cdots (a_n + b_n) &= \left(\prod_{i=1}^{n} b_i\right) \sum_{m=0}^{n} \sum_{1 \le N_1 < \cdots < N_m \le n} \frac{a_{N_m} \cdots a_{N_1}}{b_{N_m} \cdots b_{N_1}} \\ &= \left(\prod_{i=1}^{n} b_i\right) \sum_{m=0}^{n} (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_i}}{y_i!} \left(\frac{1}{i} \sum_{N=1}^{n} \left(\frac{a_N}{b_N}\right)^i\right)^{y_i}.\end{aligned}$$

Proof.

$$\prod_{i=1}^{n} (a_i + b_i) = \prod_{i=1}^{n} (b_i) \left(\frac{a_i}{b_i} + 1\right) = \left(\prod_{i=1}^{n} b_i\right) \prod_{i=1}^{n} \left(\frac{a_i}{b_i} + 1\right).$$

Letting $r_i = (a_i/b_i)$ and applying Corollary 4.3.2, we obtain this theorem. □

Corollary 4.3.2 can be used to derive several identities related to MZVs. First, let us note the following notation: $\zeta(\{p\}_m)$ represents $\zeta(p, \ldots, p)$ where the multiplicity of p is m.

Corollary 4.4.1. *For $z \in \mathbb{C}$, the sum of MZVs can be rewritten as a product as follows:*

$$\sum_{m=0}^{n} z^m \zeta_n(\{p\}_m) = \prod_{N=1}^{n} \left(1 + \frac{z}{N^p}\right) \quad \textit{and} \quad \sum_{m=0}^{\infty} z^m \zeta(\{p\}_m) = \prod_{N=1}^{\infty} \left(1 + \frac{z}{N^p}\right).$$

Proof. Applying Corollary 4.3.2 with $r_N = (z/N^p)$, we find this identity. □

Corollary 4.4.2. *The sum of MZVs converges to 2 as the arguments of the MZVs go to infinity.*

$$\lim_{p \to \infty} \sum_{m=0}^{n} \zeta_n(\{p\}_m) = \lim_{p \to \infty} \sum_{m=0}^{\infty} \zeta(\{p\}_m) = 2.$$

Proof.

$$\lim_{p \to \infty} \sum_{m=0}^{n} \zeta_n(\{p\}_m) = \lim_{p \to \infty} \prod_{N=1}^{n} \left(1 + \frac{1}{N^p}\right) = \lim_{p \to \infty} (1 + 1) \prod_{N=2}^{n} \left(1 + \frac{1}{N^p}\right) = (2)(1) = 2.$$

Letting $n \to \infty$, we obtain the second part of the identity. □

Now we will use some important product identities such as Euler products to derive some special sums of multiple zeta values.

First, we will derive a formula for the sum of multiple prime zeta functions in terms of the zeta function. Let $\mathbb{P} = \{p \in \mathbb{N} \mid p \text{ is prime}\}$ and let P_N represent the N-th prime number, we define the multiple prime zeta function as follows:

$$\zeta_{\mathbb{P}}(s_1, \cdots, s_m) = \sum_{1 \le N_1 < \cdots < N_m} \frac{1}{P_{N_1}^{s_1} \cdots P_{N_m}^{s_m}}. \tag{14}$$

Corollary 4.4.3. *For any $s \in \mathbb{C}$,*

$$\sum_{m=0}^{\infty} \zeta_{\mathbb{P}}(\{s\}_m) = \frac{\zeta(s)}{\zeta(2s)}, \qquad \sum_{m=0}^{\infty} (-1)^m \zeta_{\mathbb{P}}(\{s\}_m) = \frac{1}{\zeta(s)}.$$

$$\sum_{m=0}^{\infty} \zeta_{\mathbb{P}}(\{s\}_{2m}) = \frac{\zeta^2(s)+\zeta(2s)}{2\zeta(s)\zeta(2s)} = \frac{\zeta^{\star}(s,s)}{\zeta(s)\zeta(2s)}, \sum_{m=0}^{\infty} \zeta_{\mathbb{P}}(\{s\}_{2m+1}) = \frac{\zeta^2(s)-\zeta(2s)}{2\zeta(s)\zeta(2s)} = \frac{\zeta(s,s)}{\zeta(s)\zeta(2s)}.$$

Proof. We have the following Euler products,

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_{p\in\mathbb{P}} \left(1+p^{-s}\right) = \prod_{N=1}^{\infty} \left(1+\frac{1}{(P_N)^s}\right), \quad \frac{1}{\zeta(s)} = \prod_{p\in\mathbb{P}} \left(1-p^{-s}\right) = \prod_{N=1}^{\infty} \left(1-\frac{1}{(P_N)^s}\right).$$

Letting $r_N = 1/(P_N)^s$ respectively in Corollary 4.3.2 and Corollary 4.3.1, we get this corollary. The odd and even cases are obtained by adding and subtracting the first two equations. □

An interesting product identity that could be used in combination with Corollary 4.3.2 is Euler's expression for the sine function as a product. From this identity, we can derive the following corollary.

Corollary 4.4.4. *For any $z \in \mathbb{C}$,*

$$\sum_{m=0}^{\infty} (-1)^m z^{2m} \zeta(\{2\}_m) = \frac{\sin(\pi z)}{\pi z} \overset{\text{if } z\notin\mathbb{Z}}{=} \frac{1}{\Gamma(1+z)\Gamma(1-z)}.$$

Proof. From the product expansion of sine and Euler's reflection formula, we know that

$$\prod_{N=1}^{\infty} \left(1-\frac{z^2}{N^2}\right) = \frac{\sin(\pi z)}{\pi z} \overset{\text{if } z\notin\mathbb{Z}}{=} \frac{1}{\Gamma(1+z)\Gamma(1-z)}. \tag{15}$$

Letting $r_N = -z^2/N^2$ in Corollary 4.3.2, we obtain this corollary. □

From this corollary, we can deduce two extremely interesting particular cases:
Case 1: If $z \in \mathbb{Z}$,

$$\sum_{m=0}^{\infty} (-1)^m z^{2m} \zeta(\{2\}_m) = 0. \tag{16}$$

Case 2: If $z = i\chi$, $\chi \in \mathbb{R}$,

$$\sum_{m=0}^{\infty} \chi^{2m} \zeta(\{2\}_m) = \sum_{m=0}^{\infty} \frac{(\chi\pi)^{2m}}{(2m+1)!} = \prod_{N=1}^{\infty} \left(1+\frac{\chi^2}{N^2}\right) = \frac{e^{\chi\pi}-e^{-\chi\pi}}{2\pi\chi} = \frac{\sinh(\pi\chi)}{\pi\chi}. \tag{17}$$

The fourth and fifth terms are obtained from Corollary 4.4.4. The third term is obtained from Eq. (15). The second term is obtained from the Taylor series of $\sinh(\pi\chi)$. Using $\sqrt{\chi}$ instead of χ, we get a generating function for $\zeta(\{2\}_m)$ such that

$$\zeta(\{2\}_m) = \frac{1}{m!}\frac{d^m}{d\chi^m}\left(\frac{e^{\sqrt{\chi}\pi}-e^{-\sqrt{\chi}\pi}}{2\pi\sqrt{\chi}}\right)_{\chi=0} = \frac{1}{m!}\frac{d^m}{d\chi^m}\left(\frac{\sinh(\pi\sqrt{\chi})}{\pi\sqrt{\chi}}\right)_{\chi=0} = \frac{\pi^{2m}}{(2m+1)!}. \tag{18}$$

Using a similar procedure, we can produce generating functions for MZVs of any even arguments using product expansions. But to do so, we need to first derive the following product expansion.

Lemma 4.5. *For any $p \in \mathbb{N}^*$, $z \in \mathbb{C}$,*

$$\prod_{k=1}^{p}\left(1+(-1)^{\frac{2k-1}{p}}z\right) = (1+(-1)^p z^p), \quad \prod_{k=1}^{p}\left(1+(-1)^{\frac{2k-2}{p}}z\right) = (1-(-1)^p z^p).$$

Proof. A polynomial can be defined by its leading coefficient and roots. It is trivial to show that both sides of the equality have the same roots and the same leading coefficient. □

We define the following notation: $\chi_k = (-1)^{\frac{2k-1}{p}}$ if p is even and $\chi_k = (-1)^{\frac{2k-2}{p}}$ if p is odd.

Corollary 4.5.1. *For any $p \in \mathbb{N}^*$, we have the following generating function for $\zeta(\{p\}_m)$,*

$$\sum_{m=0}^{\infty} x^{pm}\zeta(\{p\}_m) = \prod_{N=1}^{\infty}\left(1+\frac{x^p}{N^p}\right) = \prod_{k=1}^{p}\frac{1}{\Gamma(1+\chi_k x)}, \; p \geq 2.$$

$$\sum_{m=0}^{\infty} x^{2pm}\zeta(\{2p\}_m) = \prod_{N=1}^{\infty}\left(1+\frac{x^{2p}}{N^{2p}}\right) = \frac{1}{(i\pi x)^p}\prod_{k=1}^{p}\sin\left((-1)^{\frac{2k-1}{2p}}\pi x\right), \; 2p \geq 2.$$

Proof. From Lemma 4.5, $\prod_{k=1}^{p}(1+\chi_k x) = 1 + x^p$. From Viète's formula, $\sum_{k=1}^{p}\chi_k = 0$ for $p > 1$. Hence, $\prod_{k=1}^{p} n^{-\chi_k x} = 1$. We have the following gamma function infinite product,

$$\Gamma(z) = \lim_{n\to\infty}\frac{n^z}{z}\prod_{N=1}^{n}\left(1+\frac{z}{N}\right)^{-1}, \quad \frac{1}{\Gamma(1+z)} = \lim_{n\to\infty} n^{-z}\prod_{N=1}^{n}\left(1+\frac{z}{N}\right).$$

For $z = \chi_k x$,

$$\prod_{k=1}^{p}\frac{1}{\Gamma(1+\chi_k x)} = \lim_{n\to\infty}\left(\prod_{k=1}^{p} n^{-\chi_k x}\right)\prod_{N=1}^{n}\prod_{k=1}^{p}\left(1+\chi_k\frac{x}{N}\right) = \lim_{n\to\infty}(1)\prod_{N=1}^{n}\left(1+\frac{x^p}{N^p}\right).$$

For even p, Euler's reflection formula is used to obtain the expression in terms of sines. □

For the finite case, we can use the following corollary.

Corollary 4.5.2. *For any $n, p \in \mathbb{N}^*$,*

$$\sum_{m=0}^{n} x^{pm}\zeta_n(\{p\}_m) = \prod_{N=1}^{n}\left(1+\frac{x^p}{N^p}\right) = \frac{1}{(n!)^p}\prod_{k=1}^{p}(1+\chi_k x)_n = \frac{1}{(n!)^p}\prod_{k=1}^{p}\frac{\Gamma(n+1+\chi_k x)}{\Gamma(1+\chi_k x)}$$

where $(z)_n = z\cdots(z+n-1)$ is the Pochhammer symbol (rising factorial).

Proof. For conciseness, let I represent the right-hand side term,

$$I = \frac{1}{(n!)^p}\prod_{k=1}^{p}\prod_{N=1}^{n}(N+\chi_k x) = \frac{1}{(n!)^p}\prod_{k=1}^{p}\prod_{N=1}^{n} N\left(1+\chi_k\frac{x}{N}\right) = \frac{(n!)^p}{(n!)^p}\prod_{N=1}^{n}\prod_{k=1}^{p}\left(1+\chi_k\frac{x}{N}\right).$$

Applying Lemma 4.5 with $z = x/N$, we obtain the 2nd term. Applying Corollary 4.3.2, we get the first term. □

This allows us to obtain results such as

$$\zeta_n(2) = \frac{\pi^2}{6} - \psi^{(1)}(n+1), \quad \zeta_n(4) = \frac{\pi^4}{90} - \psi^{(3)}(n+1), \quad where \quad \psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}} \ln\Gamma(z).$$

$$\zeta_n(2,2) = \frac{\pi^4}{120} + \frac{1}{2}\left[\psi^{(1)}(n+1)\right]^2 - \frac{\pi^2}{6}\psi^{(1)}(n+1) + \frac{1}{12}\psi^{(3)}(n+1).$$

We can obtain a more general version of these results by applying Theorem 3.3 to $\zeta_n(\{p\}_m)$, then using the following polygamma function identity $\zeta_n(p) = \zeta(p) + (-1)^{p-1}\psi^{(p-1)}(n+1)/(p-1)!$.

Corollary 4.5.3. *For any $m, n, p \in \mathbb{N}^*$,*

$$\zeta_n(\{p\}_m) = (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} \left(\zeta(ip) + (-1)^{ip-1}\frac{\psi^{(ip-1)}(n+1)}{(ip-1)!}\right)^{y_{k,i}}.$$

Similarly, we can do the same for sums of powers.

Corollary 4.5.4. *For any $n, p \in \mathbb{N}^*$, we get the following generating function*

$$\sum_{m=0}^{n} x^{pm} \sum_{1\le N_1<\cdots<N_m\le n} {N_m}^p \cdots {N_1}^p = \prod_{N=1}^{n}(1+x^pN^p) = x^{pn}\prod_{k=1}^{p}\frac{\Gamma(n+1+\chi_k/x)}{\Gamma(1+\chi_k/x)}.$$

Proof. Let I represent the right-hand side term of the equation.

$$I = \prod_{k=1}^{p}\prod_{N=1}^{n} x\left(N + \frac{\chi_k}{x}\right) = \prod_{N=1}^{n} x^pN^p \prod_{k=1}^{p}\left(1+\frac{\chi_k}{xN}\right) = \prod_{N=1}^{n}(x^pN^p+1).$$

Applying Corollary 4.3.2, we get this corollary. □

Seeing the usefulness of Corollary 4.3.2, one might be tempted to ask if there is an analogous result for recurrent sums (presented in Part 1 of this study [22])? To the author's best efforts, he was not able to derive such a formula except for the particular case of multiple zeta star values $\zeta^\star(\{p\}_m)$. In fact, this result was rediscovered by the author as it had already been shown in [2, 39]. This result is the following:

$$\sum_{m=0}^{\infty} x^m \zeta^\star(\{p\}_m) = \prod_{N=1}^{\infty}\left(1-\frac{x}{N^p}\right)^{-1}, \; x \ne 1. \tag{19}$$

From Eq. (19), we can find an analogous version of Corollary 4.5.1.

Theorem 4.6. *For any $p \in \mathbb{N}^*$, we have the following generating function for $\zeta^\star(\{p\}_m)$,*

$$\sum_{m=0}^{\infty} x^{pm}\zeta^\star(\{p\}_m) = \prod_{N=1}^{\infty}\left(1-\frac{x^p}{N^p}\right)^{-1} = \prod_{k=1}^{p}\Gamma(1-\chi_k x), \; x \ne 1, p \ge 2.$$

$$\sum_{m=0}^{\infty} x^{2pm}\zeta^\star(\{2p\}_m) = \prod_{N=1}^{\infty}\left(1-\frac{x^{2p}}{N^{2p}}\right)^{-1} = (\pi x)^p i^{p-1}\prod_{k=0}^{p-1}\frac{1}{\sin((-1)^{\frac{k}{p}}\pi x)}, \; x \ne 1, 2p \ge 2.$$

Remark 13. Corollary 4.5.1 and Theorem 4.6 lead the author to believe that the Gamma function is key for developping an expression for zeta values of odd arguments, e.g., $\zeta(3)$, $\zeta(5)$, etc.

Remark 14. For any arbitrary function $f(x)$, $\prod_{k=1}^{p} f(\chi_k) = \prod_{k=1}^{p} f(-(-1)^{\frac{2k-1}{p}})$ because $\{\chi_k \,|\, 1 \le k \le p\} = \{-(-1)^{\frac{2k-1}{p}} \,|\, 1 \le k \le p\}$. Hence, we can replace χ_k by $-(-1)^{\frac{2k-1}{p}}$ in many of the previous lemmas, corollaries, and theorems.

Using this theorem, we can easily reobtain Schneider's formula for $\zeta^\star(\{2\}_m)$ [35] as well as we can find formulas for even more general cases such as the following.

Example 4.1. *From Theorem 4.6, $(\pi x)^2 csc(\pi x) csch(\pi x)$ is a generating function for $\zeta^\star(\{4\}_m)$. Using the Cauchy product to obtain its Taylor series, we get that*

$$\zeta^\star(\{4\}_m) = \pi^{4m} \sum_{\ell=0}^{2m} \frac{(-1)^\ell (4^\ell - 2)(4^{2m-\ell} - 2) B_{2\ell} B_{4m-2\ell}}{(2\ell)!\,(4m-2\ell)!} = \sum_{\ell=0}^{2m} (-1)^\ell \zeta^\star(\{2\}_\ell)\zeta^\star(\{2\}_{2m-\ell}).$$

We can further simplify it by noting that the sum from 0 till $m-1$ is equal to from $m+1$ till $2m$. Finally, from Corollary 4.5.1 and Theorem 4.6, we obtain a formula for $\zeta(\{2p\}_m)$ and $\zeta^\star(\{2p\}_m)$. But, first, we define the condition $|\mathbf{n}| = pm$ to mean $n_1 + \cdots + n_p = pm$, $n_j \ge 0$.

Theorem 4.7. *For any $m \in \mathbb{N}$, $p \in \mathbb{N}^*$, and where $\ell_k = \alpha_k - \alpha_{k-1}$,*

$$\frac{\zeta(\{2p\}_m)}{(i\pi)^{2pm}} = (-1)^m \sum_{|\mathbf{n}|=pm} \prod_{k=1}^{p} \frac{(-1)^{\frac{k}{p}(2n_k)}}{(2n_k+1)!} = \frac{(-1)^{m(p+1)}}{(i\pi)^{2pm}} \sum_{|\mathbf{n}|=pm} \prod_{k=1}^{p} (-1)^{\frac{k}{p}(2n_k)} \zeta(\{2\}_{n_k}).$$

$$\frac{\zeta(\{2p\}_m)}{(i\pi)^{2pm}} = (-1)^m \sum_{\alpha_{p-1}=0}^{\alpha_p} \cdots \sum_{\alpha_1=0}^{\alpha_2} \prod_{k=1}^{p} \frac{(-1)^{\frac{k}{p}(2\ell_k)}}{(2\ell_k+1)!}, \quad \text{with } \alpha_p = pm, \alpha_0 = 0.$$

Proof. From Corollary 4.5.1 and using the Taylor series of the sine function,

$$\sum_{m=0}^{\infty} x^{2pm} \zeta(\{2p\}_m) = \prod_{k=1}^{p} \frac{\sin\left((-1)^{\frac{2k-1}{2p}} \pi x\right)}{(-1)^{\frac{2k-1}{2p}} \pi x} = \prod_{k=1}^{p} \sum_{n_k=0}^{\infty} \frac{(-1)^{n_k}}{(2n_k+1)!} (-1)^{\frac{2k-1}{p} n_k} \pi^{2n_k} x^{2n_k}. \quad (20)$$

We note the coefficients in each Taylor series as $a_{(k);n_k}$ and apply Lemma 4.5 of [22] then by looking at the left term of Eq. (20), we notice that the indices must sum to a multiple of p (note that otherwise, the corresponding coefficient will be zero).

$$\sum_{m=0}^{\infty} x^{2pm} \zeta(\{2p\}_m) = \sum_{n_p=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} \prod_{k=1}^{p} a_{(k);n_k} x^{2n_k} = \sum_{m=0}^{\infty} x^{2pm} \sum_{\substack{n_1+\cdots+n_p=pm \\ n_j \ge 0}} \prod_{k=1}^{p} a_{(k);n_k}.$$

After simplification and by identification, we obtain the theorem. The second expression is obtain by applying the generalized Cauchy product to Eq (20) instead. □

Using the same procedure, we prove the following theorem.

Theorem 4.8. *For any $m \in \mathbb{N}$, $p \in \mathbb{N}^*$, and where $\ell_k = \alpha_k - \alpha_{k-1}$,*

$$\frac{\zeta^\star(\{2p\}_m)}{(i\pi)^{2pm}} = \sum_{|\mathbf{n}|=pm} \prod_{k=1}^{p} \frac{(-1)^{\frac{k}{p}(2n_k)}(2 - 2^{2n_k}) B_{2n_k}}{(2n_k)!} = \frac{1}{(i\pi)^{2pm}} \sum_{|\mathbf{n}|=pm} \prod_{k=1}^{p} (-1)^{\frac{k}{p}(2n_k)} \zeta^\star(\{2\}_{n_k}).$$

$$\frac{\zeta^\star(\{2p\}_m)}{(i\pi)^{2pm}} = \sum_{\alpha_{p-1}=0}^{\alpha_p} \cdots \sum_{\alpha_1=0}^{\alpha_2} \prod_{k=1}^{p} \frac{(-1)^{\frac{k}{p}(2\ell_k)}(2 - 2^{2\ell_k}) B_{2\ell_k}}{(2\ell_k)!}, \quad \text{with } \alpha_p = pm, \alpha_0 = 0.$$

Remark 15. Note that the expression in the second equation of Theorems 4.7, 4.8 has a "recurrent sum"-like structure but more general as sequences cannot be isolated for each index.

Remark 16. Let $\lambda(\mathbf{n}) = \sum_{k=1}^{p} k \cdot n_k$ and $\lambda(\boldsymbol{\ell}) = \sum_{k=1}^{p} k \cdot \ell_k = (p+1)pm - |\boldsymbol{\alpha}|$. Noting that $\zeta(\{2p\}_m), \zeta^\star(\{2p\}_m) \in \mathbb{R}$, we can replace the product of (-1)'s by its real part, e.g.,

$$\frac{\zeta(\{2p\}_m)}{(-1)^m (i\pi)^{2pm}} = \sum_{|\mathbf{n}|=pm} \frac{\cos\left(\frac{2\pi}{p}\lambda(\mathbf{n})\right)}{(2n_1+1)!\cdots(2n_p+1)!} = \sum_{\alpha_{p-1}=0}^{\alpha_p} \cdots \sum_{\alpha_1=0}^{\alpha_2} \frac{\cos\left(\frac{2\pi}{p}|\boldsymbol{\alpha}|\right)}{(2\ell_1+1)!\cdots(2\ell_p+1)!}. \tag{21}$$

$$\frac{\zeta^\star(\{2p\}_m)}{(i\pi)^{2pm}} = \sum_{|\mathbf{n}|=pm} \cos\left(\frac{2\pi\lambda(\mathbf{n})}{p}\right) \prod_{k=1}^{p} \frac{(2-2^{2n_k})B_{2n_k}}{(2n_k)!}. \tag{22}$$

Equivalently, the imaginary part is zero.

$$\sum_{|\mathbf{n}|=pm} \frac{\sin\left(\frac{2\pi}{p}\lambda(\mathbf{n})\right)}{(2n_1+1)!\cdots(2n_p+1)!} = 0, \sum_{|\mathbf{n}|=pm} \sin\left(\frac{2\pi\lambda(\mathbf{n})}{p}\right) \prod_{k=1}^{p} \frac{(2-2^{2n_k})B_{2n_k}}{(2n_k)!} = 0. \tag{23}$$

Applying the generalized formula for the product of sines (which can be easily proven by induction) to Corollary 4.5.1 and using the Taylor series, we get Theorem 4.9.

$$\prod_{k=1}^{p} \sin a_k = \frac{(-1)^{\lfloor p/2 \rfloor}}{2^p} \sum_{\ell_k \in \{0,1\}} (-1)^{\sum \ell_k} \begin{cases} \sin\left(\sum_{k=1}^{p} (-1)^{\ell_k} a_k\right), & \text{if } p \text{ is odd,} \\ \cos\left(\sum_{k=1}^{p} (-1)^{\ell_k} a_k\right), & \text{if } p \text{ is even.} \end{cases} \tag{24}$$

Theorem 4.9. *For any $m \in \mathbb{N}$, $p \in \mathbb{N}^*$,*

$$\zeta(\{2p\}_m) = \frac{(i\pi)^{2pm}}{(2i)^p (2pm+p)!} \sum_{\ell_k \in \{0,1\}} (-1)^{\sum \ell_k} \left(\sum_{k=1}^{p} (-1)^{\ell_k + \frac{2k-1}{2p}}\right)^{2pm+p}.$$

$$\zeta(\{2p\}_m) = \frac{(2i\pi)^{2pm}(-1)^m}{(2pm+p)!\, i^{p+1}} \sum_{\ell_k \in \{0,1\}} (-1)^{\sum \ell_k} \left(\frac{1}{2}\sum_{k=1}^{p} (-1)^{\ell_k} e^{\frac{i\pi k}{p}}\right)^{2pm+p}.$$

The advantage of this theorem is that given a specific p, after some calculations and simplifications, we can find the explicit expression of $\zeta(\{2p\}_m)$ for this p. Doing so for some values of p, we notice the following.

Conjecture 4.1. *Among the 2^p terms of the sum over ℓ's, there are $\lfloor 2^p/2p \rfloor$ distinct terms each of which is repeated $2p$ times and the remaining terms are zeros.*

Example 4.2. *For $p = 1, 2, 3$, $\lfloor 2^p/2p \rfloor = 1$ and we can easily verify by doing the calculation that*

$$\sum_{\ell_k \in \{0,1\}} (-1)^{\sum \ell_k} \left(\frac{1}{2}\sum_{k=1}^{p} (-1)^{\ell_k} e^{\frac{i\pi k}{p}}\right)^{2pm+p} = (2p)\left(\frac{1}{2}\sum_{k=1}^{p} e^{\frac{i\pi k}{p}}\right)^{2pm+p} = \frac{-(2p)}{\left(1-e^{\frac{i\pi}{p}}\right)^{2pm+p}}.$$

$$\zeta(\{2p\}_m) = \frac{(2i\pi)^{2pm}(-1)^{m+1}(2p)}{(2pm+p)!\, i^{p+1}\left(1-e^{\frac{i\pi}{p}}\right)^{2pm+p}} = \frac{(2\pi)^{2pm}(-1)^m(2ip)}{(2pm+p)!\, i^{2pm+p}\left(1-e^{\frac{i\pi}{p}}\right)^{2pm+p}}.$$

Example 4.3. *For $p = 4$, $\lfloor 2^p/2p \rfloor = 2$ and we can easily verify by doing the calculation that*

$$\zeta(\{2p\}_m) = \frac{(2\pi)^{2pm}(-1)^m(2ip)}{(2pm+p)!\, i^{2pm+p}}\left[\frac{1}{\left(1-e^{\frac{i\pi}{p}}\right)^{2pm+p}} + \frac{1}{\left(1+e^{\frac{i\pi}{p}}\right)^{2pm+p}}\right].$$

5 Applications to special sums

In this section, we will apply the reduction formula presented in Theorem 3.3 to simplify certain special multiple sums. The first special sum that we will simplify is the multiple sum of powers. The second special sum is the multiple zeta values for positive even arguments.

5.1 Multiple power sums

The Faulhaber formula is a formula developed by Faulhaber in a 1631 edition of Academia Algebrae [18] to calculate sums of powers. The Faulhaber formula is as follows

$$\sum_{N=1}^{n} N^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j n^{p+1-j}$$

where B_j are the Bernoulli numbers of the first kind [28].

In this section, we will use the reduction formula for multiple sums to develop a more general form of the Faulhaber formula.

Theorem 5.1. *For any $m, n, p \in \mathbb{N}$ such that $n \geq m$, we have that*

$$\sum_{1\leq N_1<\cdots<N_m\leq n} {N_m}^p \cdots {N_1}^p = (-1)^m \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}}\left(\sum_{N=1}^{n} N^{ip}\right)^{y_{k,i}}$$
$$= (-1)^m \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}}\left(\frac{n^{ip+1}}{ip+1}\sum_{j=0}^{ip}(-1)^j\binom{ip+1}{j}\frac{B_j}{n^j}\right)^{y_{k,i}}.$$

Proof. By applying Theorem 3.3 and then applying Faulhaber's formula, we get the theorem. □

Corollary 5.1.1. *For any $m, n \in \mathbb{N}$, we have that*

$$\begin{bmatrix} n+1 \\ n-m+1 \end{bmatrix} = (-1)^m \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}}\left(\sum_{N=1}^{n} N^{i}\right)^{y_{k,i}}$$
$$= (-1)^m \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}}\left(\frac{n^{i+1}}{i+1}\sum_{j=0}^{i}(-1)^j\binom{i+1}{j}\frac{B_j}{n^j}\right)^{y_{k,i}}.$$

Proof. Knowing that

$$\sum_{1\leq N_1<\cdots<N_m\leq n} N_m \cdots N_1 = \begin{bmatrix} n+1 \\ n-m+1 \end{bmatrix}.$$

By applying Theorem 5.1 for $p = 1$, the theorem is proven. □

Let us consider the following special cases:

- <u>Case 1: $m = 2$</u>

$$\sum_{1\leq N_1<N_2\leq n} N_2^p N_1^p = \frac{1}{2}\left(\sum_{N=1}^{n} N^p\right)^2 - \frac{1}{2}\left(\sum_{N=1}^{n} N^{2p}\right)$$
$$= \frac{1}{2}\left(\frac{n^{p+1}}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}\frac{B_j}{n^j}\right)^2 - \frac{1}{2}\left(\frac{n^{2p+1}}{2p+1}\sum_{j=0}^{2p}(-1)^j\binom{2p+1}{j}\frac{B_j}{n^j}\right).$$

Example 5.1. By using the previous equation, we can get the following formulas

$$\sum_{1\le N_1<N_2\le n} N_2^2N_1^2 = \frac{n(n-1)(n+1)(2n-1)(2n+1)(5n+6)}{360} = \binom{2n+2}{5}\frac{5n+6}{4!}.$$

$$\sum_{1\le N_1<N_2\le n} N_2^3N_1^3 = \frac{n(n-1)(n+1)(21n^5+36n^4-21n^3-48n^2+8)}{672}.$$

- <u>Case 2: $m=3$</u>

$$\sum_{1\le N_1<N_2<N_3\le n} {N_3}^p{N_2}^p{N_1}^p = \frac{1}{6}\left(\sum_{N=1}^{n} N^p\right)^3 - \frac{1}{2}\left(\sum_{N=1}^{n} N^p\right)\left(\sum_{N=1}^{n} N^{2p}\right) + \frac{1}{3}\left(\sum_{N=1}^{n} N^{3p}\right).$$

Example 5.2. For $p=2$, by applying this theorem with Faulhaber's formula, we have

$$\sum_{1\le N_1<N_2<N_3\le n} N_3^2N_2^2N_1^2 = \binom{2n+2}{7}\frac{35n^2+91n+60}{144}.$$

Remark 17. Some of these sequences as well as additional ones were added by the author to the OEIS:

- A347107 (`https://oeis.org/A347107`),
- A351760 (`https://oeis.org/A351760`),
- A351805 (`https://oeis.org/A351805`).

5.2 Multiple zeta values

In this section, using the formula developed by Euler and the reduction theorem, we prove an expression which can be used to calculate multiple zeta values for positive even values. Then we present new identities based on solutions for some more general forms of the Basel problem.

We start by using Theorem 3.3 to simplify the expression of MZVs for positive even values.

Theorem 5.2. *For any $m\in\mathbb{N}$, $p\in\mathbb{N}^*$, we have that*

$$\frac{\zeta(\{2p\}_m)}{(-1)^m} = \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!\,i^{y_{k,i}}}\left(\zeta(2ip)\right)^{y_{k,i}} = (2i\pi)^{2pm}\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{B_{2ip}}{(2i)(2ip)!}\right)^{y_{k,i}}.$$

$$\zeta(\{2p\}_m) = \frac{(-1)^{(p+1)m}(2\pi)^{2pm}}{(2pm)!}\sum_{\pi(m)}\binom{2pm}{\boldsymbol{\beta}}\prod_{i=1}^{m}\frac{(B_{2ip})^{y_{k,i}}}{(2i)^{y_{k,i}}y_{k,i}!},\quad \boldsymbol{\beta} = (\{2p\}_{y_1},\ldots,\{2pm\}_{y_m}).$$

Proof. Applying Theorem 3.3, we obtain the first equality. Euler proved that, for $m\ge 1$ [3],

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!}B_{2m}. \tag{25}$$

Hence,

$$\sum_{1\le N_1<\cdots<N_m}\frac{1}{N_m^{2p}\cdots N_1^{2p}} = (-1)^m\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!\,i^{y_{k,i}}}\left((-1)^{ip+1}\frac{B_{2ip}(2\pi)^{2ip}}{2(2ip)!}\right)^{y_{k,i}}$$

$$= (-1)^{(p+1)m}(2\pi)^{2pm}\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{B_{2ip}}{(2i)(2ip)!}\right)^{y_{k,i}}. \qquad \square$$

Example 5.3. *For $m = 2$, Theorem 5.2 gives*

$$\zeta(2p, 2p) = \frac{1}{2}\left[(\zeta(2p))^2 - \zeta(4p)\right] = \frac{(2\pi)^{4p}}{(4p)!}\left[\binom{4p}{2p}\frac{(B_{2p})^2}{2^2 2!} + \frac{B_{4p}}{4}\right].$$

Remark 18. In this article, we presented three formulas for $\zeta(\{2p\}_m)$ and $\zeta^\star(\{2p\}_m)$. One factor to consider when choosing the formula to use is the number of terms summed. The number of terms for the first equation of Theorems 4.7, 4.8, that for the second equation of Theorems 4.7, 4.8, and that for Theorem 5.2 (and Theorem 4.9 of [22]) are respectively

$$\sum_{\substack{\pi(pm)\\ \#(\pi)\le p}} \frac{(p)!}{y_1!\cdots y_{pm}!\,(p - \sum y_i)!} = \sum_{\pi(pm)} \binom{p}{y_1, \ldots, y_{pm}}, \quad \binom{pm+p-1}{p-1}, \quad p(m), \tag{26}$$

where the multinomial coefficient is defined to be 0 if $\#(\pi) = \sum y_i > p$. Additionally, note that from the way the first and second equation of Theorems 4.7, 4.8 were derived, they should sum the same number of terms (their sums are over the same condition, for the first, $|\mathbf{n}|= pm$, for the second, $|\boldsymbol{\ell}|= pm$). Hence, the first and second expression of Eq. (26) must be equal.

$$\sum_{\pi(pm)} \binom{p}{y_1, \ldots, y_{pm}} = \sum_{\pi(pm)} \binom{p}{\sum y_i}\binom{\sum y_i}{y_1, \ldots, y_{pm}} = \binom{pm+p-1}{p-1}. \tag{27}$$

Using Theorem 5.2, we will present special cases for $2p = 2, 4, 6$.

Theorem 5.3. *For any $m \in \mathbb{N}$, we have that*

$$\zeta(\{2\}_m) = \sum_{1\le N_1<\cdots<N_m} \frac{1}{N_m^2\cdots N_1^2} = \sum_{\pi(m)}\prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}}\,(\zeta(2i))^{y_{k,i}} = \frac{\pi^{2m}}{(2m+1)!},$$

$$\sum_{\pi(m)}\prod_{i=1}^{m} \frac{1}{(y_{k,i})!}\left(\frac{B_{2i}}{(2i)(2i)!}\right)^{y_{k,i}} = \frac{1}{2^{2m}(2m+1)!}.$$

Proof. In [24], [29], and [35] (as well as in Eq. (18)), the following relation was proven,

$$\sum_{1\le N_1<\cdots<N_m} \frac{1}{N_m^2\cdots N_1^2} = \frac{\pi^{2m}}{(2m+1)!}.$$

By applying Theorem 3.3, we get

$$\sum_{\pi(m)}\prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}}\,(\zeta(2i))^{y_{k,i}} = \frac{\pi^{2m}}{(2m+1)!}.$$

Applying Theorem 5.2 with $p = 1$, we obtain the second equation. □

Example 5.4. For $m = 4$, we have

$$\zeta(\{2\}_4) = \frac{1}{24}(\zeta(2))^4 - \frac{1}{4}(\zeta(2))^2\zeta(4) + \frac{1}{3}\zeta(2)\zeta(6) + \frac{1}{8}(\zeta(4))^2 - \frac{1}{4}\zeta(8) = \frac{\pi^8}{9!}(\approx 0.02614784782).$$

Using Theorem 5.3, we will prove that the multiple sum of $\frac{1}{N^p}$ will converge to 0 as the number of summations m goes to infinity for any $p \ge 2$.

Theorem 5.4. *For any $p \geq 2$, we have that*

$$\lim_{m\to\infty} \zeta(\{p\}_m) = \lim_{m\to\infty} \left(\sum_{1\leq N_1 < \cdots < N_m} \frac{1}{N_m^p \cdots N_1^p} \right) = 0.$$

Proof. Knowing that for any $p \geq 2$, we have $0 \leq \frac{1}{N_i^p} \leq \frac{1}{N_i^2}$, therefore, $0 \leq \frac{1}{N_m^p \cdots N_1^p} \leq \frac{1}{N_m^2 \cdots N_1^2}$, which then implies that

$$0 \leq \sum_{1\leq N_1 < \cdots < N_m} \frac{1}{N_m^p \cdots N_1^p} \leq \sum_{1\leq N_1 < \cdots < N_m} \frac{1}{N_m^2 \cdots N_1^2}.$$

By taking the limit as m goes to infinity and applying Theorem 5.3, we get

$$0 \leq \lim_{m\to\infty} \left(\sum_{1\leq N_1 < \cdots < N_m} \frac{1}{N_m^p \cdots N_1^p} \right) \leq \lim_{m\to\infty} \left(\frac{\pi^{2m}}{(2m+1)!} \right) = 0.$$

Hence, the theorem is proven. □

Theorem 5.5. *For any $m \in \mathbb{N}$, we have that*

$$\zeta(\{4\}_m) = \sum_{1\leq N_1 < \cdots < N_m} \frac{1}{N_m^4 \cdots N_1^4} = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} (\zeta(4i))^{y_{k,i}} = \frac{2(2^{2m})\pi^{4m}}{(4m+2)!} = \frac{2(\sqrt{2}\pi)^{4m}}{(4m+2)!},$$

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!} \left(\frac{B_{4i}}{(2i)(4i)!} \right)^{y_{k,i}} = \frac{2(-1)^m}{2^{2m}(4m+2)!}.$$

Proof. Using Corollary 4.5.1, we obtain the following expression whose Taylor series gives the well-known expression of $\zeta(\{4\}_m)$.

$$\sum_{m=0}^{\infty} \chi^{4m} \zeta(\{4\}_m) = \frac{\sin(\sqrt{i}\pi\chi)\sinh(\sqrt{i}\pi\chi)}{(\sqrt{i}\pi\chi)^2} = \frac{\cosh(\sqrt{2}\pi\chi) - \cos(\sqrt{2}\pi\chi)}{(\sqrt{2}\pi\chi)^2} = \sum_{m=0}^{\infty} \frac{2(\sqrt{2}\pi\chi)^{4m}}{(4m+2)!}.$$

The first equation is obtained from Theorem 3.3 and the second from Theorem 5.2 for $p = 2$. □

Example 5.5. For $m = 3$, we have

$$\sum_{1\leq N_1 < N_2 < N_3} \frac{1}{N_3^4 N_2^4 N_1^4} = \frac{1}{6}(\zeta(4))^3 - \frac{1}{2}\zeta(4)\zeta(8) + \frac{1}{3}\zeta(12) = \frac{\pi^{12}}{681080400} (\approx 0.001357063251).$$

Theorem 5.6. *For any $m \in \mathbb{N}$, we have that*

$$\zeta(\{6\}_m) = \sum_{1\leq N_1 < \cdots < N_m} \frac{1}{N_m^6 \cdots N_1^6} = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} (\zeta(6i))^{y_{k,i}} = \frac{6(2\pi)^{6m}}{(6m+3)!},$$

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!} \left(\frac{B_{6i}}{(2i)(6i)!} \right)^{y_{k,i}} = \frac{6}{(6m+3)!}.$$

Proof. Similarly, using Corollary 4.5.1, we can obtain the well-known expression of $\zeta(\{6\}_m)$,

$$\zeta(\{6\}_m) = \frac{6(2\pi)^{6m}}{(6m+3)!}.$$

The first equation is obtained from Theorem 3.3 and the second from Theorem 5.2 for $p = 3$. □

Example 5.6. For $m = 3$, we have

$$\sum_{1\le N_1<N_2<N_3} \frac{1}{N_3^6 N_2^6 N_1^6} = \frac{1}{6}\left(\zeta(6)\right)^3 - \frac{1}{2}\zeta(6)\zeta(12) + \frac{1}{3}\zeta(18) = \frac{6(2\pi)^{6(3)}}{(6(3)+3)!} (\approx 0.0000273555196).$$

6 Relation to recurrent sums and odd-even partition identities

Recurrent sums and multiple sums have been studied separately respectively in [22] and in this paper. In this section. we compare these types of sums and show their similarities and the link between them. Then by combining the individual relations of each of these sums, we will produce new results. In particular, we obtain new relations governing odd partitions and even partitions.

6.1 Relations between recurrent sums and multiple sums

In this section, we develop the relation linking recurrent and multiple sums. Recurrent sums and multiple sums can be related by the following theorem.

Theorem 6.1. *For any $m, q, n \in \mathbb{N}$ and for any sequence a_N defined in the interval $[q, n]$, we have that:*

$$\sum_{q\le N_1\le\cdots\le N_m\le n} a_{N_m}\cdots a_{N_1} + (-1)^m \sum_{q\le N_1<\cdots<N_m\le n} a_{N_m}\cdots a_{N_1} = 2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

$$\sum_{q\le N_1\le\cdots\le N_m\le n} a_{N_m}\cdots a_{N_1} - (-1)^m \sum_{q\le N_1<\cdots<N_m\le n} a_{N_m}\cdots a_{N_1} = 2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is odd}}} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

Proof. We can notice that

$$\begin{aligned}
&\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}} + \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}\\
&\qquad = 2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.
\end{aligned}$$

$$\begin{aligned}
&\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}} - \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}\\
&\qquad = 2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is odd}}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.
\end{aligned}$$

From Part 1 of this study [22], we have:

$$\sum_{q\le N_1\le\cdots\le N_m\le n} a_{N_m}\cdots a_{N_1} = \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

From Theorem 3.3, we have:

$$\sum_{q\le N_1<\cdots<N_m\le n} a_{N_m}\cdots a_{N_1} = (-1)^m \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{1}{i}\sum_{N=q}^{n} (a_N)^i\right)^{y_{k,i}}.$$

Hence, by combining these relations, we obtain the theorem. □

Example 6.1. For $m = 2$, Theorem 6.1 gives

$$\sum_{1\le N_1\le N_2\le n} a_{N_2}a_{N_1} + \sum_{1\le N_1<N_2\le n} a_{N_2}a_{N_1} = \left(\sum_{N=1}^{n} a_N\right)^2.$$

$$\sum_{1\le N_1\le N_2\le n} a_{N_2}a_{N_1} - \sum_{1\le N_1<N_2\le n} a_{N_2}a_{N_1} = \left(\sum_{N=1}^{n} (a_N)^2\right).$$

Example 6.2. For $m = 3$, Theorem 6.1 gives

$$\sum_{1\le N_1\le N_2\le N_3\le n} a_{N_3}a_{N_2}a_{N_1} + \sum_{1\le N_1<N_2<N_3\le n} a_{N_3}a_{N_2}a_{N_1} = \frac{1}{3}\left(\sum_{N=1}^{n} a_N\right)^3 + \frac{2}{3}\left(\sum_{N=1}^{n} (a_N)^3\right).$$

$$\sum_{1\le N_1\le N_2\le N_3\le n} a_{N_3}a_{N_2}a_{N_1} - \sum_{1\le N_1<N_2<N_3\le n} a_{N_3}a_{N_2}a_{N_1} = \left(\sum_{N=1}^{n} a_N\right)\left(\sum_{N=1}^{n} (a_N)^2\right).$$

6.2 Odd and even partition identities

In Part 1 of this study [22] and in the present paper, we have produced multiple partition identities. Combining these identities, we are able to produce several identities for even and odd partitions. Note that a partition $\pi \equiv (y_{k,1},\ldots,y_{k,m})$ is odd if $\#(\pi) = \sum y_{k,i}$ is odd and even if $\#(\pi) = \sum y_{k,i}$ is even.

Theorem 6.2. *Let m be a non-negative integer,*

$$\sum_{\substack{\pi(m)\\ \#(\pi)\ is\ even}} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = \begin{cases} 1 & \textit{for } m = 0,\\ 0 & \textit{for } m = 1,\\ \frac{1}{2} & \textit{for } m \ge 2.\end{cases}$$

$$\sum_{\substack{\pi(m)\\ \#(\pi)\ is\ odd}} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = \begin{cases} 0 & \textit{for } m = 0,\\ 1 & \textit{for } m = 1,\\ \frac{1}{2} & \textit{for } m \ge 2.\end{cases}$$

Proof. We can notice that:

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!} + \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{i^{y_{k,i}}(y_{k,i})!} = 2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!}.$$

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!} - \sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{i^{y_{k,i}}(y_{k,i})!} = 2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is odd}}}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!}.$$

From Part 1 of this study [22], we have

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!}=1.$$

From Lemma 3.1, we have

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{i^{y_{k,i}}(y_{k,i})!}=\begin{cases}(-1)^m & \text{for } 0\le m\le 1,\\ 0 & \text{for } m\ge 2.\end{cases}$$

Hence, by combining these relations, we obtain the theorem. □

Theorem 6.3. *Let* $(y_{k,1},\dots,y_{k,m})=\{(y_{1,1},\dots,y_{1,m}),(y_{2,1},\dots,y_{2,m}),\dots\}$ *be the set of all partitions of* m. *Let* $(v_1,\dots,v_m)$ *be a partition of* $r\le m$. *We have that*

$$\begin{aligned}\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}&=\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}\\ y_{k,i}\ge v_i}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}\\ &=\begin{cases}\frac{1+(-1)^{\sum v_i}}{2}\prod_{i=1}^{m}\frac{1}{i^{v_i}(v_i)!} & \text{for } m-r=0,\\ \frac{1-(-1)^{\sum v_i}}{2}\prod_{i=1}^{m}\frac{1}{i^{v_i}(v_i)!} & \text{for } m-r=1,\\ \frac{1}{2}\prod_{i=1}^{m}\frac{1}{i^{v_i}(v_i)!} & \text{for } m-r\ge 2.\end{cases}\end{aligned}$$

$$\begin{aligned}\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is odd}}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}&=\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is odd}\\ y_{k,i}\ge v_i}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}\\ &=\begin{cases}\frac{1-(-1)^{\sum v_i}}{2}\prod_{i=1}^{m}\frac{1}{i^{v_i}(v_i)!} & \text{for } m-r=0,\\ \frac{1+(-1)^{\sum v_i}}{2}\prod_{i=1}^{m}\frac{1}{i^{v_i}(v_i)!} & \text{for } m-r=1,\\ \frac{1}{2}\prod_{i=1}^{m}\frac{1}{i^{v_i}(v_i)!} & \text{for } m-r\ge 2.\end{cases}\end{aligned}$$

Proof. We can notice that

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}+\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}.$$

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}-\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}.$$

From Part 1 of this study [22], we have

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ y_{k,i}\ge v_i}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\prod_{i=1}^{m}\frac{1}{i^{v_i}(v_i)!}.$$

From Lemma 3.2, we have

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ y_{k,i}\geq v_i}}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}\binom{y_{k,i}}{v_i}}{i^{y_{k,i}}(y_{k,i})!}=\begin{cases}(-1)^{m-r}\prod_{i=1}^{m}\frac{(-1)^{v_i}}{i^{v_i}(v_i)!} & \text{for } 0\leq m-r\leq 1,\\ 0 & \text{for } m-r\geq 2.\end{cases}$$

Hence, by combining these relations, we obtain the theorem. □

Theorem 6.4. *Let* $m,n\in\mathbb{N}$,

$$\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}=\frac{1}{2}\left[\binom{n-m+1}{m}+(-1)^m\binom{n}{m}\right].$$

$$\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is odd}}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}=\frac{1}{2}\left[\binom{n-m+1}{m}-(-1)^m\binom{n}{m}\right].$$

Proof. We can notice that

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}+\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}=2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is even}}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}.$$

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}-\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}=2\sum_{\substack{\pi(m)\\ \#(\pi)\text{ is odd}}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}.$$

From Part 1 of this study [22], we have

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}=\binom{n+m-1}{m}.$$

From Corollary 3.3.1, we have

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{(-1)^{y_{k,i}}}{(y_{k,i})!}\left(\frac{n}{i}\right)^{y_{k,i}}=(-1)^m\binom{n}{m}.$$

Hence, by combining these relations, we obtain the theorem. □

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