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A generalization of multiple zeta values. Part 1: Recurrent sums

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Abstract: Multiple zeta star values have become a central concept in number theory with a wide variety of applications. In this article, we propose a generalization, which we will refer to as *recurrent sums*, where the reciprocals are replaced by arbitrary sequences. We introduce a toolbox of formulas for the manipulation of such sums. We begin by developing variation formulas that allow the variation of a recurrent sum of order m to be expressed in terms of lower order recurrent sums. We then proceed to derive theorems (which we will call inversion formulas) which show how to interchange the order of summation in a multitude of ways. Later, we introduce a set of new partition identities in order to then prove a reduction theorem which permits the expression of a recurrent sum in terms of a combination of non-recurrent sums. Finally, we use these theorems to derive new results for multiple zeta star values and recurrent sums of powers.
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1 Introduction and notation

The harmonic series was first studied and proven to diverge in the 14-th century by Nicole Oresme [32]. Later, in the 17th century, new proofs for this divergence were provided by Pietro Mengoli [29], Johann Bernoulli [5], and Jacob Bernoulli [3,4]. However, a more general form of this series does converge. Euler was the first to study such sums of the form:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

where s is a real number. In the famous Basel problem, Euler proved that $\zeta(2)=\dfrac{\pi^2}{6}$ (see [12, 13, 16]. Fourteen additional proofs can be found in [11]). He later provided a general formula for this zeta function for positive even values of s.

Euler's definition was then extended to a complex variable s by Riemann in his 1859 article "On the Number of Primes Less Than a Given Magnitude". More recently, the multiple harmonic series, an even more general form of the zeta function, has been introduced and studied. Note that Euler was the first to study these multiple harmonic series for length 2 in [15]. A multiple harmonic series (MHS) or multiple zeta values (MZV) is defined as:

$$\zeta(s_1,s_2,\ldots,s_k)=\sum_{1\leq N_1<N_2<\cdots<N_k}\frac{1}{N_1^{s_1}N_2^{s_2}\cdots N_k^{s_k}}.$$

A very important variant of the MHS (see [25,27,30]) often referred to as multiple zeta star values MZSV or multiple harmonic star series MHSS (or simply multiple zeta values) is defined by:

$$\zeta^{\star}(s_1,s_2,\ldots,s_k)=\sum_{1\leq N_1\leq N_2\leq\cdots\leq N_k}\frac{1}{N_1^{s_1}N_2^{s_2}\cdots N_k^{s_k}}.$$

This variant of the MHS is directly related to the Riemann zeta function $\zeta(s)$ [18,23]. Additionally, it is involved in a variety of sums and series including the Arakawa–Kaneko zeta function [37] and Euler sums.

Such sums have tremendous importance in number theory. They have been of interest to mathematicians for a long time and have been systematically studied since the 1990s with the work of Hoffman [23, 24] and Zagier [38]. However, their importance is not limited to Number Theory. In fact, such sums/series have appeared in physics even before the phrase "multiple zeta values" had been coined. As an example, the number $\zeta(\overline{6},\overline{2})$ appeared in the quantum field theory literature in 1986 [8]. They play a major role in the connection of knot theory with quantum field theory [9, 26]. MZVs and MZSVs became even more important after they became needed for higher order calculations in quantum electrodynamics (QED) and quantum chromodynamics (QCD) [6, 7].

These sums are a particular case of what we called recurrent sums as they are of the form

$$\sum_{1\leq N_1\leq\cdots\leq N_m\leq n}a_{(m);N_m}\cdots a_{(1);N_1}$$

with $a_{(i);N_i}=\frac{1}{N_i^{s_i}}$ for all i. The particular case has been extensively studied while the general case received much less interest. Although there are hundreds if not thousands of formulae to help in the study of multiple harmonic star sums and multiple zeta star values, barely any formulae can be found for its general counterpart. In this article, we are interested in studying this more general form which is expressed as follows:

$$\sum_{1\leq N_1\leq\cdots\leq N_m\leq n}a_{(m);N_m}\cdots a_{(1);N_1}.$$

We will also consider the particular case where all sequences are the same:

$$\sum_{1\leq N_1\leq\cdots\leq N_m\leq n} a_{N_m}\cdots a_{N_1}.$$

This structure of sums appears in a variety of areas of mathematics. The objective is to develop formulae to improve and facilitate the way we work with recurrent sums. This includes deriving formulae to calculate the variation of such sums, formulae to interchange the order of summation as well as formulae to represent recurrent sums in terms of a combination of non-recurrent sums. Note that this type of sums is intimately related to partitions as they appear in the representation of recurrent sums as a combination of simple non-recurrent sums. Therefore, this article will also focus on partition identities that are needed to prove the previously stated theorems as well as the ones that can be derived from these same theorems. Among these partition identities that can be found through these theorems, a definition of binomial coefficients in terms of a sum over partitions will be presented. Similarly, we produce some identities involving special sums, over partitions, of Bernoulli numbers. Furthermore, we are also interested in applying the formulae developed for the general case to some particular cases. First, we will apply our results to the multiple sums of powers in order to generalize Faulhaber's formula.

Then, we will go back to the most famous particular case which is the MZSV and show how our results on the general case can improve in this case. A particularly beautiful identity that we will present is the following which relates the recurrent sum of $\frac{1}{N^2}$ to the zeta function for positive even values (this heavily relies on Schneider's work [35]):

$$\sum_{N_m=1}^{\infty}\cdots\sum_{N_1=1}^{N_2}\frac{1}{N_m^2\cdots N_1^2}=\sum_{\sum i.y_{k,i}=m}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!\,i^{y_{k,i}}}\left(\zeta(2i)\right)^{y_{k,i}}=\left(2-\frac{1}{2^{2(m-1)}}\right)\zeta(2m).$$

Although this paper focuses on the generalized version of the multiple zeta star values, the multiple zeta values itself is a particular form of a type of sums presented in [21] and which is closely related to the recurrent sums by the relations also presented in that cited article.

The main theorems of this paper have potential applications such as the following: Surprisingly, this form appears in the general formula for the n-th integral of $x^m(\ln x)^{m'}$. In the paper [20], the relations presented in this paper are used to derive and prove this general formula for the n-th integral of $x^m(\ln x)^{m'}$. In the Part 2 of this study [21], the partition identities here presented are combined with additional partition identities in order to produce identities for odd and even partitions.

Let us now introduce some notation in order to facilitate the representation of such sums in this paper. For any $m, q, n \in \mathbb{N}$ where $n \geq q$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n]$, let $R_{m,q,n}(a_{(1);N_1}, \ldots, a_{(m);N_m})$ represent the general recurrent sum of order m for the sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ with lower and upper bounds, respectively q and n. For simplicity, however, we will denote it simply as $R_{m,q,n}$.

$$
\begin{aligned}
R_{m,q,n} &= \sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} \\
&= \sum_{N_m=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(2);N_2} a_{(1);N_1} \\
&= \sum_{q \le N_1 \le \cdots \le N_m \le n} a_{(m);N_m} \cdots a_{(2);N_2} a_{(1);N_1}.
\end{aligned} \tag{1}
$$

The most common case of a recurrent sum is that where all sequences are the same,

$$
\begin{aligned}
R_{m,q,n}(a_{N_1}, \ldots, a_{N_m}) &= \sum_{N_m=q}^{n} a_{N_m} \cdots \sum_{N_2=q}^{N_3} a_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} \\
&= \sum_{N_m=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{N_1=q}^{N_2} a_{N_m} \cdots a_{N_2} a_{N_1} \\
&= \sum_{q \le N_1 \le \cdots \le N_m \le n} a_{N_m} \cdots a_{N_2} a_{N_1}.
\end{aligned} \tag{2}
$$

For simplicity, we will denote it as $\hat{R}_{m,q,n}$. We could also denote it as $R_{m,q,n}(a_N)$.
This type of sums is described as recurrent because they can also be expressed using the following recurrent form:

$$
\begin{cases}
R_{m,q,n} &= \sum_{N_m=q}^{n} a_{(m);N_m} R_{m-1,q,N_m} \\
R_{0,i,j} &= 1 \ \forall i, j \in \mathbb{N}.
\end{cases} \tag{3}
$$

Remark 1. A recurrent sum of order 0 is always equal to 1. It is not equivalent to an empty sum (which is equal to 0).

In this paper, recurrent sums will be studied. In Section 2, formulas for the calculation of variation of these sums in terms of lower order recurrent sums will be presented. Then, in Section 3, inversion formulas will be presented, which will allow the interchange of the order of summation in such sums. Finally, in Section 4, we will present a set of partition identities as well as a reduction formula that allows the representation of a recurrent sum as a combination of simple (non-recurrent) sums. These relations will be, then, used to simplify certain special sums/series such as the recurrent sum of powers and the multiple zeta star values.

2 Variation formulas

In this section, we will develop formulas to express the variation of a recurrent sum of order m $(R_{m,q,n+1} - R_{m,q,n})$ in terms of lower order recurrent sums. Equivalently, these formulas can be used to express $R_{m,q,n+1}$ in terms of $R_{m,q,n}$ and lower order recurrent sums.

2.1 Simple expression

We start by proving the most basic form for the variation formula as illustrated by the following lemma. This is needed in order to prove the general form of this formula.

Remark 2. Although the formulas presented in this section will be for recurrent sums with a common lower bound q, they hold for recurrent sums with distinct lower bounds. It suffices to replace $R_{m,q,n}$ by $R_{m,\mathbf{q}_m,n}$ where $\mathbf{q}_m = (q_1, \ldots, q_m)$.

Lemma 2.1. *For any $m, q, n \in \mathbb{N}$, we have that*

$$R_{m+1,q,n+1} = a_{(m+1);n+1} R_{m,q,n+1} + R_{m+1,q,n}.$$

Proof.

$$\begin{aligned} R_{m+1,q,n+1} &= \sum_{N_{m+1}=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(m+1);N_{m+1}} \cdots a_{(1);N_1} \\ &= a_{(m+1);n+1} \sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(1);N_1} + \sum_{N_{m+1}=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(m+1);N_{m+1}} \cdots a_{(1);N_1}. \end{aligned}$$

Substituting the recurrent sums with the notation, we get the lemma. □

Now we apply the basic case from Lemma 2.1 to show the general variation formula that allows $R_{m,q,n+1}$ to be expressed in terms of $R_{m,q,n}$ and of recurrent sums of order going from 0 to $(m-1)$.

Theorem 2.2. *For any $m, q, n \in \mathbb{N}$ where $n \geq q$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n+1]$, we have that*

$$\sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(1);N_1} = \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) \left(\sum_{N_k=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(k);N_k} \cdots a_{(1);N_1} \right).$$

Using the notation from Eq. (1)*, this theorem can be written as*

$$R_{m,q,n+1} = \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n}.$$

Proof. <u>Base case (for $m = 1$):</u>

$$\sum_{k=0}^{1} \left(\prod_{j=0}^{-k} a_{(1-j);n+1} \right) R_{k,q,n} = (a_{(1);n+1})(1) + (1) \left(\sum_{N_1=q}^{n} a_{(1);N_1} \right) = R_{1,q,n+1}.$$

<u>Induction hypothesis:</u>

$$R_{m,q,n+1} = \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n}.$$

Induction step: From Lemma 2.1,

$$R_{m+1,q,n+1} = a_{(m+1);n+1} R_{m,q,n+1} + R_{m+1,q,n}.$$

By applying the induction hypothesis,

$$\begin{aligned} R_{m+1,q,n+1} &= a_{(m+1);n+1} \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n} + R_{m+1,q,n} \\ &= \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k} a_{(m+1-j);n+1} \right) R_{k,q,n} + R_{m+1,q,n}. \end{aligned}$$

Noticing that

$$\sum_{k=m+1}^{m+1} \left(\prod_{j=0}^{m-k} a_{(m+1-j);n+1} \right) R_{k,q,n} = R_{m+1,q,n},$$

hence, the case for $(m+1)$ is proven. Thus, the theorem is proven by induction. ☐

Corollary 2.2.1. *If all sequences are the same, Theorem 2.2 reduces to the following form:*

$$\sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{N_m} \cdots a_{N_1} = \sum_{k=0}^{m} (a_{n+1})^{m-k} \left(\sum_{N_k=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{N_k} \cdots a_{N_1} \right).$$

Using the notation from Eq. (2)*, this theorem can be written as*

$$\hat{R}_{m,q,n+1} = \sum_{k=0}^{m} (a_{n+1})^{m-k} \hat{R}_{k,q,n}.$$

Example 2.1. Consider that $m = 2$, we have the two following cases:

- If all sequences are distinct,

$$\sum_{N_2=q}^{n+1} b_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} - \sum_{N_2=q}^{n} b_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} = (b_{n+1}) \sum_{N_1=q}^{n} a_{N_1} + (b_{n+1})(a_{n+1}).$$

- If all sequences are the same,

$$\sum_{N_2=q}^{n+1} a_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} - \sum_{N_2=q}^{n} a_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} = (a_{n+1}) \sum_{N_1=q}^{n} a_{N_1} + (a_{n+1})^2.$$

Remark 3. Set $a_{(m);N} = \cdots = a_{(2);N} = 1$, Theorem 2.2 becomes

$$\sum_{N_m=q}^{n+1} \sum_{N_{m-1}=q}^{N_m} \cdots \sum_{N_1=q}^{N_2} a_{N_1} = \sum_{k=1}^{m} \left(\sum_{N_k=q}^{n} \sum_{N_{k-1}=q}^{N_k} \cdots \sum_{N_1=q}^{N_2} a_{N_1} \right) + a_{n+1}.$$

2.2 Simple recurrent expression

A recursive form of Theorem 2.2 can be obtained by expanding and factoring the theorem's expression.

Theorem 2.3. *For any $m, q, n \in \mathbb{N}$ where $n \geq q$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n+1]$, we have that*

$$\sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(1);N_1} - \sum_{N_m=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(1);N_1}$$
$$= a_{(m);n+1} \left\{ a_{(m-1);n+1} \left[\cdots a_{(2);n+1} \left(a_{(1);n+1}(1) + \sum_{N_1=q}^{n} a_{(1);N_1} \right) + \sum_{N_2=q}^{n} \sum_{N_1=q}^{N_2} a_{(2);N_2} a_{(1);N_1} \right] + \sum_{N_{m-1}=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(m-1);N_{m-1}} \cdots a_{(1);N_1} \right\}.$$

Using the notation from Eq. (1)*, this theorem can be written as*

$$R_{m,q,n+1} = a_{(m);n+1} \left\{ a_{(m-1);n+1} \left[\cdots a_{(2);n+1} \left(a_{(1);n+1} \left(R_{0,q,n} \right) + R_{1,q,n} \right) + R_{2,q,n} \right] + R_{m-1,q,n} \right\} + R_{m,q,n}$$

where $R_{0,q,n} = 1$.

Proof. Base case (for $m = 1$): From Lemma 2.1,

$$R_{1,q,n+1} = a_{(1);n+1}(R_{0,q,n+1}) + R_{1,q,n} = a_{(1);n+1}(R_{0,q,n}) + R_{1,q,n}.$$

Induction hypothesis:

$$R_{m,q,n+1} = a_{(m);n+1} \left\{ a_{(m-1);n+1} \left[\cdots a_{(2);n+1} \left(a_{(1);n+1} \left(R_{0,q,n} \right) + R_{1,q,n} \right) + R_{2,q,n} \right] + R_{m-1,q,n} \right\} + R_{m,q,n}.$$

Induction step: From Lemma 2.1, $R_{m+1,q,n+1} = a_{(m+1);n+1} R_{m,q,n+1} + R_{m+1,q,n}$.
By applying the induction hypothesis,

$$R_{m+1,q,n+1} = a_{(m+1);n+1} \left\{ a_{(m);n+1} \left[\cdots a_{(2);n+1} \left(a_{(1);n+1} \left(R_{0,q,n} \right) + R_{1,q,n} \right) + R_{2,q,n} \right] + R_{m,q,n} \right\} + R_{m+1,q,n}.$$

Hence, the theorem is proven by induction. □

Corollary 2.3.1. *If all sequences are the same, Theorem 2.3 reduces to the following form:*

$$\sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{N_m} \cdots a_{N_1} - \sum_{N_m=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{N_m} \cdots a_{N_1}$$
$$= a_{n+1} \left\{ a_{n+1} \left[\cdots a_{n+1} \left(a_{n+1}(1) + \sum_{N_1=q}^{n} a_{N_1} \right) + \sum_{N_2=q}^{n} \sum_{N_1=q}^{N_2} a_{N_2} a_{N_1} \right] + \sum_{N_{m-1}=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{N_{m-1}} \cdots a_{N_1} \right\}.$$

Using the notation from Eq. (2)*, this theorem can be written as*

$$\hat{R}_{m,q,n+1} = a_{n+1} \left\{ a_{n+1} \left[\cdots a_{n+1} \left(a_{n+1} \left(\hat{R}_{0,q,n} \right) + \hat{R}_{1,q,n} \right) + \hat{R}_{2,q,n} \right] + \hat{R}_{m-1,q,n} \right\} + \hat{R}_{m,q,n}$$

where $\hat{R}_{0,q,n} = 1$.

Example 2.2. Consider that $m = 2$, we have the two following cases:

- If all sequences are distinct,

$$\sum_{N_2=q}^{n+1} b_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} - \sum_{N_2=q}^{n} b_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} = (b_{n+1}) \left\{ a_{n+1}(1) + \sum_{N_1=q}^{n} a_{N_1} \right\}.$$

- If all sequences are the same,

$$\sum_{N_2=q}^{n+1} a_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} - \sum_{N_2=q}^{n} a_{N_2} \sum_{N_1=q}^{N_2} a_{N_1} = (a_{n+1}) \left\{ a_{n+1}(1) + \sum_{N_1=q}^{n} a_{N_1} \right\}.$$

2.3 General expression

The variation of a recurrent sum can also be expressed in terms of only a certain range of lower order recurrent sums. In other words, $R_{m,q,n+1}$ can be expressed in terms of $R_{m,q,n}$ and of recurrent sums of order going only from p to $(m-1)$. To do so, we develop the following theorem.

Theorem 2.4. *For any $m, q, n \in \mathbb{N}$ where $n \geq q$, for any $p \in [0, m]$, and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n+1]$, we have that*

$$\sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(1);N_1} = \sum_{k=p+1}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) \left(\sum_{N_k=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(k);N_k} \cdots a_{(1);N_1} \right) + \left(\prod_{j=0}^{m-p-1} a_{(m-j);n+1} \right) \left(\sum_{N_p=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(p);N_p} \cdots a_{(1);N_1} \right).$$

Using the notation from Eq. (1)*, this theorem can be written as*

$$R_{m,q,n+1} = \sum_{k=p+1}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n} + \left(\prod_{j=0}^{m-p-1} a_{(m-j);n+1} \right) R_{p,q,n+1}.$$

Proof. By applying Theorem 2.2,

$$\begin{aligned} R_{m,q,n+1} &= \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n} \\ &= \sum_{k=p+1}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n} + \sum_{k=0}^{p} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n} \\ &= \sum_{k=p+1}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n} + \left(\prod_{j=0}^{m-p-1} a_{(m-j);n+1} \right) \sum_{k=0}^{p} \left(\prod_{j=m-p}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n}. \end{aligned}$$

From Theorem 2.2, with m substituted by p, we have

$$R_{p,q,n+1} = \sum_{k=0}^{p} \left(\prod_{j=0}^{p-k-1} a_{(p-j);n+1} \right) R_{k,q,n} = \sum_{k=0}^{p} \left(\prod_{j=m-p}^{m-k-1} a_{(m-j);n+1} \right) R_{k,q,n}.$$

Hence, by substituting, we get the theorem. □

Corollary 2.4.1. *If all sequences are the same, Theorem 2.4 reduces to the following form:*

$$\sum_{N_m=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1} = \sum_{k=p+1}^{m}(a_{n+1})^{m-k}\left(\sum_{N_k=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_k}\cdots a_{N_1}\right) + (a_{n+1})^{m-p}\left(\sum_{N_p=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_p}\cdots a_{N_1}\right).$$

Using the notation from Eq. (2), *this theorem can be written as*

$$\hat{R}_{m,q,n+1} = \sum_{k=p+1}^{m}(a_{n+1})^{m-k}\,\hat{R}_{k,q,n} + (a_{n+1})^{m-p}\,\hat{R}_{p,q,n+1}.$$

Example 2.3. For $p = 2$ and if the sequences are the same:

$$\sum_{N_m=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1} = \sum_{k=3}^{m}(a_{n+1})^{m-k}\left(\sum_{N_k=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_k}\cdots a_{N_1}\right) + (a_{n+1})^{m-2}\left(\sum_{N_2=q}^{n+1}\sum_{N_1=q}^{N_2} a_{N_2}a_{N_1}\right).$$

Example 2.4. For $p = m - 2$ and if the sequences are the same:

$$\sum_{N_m=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1} - \sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1}$$
$$= (a_{n+1})\left(\sum_{N_{m-1}=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_{m-1}}\cdots a_{N_1}\right) + (a_{n+1})^2\left(\sum_{N_{m-2}=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_{m-2}}\cdots a_{N_1}\right).$$

Remark 4. Set $a_{(m);N} = \cdots = a_{(2);N} = 1$, Theorem 2.4 becomes

$$\sum_{N_m=q}^{n+1}\sum_{N_{m-1}=q}^{N_m}\cdots\sum_{N_1=q}^{N_2} a_{N_1} = \sum_{k=p+1}^{m}\left(\sum_{N_k=q}^{n}\sum_{N_{k-1}=q}^{N_k}\cdots\sum_{N_1=q}^{N_2} a_{N_1}\right) + \sum_{N_p=q}^{n+1}\sum_{N_{p-1}=q}^{N_p}\cdots\sum_{N_1=q}^{N_2} a_{N_1}.$$

2.4 General recurrent expression

Similarly, the theorem introduced in the previous section can be reformulated in a recursive form by expanding and factoring the expression of Theorem 2.4 to obtain the following expression.

Theorem 2.5. *For any* $m, q, n \in \mathbb{N}$ *where* $n \geq q$, *for any* $p \in [0, m]$, *and for any set of sequences* $a_{(1);N_1}, \ldots, a_{(m);N_m}$ *defined in the interval* $[q, n+1]$, *we have that*

$$R_{m,q,n+1} = a_{(m);n+1}\left\{a_{(m-1);n+1}\left[\cdots a_{(p+2);n+1}\left(a_{(p+1);n+1}\left(R_{p,q,n+1}\right) + R_{p+1,q,n}\right) + R_{p+2,q,n}\right] + R_{m-1,q,n}\right\} + R_{m,q,n}.$$

Proof. From Theorem 2.3, with m substituted by p, we have

$$R_{p,q,n+1} = a_{(p);n+1}\left\{a_{(p-1);n+1}\left[\cdots a_{(2);n+1}\left(a_{(1);n+1}\left(R_{0,q,n}\right) + R_{1,q,n}\right) + R_{2,q,n}\right] + R_{p-1,q,n}\right\} + R_{p,q,n}.$$

Substituting into the expression of Theorem 2.3, the inner part becomes $R_{p,q,n+1}$ and we get the desired formula. □

Corollary 2.5.1. *If all sequences are the same, Theorem 2.5 reduces to the following form:*
$\hat{R}_{m,q,n+1} = a_{n+1}\left\{a_{n+1}\left[\cdots a_{n+1}\left(a_{n+1}\left(\hat{R}_{p,q,n+1}\right)+\hat{R}_{p+1,q,n}\right)+\hat{R}_{p+2,q,n}\right]+\hat{R}_{m-1,q,n}\right\}+\hat{R}_{m,q,n}.$

Example 2.5. For $p = 1$ and if the sequences are the same:

$$\sum_{N_m=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1} - \sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1}$$
$$= a_{n+1}\left\{a_{n+1}\left[\cdots a_{n+1}\left(\sum_{N_1=q}^{n+1} a_{N_1}\right)+\sum_{N_2=q}^{n}\sum_{N_1=q}^{N_2} a_{N_2}a_{N_1}\right]+\sum_{N_{m-1}=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_{m-1}}\cdots a_{N_1}\right\}.$$

Example 2.6. For $p = m-2$ and if the sequences are the same:

$$\sum_{N_m=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1} - \sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1}$$
$$= a_{n+1}\left\{a_{n+1}\left[\sum_{N_{m-2}=q}^{n+1}\cdots\sum_{N_1=q}^{N_2} a_{N_{m-2}}\cdots a_{N_1}\right]+\sum_{N_{m-1}=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_{m-1}}\cdots a_{N_1}\right\}.$$

3 Inversion formulas

In this section, we develop formulas to interchange the order of summation in a recurrent sum.

3.1 Particular case (for two sequences)

We start by proving the inversion formula with two sequences which is required in order to prove the more general inversion formula with m sequences.

Theorem 3.1. *For $q, n \in \mathbb{N}$ where $n \geq q$ and for any two sequences a_{N_1} and b_{N_2} defined in the interval $[q, n]$, we have that*

$$\sum_{N_2=q}^{n} b_{N_2}\sum_{N_1=q}^{N_2} a_{N_1} = \sum_{N_1=q}^{n} a_{N_1}\sum_{N_2=N_1}^{n} b_{N_2}.$$

Proof. Let us note the left term of the theorem as B. By expanding the sum, we get

$$B = b_q\left(\sum_{N_1=q}^{q} a_{N_1}\right)+b_{q+1}\left(\sum_{N_1=q}^{q+1} a_{N_1}\right)+\cdots+b_{n-1}\left(\sum_{N_1=q}^{n-1} a_{N_1}\right)+b_n\left(\sum_{N_1=q}^{n} a_{N_1}\right)$$
$$= b_q\left(a_q\right)+b_{q+1}\left(a_q+a_{q+1}\right)+\cdots+b_{n-1}\left(a_q+\cdots+a_{n-1}\right)+b_n\left(a_q+\cdots+a_n\right).$$

By regrouping the b_N terms instead of the a_N terms, the expression becomes

$$B = a_q\left(b_q+\cdots+b_n\right)+a_{q+1}\left(b_{q+1}+\cdots+b_n\right)+\cdots+a_{n-1}\left(b_{n-1}+b_n\right)+a_n\left(b_n\right)$$
$$= a_q\left(\sum_{N_2=q}^{n} b_{N_2}\right)+a_{q+1}\left(\sum_{N_2=q+1}^{n} b_{N_2}\right)+\cdots+a_{n-1}\left(\sum_{N_2=n-1}^{n} b_{N_2}\right)+a_n\left(\sum_{N_2=n}^{n} b_{N_2}\right)$$
$$= \sum_{N_1=q}^{n} a_{N_1}\sum_{N_2=N_1}^{n} b_{N_2}.$$

This completes the proof. □

3.2 General case (for m sequences)

We now prove the more general inversion formula with m sequences which allows us to invert the order of summation for a recurrent sum of order m.

Theorem 3.2. *For any $m, q, n \in \mathbb{N}$ where $n \geq q$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n]$, we have that*

$$\sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} = \sum_{N_1=q}^{n} a_{(1);N_1} \sum_{N_2=N_1}^{n} a_{(2);N_2} \cdots \sum_{N_m=N_{m-1}}^{n} a_{(m);N_m}.$$

Proof. Base case (for $m = 2$): This statement is true as proven in Theorem 3.1.
Induction hypothesis:

$$\sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} = \sum_{N_1=q}^{n} a_{(1);N_1} \sum_{N_2=N_1}^{n} a_{(2);N_2} \cdots \sum_{N_m=N_{m-1}}^{n} a_{(m);N_m}.$$

Induction step: To simplify the proof, we use the notation from Eq. (1) to represent the left hand side term:

$$\begin{aligned} R_{m+1,q,n} &= \sum_{N_{m+1}=q}^{n} a_{(m+1);N_{m+1}} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} \\ &= \sum_{N_{m+1}=q}^{n} a_{(m+1);N_{m+1}} \left(\sum_{N_m=q}^{N_{m+1}} a_{(m);N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} \right). \end{aligned}$$

Let b_{N_m} be the following sequence (that dependents only on N_m),

$$b_{N_m} = a_{(m);N_m} \sum_{N_{m-1}=q}^{N_m} a_{(m-1);N_{m-1}} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1}.$$

By applying this substitution in the previous expression, we obtain a recurrent sum of order 2 that contains the two sequences $a_{(m+1);N_{m+1}}$ and b_{N_m}. Then, we apply the inversion formula for the case of two sequences (Theorem 3.1) to get the following,

$$\begin{aligned} R_{m+1,q,n} &= \sum_{N_{m+1}=q}^{n} a_{(m+1);N_{m+1}} \left(\sum_{N_m=q}^{N_{m+1}} b_{N_m} \right) \\ &= \sum_{N_m=q}^{n} b_{N_m} \left(\sum_{N_{m+1}=N_m}^{n} a_{(m+1);N_{m+1}} \right) \\ &= \sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} \left(\sum_{N_{m+1}=N_m}^{n} a_{(m+1);N_{m+1}} \right). \end{aligned}$$

The sum of $a_{(m+1);N_{m+1}}$ has N_m and n as lower and upper bounds. Thus, knowing that n is a constant, the sum of $a_{(m+1);N_{m+1}}$ depends only on N_m. This allows us to extract this sum from the inner sums to get

$$R_{m+1,q,n} = \sum_{N_m=q}^{n} \left(a_{(m);N_m} \sum_{N_{m+1}=N_m}^{n} a_{(m+1);N_{m+1}} \right) \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1}.$$

Let A_{N_m} be the following sequence (that only depends on N_m),

$$A_{N_m} = a_{(m);N_m} \sum_{N_{m+1}=N_m}^{n} a_{(m+1);N_{m+1}}.$$

By substituting A_{N_m} into the previous expression, we get a recurrent sum of order m in terms of the following m sequences: $A_{N_m}, a_{(m-1);N_{m-1}}, \ldots, a_{(1);N_1}$. Then the inversion formula for the case of m sequences (which was assumed to be true in the induction hypothesis) is applied,

$$\begin{aligned} R_{m+1,q,n} &= \sum_{N_m=q}^{n} A_{N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} \\ &= \sum_{N_1=q}^{n} a_{(1);N_1} \sum_{N_2=N_1}^{n} a_{(2);N_2} \cdots \sum_{N_m=N_{m-1}}^{n} A_{N_m} \\ &= \sum_{N_1=q}^{n} a_{(1);N_1} \sum_{N_2=N_1}^{n} a_{(2);N_2} \cdots \sum_{N_m=N_{m-1}}^{n} a_{(m);N_m} \sum_{N_{m+1}=N_m}^{n} a_{(m+1);N_{m+1}}. \end{aligned}$$

We conclude that it must hold for all $m \geq 2$. □

Similarly, the innermost summation can be turned into the outermost summation as illustrated by Theorem 3.3.

Theorem 3.3. *For any $m, q, n \in \mathbb{N}$ where $n \geq q$ and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n]$, we have that*

$$\begin{aligned} &\sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} \\ &\quad = \sum_{N_1=q}^{n} a_{(1);N_1} \sum_{N_m=N_1}^{n} a_{(m);N_m} \cdots \sum_{N_3=N_1}^{N_4} a_{(3);N_3} \sum_{N_2=N_1}^{N_3} a_{(2);N_2}. \end{aligned}$$

Proof. From Theorem 3.2,

$$\sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} = \sum_{N_1=q}^{n} a_{(1);N_1} \sum_{N_2=N_1}^{n} a_{(2);N_2} \cdots \sum_{N_m=N_{m-1}}^{n} a_{(m);N_m}.$$

Applying Theorem 3.2 to the inner part of the right-hand side sum would transform it as follows

$$\sum_{N_2=N_1}^{n} a_{(2);N_2} \cdots \sum_{N_m=N_{m-1}}^{n} a_{(m);N_m} = \sum_{N_m=N_1}^{n} a_{(m);N_m} \cdots \sum_{N_3=N_1}^{N_4} a_{(3);N_3} \sum_{N_2=N_1}^{N_3} a_{(2);N_2}.$$

Hence, substituting back into Theorem 3.2 would give us the desired formula. □

3.3 Inversion of p sequences from m sequences

Finally, as we will show in this section, it is possible to partially invert the order of summation for a recurrent sum. In other words, as shown by the following theorem, it is possible to invert the order of summation of only the p innermost summations from m summations.

Theorem 3.4. *For any $m, q, n \in \mathbb{N}$ where $n \geq q$, for any $p \in [1, m]$, and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n]$, we have that*

$$\sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_p=q}^{N_{p+1}} a_{(p);N_p} \cdots \sum_{N_1=q}^{N_2} a_{(1);N_1}$$
$$= \sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_{p+1}=q}^{N_{p+2}} a_{(p+1);N_{p+1}} \sum_{N_1=q}^{N_{p+1}} a_{(1);N_1} \sum_{N_2=N_1}^{N_{p+1}} a_{(2);N_2} \cdots \sum_{N_p=N_{p-1}}^{N_{p+1}} a_{(p);N_p}.$$

Proof. By replacing m by p and n by N_{p+1} in Theorem 3.2, we get the following relation,

$$\sum_{N_p=q}^{N_{p+1}} a_{(p);N_p} \cdots \sum_{N_2=q}^{N_3} a_{(2);N_2} \sum_{N_1=q}^{N_2} a_{(1);N_1} = \sum_{N_1=q}^{N_{p+1}} a_{(1);N_1} \sum_{N_2=N_1}^{N_{p+1}} a_{(2);N_2} \cdots \sum_{N_p=N_{p-1}}^{N_{p+1}} a_{(p);N_p}.$$

Thus, substituting into the left-hand side term of the theorem, we prove the theorem. □

Similarly, the innermost summation can be pulled back to the p-th position as illustrated by Theorem 3.5.

Theorem 3.5. *For any $m, q, n \in \mathbb{N}$ where $n \geq q$, for any $p \in [1, m]$, and for any set of sequences $a_{(1);N_1}, \ldots, a_{(m);N_m}$ defined in the interval $[q, n]$, we have that*

$$\sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_p=q}^{N_{p+1}} a_{(p);N_p} \cdots \sum_{N_1=q}^{N_2} a_{(1);N_1}$$
$$= \sum_{N_m=q}^{n} a_{(m);N_m} \cdots \sum_{N_{p+1}=q}^{N_{p+2}} a_{(p+1);N_{p+1}} \sum_{N_1=q}^{N_{p+1}} a_{(1);N_1} \sum_{N_p=N_1}^{N_{p+1}} a_{(p);N_p} \sum_{N_{p-1}=N_1}^{N_p} a_{(p-1);N_{p-1}} \cdots \sum_{N_2=N_1}^{N_3} a_{(2);N_2}.$$

Proof. By applying Theorem 3.3 (with m substituted by p and n substituted by N_{p+1}) to Theorem 3.4, we get the desired theorem. □

4 Reduction formulas

The objective of this section is to introduce formulas which can be used to reduce recurrent sums from their originally recurrent form to a form containing only simple non-recurrent sums.

4.1 A brief introduction to partitions

In this paper, partitions are involved in the reduction formula for a recurrent sum. For this reason, in this section, we will present a brief introduction to partitions.

Definition. *A partition of a non-negative integer m is a set of positive integers whose sum equals m. The summands of the partition are known as parts and the number of parts is referred to as the length of the partition. The sum of the parts is called the content of the partition.*

We can represent a partition of m as an ordered set $(y_{k,1},\ldots,y_{k,m})$ that verifies

$$y_{k,1}+2y_{k,2}+\cdots+my_{k,m}=\sum_{i=1}^{m} i\,y_{k,i}=m. \tag{4}$$

The coefficient $y_{k,i}$ is the multiplicity of the integer i in the k-th partition of m. Note that $0\le y_{k,i}\le m$ while $1\le i\le m$. Also note that the number of partitions of an integer m is given by the partition function denoted $p(m)$ and hence, $1\le k\le p(m)$. In the remainder of this text, the subscript k will be added to indicate that a given parameter is associated with a given partition. The value of $p(m)$ is obtained from the generating function developed by Euler in the mid-eighteen century [14],

$$\sum_{m=0}^{\infty} p(m)x^m=\prod_{j=1}^{\infty}\frac{1}{1-x^j}. \tag{5}$$

Other ways of computing $p(m)$ include Euler's recurrent formula, Hardy and Ramanujan's asymptotic expression [22], Rademacher's formula [34], and, most recently, Ono and Bruinier's formula for $p(m)$ as a finite sum [10].

Throughout this article, we will use the following standard notation for partitions: We denote the set of all partitions of an integer m as $\mathcal{P}_m$. We use $\pi\vdash m$ for the relation "partition π sums to m". The content of a partition π is denoted by $c(\pi)$. Notice that $\pi\in\mathcal{P}_m$, $\pi\vdash m$, and $c(\pi)=m$ mean the same thing. Likewise, we use $\#(\pi)$ for the number of parts of the partition π. $\#(\pi)$ can also be seen as the sum of the multiplicities:

$$\#(\pi)=y_{k,1}+y_{k,2}+\cdots+y_{k,m}=\sum_{i=1}^{m} y_{k,i}. \tag{6}$$

Remark 5. By convention, zero has only one partition ($p(0)=1$). This partition has length zero (if $c(\pi)=0$, then $\#(\pi)=0$).

We will be dealing with sums of the form $\sum f(\pi)$, that is, sums over all partitions π of an integer. We indicate that a sum is over all partitions of m by the index $\pi(m)$. We restrict the partitions being summed over to those of length r by adding the index $\#(\pi)=r$. For simplicity, we define $\sum f(i)$ to mean $\sum_{i=1}^{m} f(i)$. In particular, $\sum i.y_{k,i}=\sum_{i=1}^{m} i.y_{k,i}$ and $\sum y_{k,i}=\sum_{i=1}^{m} y_{k,i}$.

Moreover, the most famous ways of representing a partition are using Ferrers diagrams or using Young diagrams. There also exists some variants of Ferrers diagrams that are used [33].

Remark 6. For readers intrested in a more detailed explanation of partitions, see [1].

Before we can proceed to the next section, we need to introduce some combinatorical and number theoretical concepts that will be crucial when working with partitions. We begin by defining the following notation: Let $[x^r]\,(P(x))$ represent the coefficient of x^r in $P(x)$. Let $x^{\overline{m}}=x(x+1)\cdots(x+m-1)$ represent the rising factorial. Let $(x)_m=x(x-1)\cdots(x-m+1)$ represent the falling factorial. Now that the needed notation has been presented, we introduce the Stirling numbers.

The original definition of Stirling numbers of the first kind $S(m, r)$ [28] was as the coefficients in the expansion of $(x)_m$:

$$(x)_m = \sum_{k=0}^{m} S(m, k)x^k \quad \text{or} \quad S(m, r) = [x^r](x)_m. \tag{7}$$

In a similar way, the unsigned Stirling numbers of the first kind, denoted $|S(m, r)|$ or $\left[{m \atop r}\right]$, can be expressed in terms of the rising factorial $x^{\overline{m}}$:

$$x^{\overline{m}} = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} x^k \quad \text{or} \quad \begin{bmatrix} m \\ r \end{bmatrix} = [x^r]\left(x^{\overline{m}}\right). \tag{8}$$

From this definition, the famous finite sum of the unsigned Stirling numbers of the first kind can be directly deduced by substituting x by 1 to get

$$\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} = 1(1+1)\cdots(1+m-1) = m!. \tag{9}$$

Note that $|S(m, r)|$ can also be defined as the number of permutations of m elements with r disjoint cycles. Similarly, the previous relation can be obtained by noticing that permutations are partitioned by number of cycles.

The final concept we need to introduce is partial Bell polynomials. A Bell polynomial is defined as

$$\begin{aligned} &B_{m,r}(x_1, x_2, \ldots, x_{m-r+1}) \\ &= \sum_{\substack{y_1+2y_2+\cdots+(m-r+1)y_{m-r+1}=m \\ y_1+y_2+\cdots+y_{m-r+1}=r}} \frac{m!}{y_1!\, y_2! \cdots y_{m-r+1}!} \left(\frac{x_1}{1!}\right)^{y_1} \left(\frac{x_2}{2!}\right)^{y_2} \cdots \left(\frac{x_{m-r+1}}{(m-r+1)!}\right)^{y_{m-r+1}}. \end{aligned}$$

These polynomials can also be rewritten more compactly as

$$B_{m,r}(x_1, x_2, \ldots, x_{m-r+1}) = m! \sum_{\substack{\pi(m) \\ \#(\pi)=r}} \prod_{i=1}^{m-r+1} \frac{1}{y_{k,i}!} \left(\frac{x_i}{i!}\right)^{y_{k,i}}. \tag{10}$$

Similarly, the complete Bell polynomial can be defined in terms of the partial Bell polynomial by the following relation:

$$B_m(x_1, \ldots, x_m) = \sum_{r=0}^{m} B_{m,r}(x_1, \ldots, x_{m-r+1}), \tag{11}$$

where $B_{0,0} = 1$, $B_{m,0} = 0$ for $m \geq 1$, and $B_{0,r} = 0$ for $r \geq 1$. Also, note that $B_0 = 1$.

4.2 Reduction Theorem and partition identities

We will start this section by proving several lemmas which are needed in order to prove the main theorem of this section (Theorem 4.6, which we will call the Reduction Theorem). However, some of these lemmas are important on their own as they provide relations governing partitions.

We start by presenting the following trivial remark.

Remark 7. *No partition of a non-negative integer m constructed from a sum of r terms (positive integers) can contain an integer larger or equal to $m - r + 2$.*

Now, we can proceed with proving the required lemmas.

Lemma 4.1. *Let m and r be two non-negative integers with $r \leq m$, the following sum over partitions of m of length r can be expressed in terms of the unsigned Stirling numbers of the first kind as follows,*

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = \frac{1}{m!}\begin{bmatrix} m \\ r \end{bmatrix} = \frac{1}{m!}[x^r]\left(x^{\overline{m}}\right).$$

Proof. A property of Bell polynomials, shown in [36], is that the value of the partial Bell polynomial on the sequence of factorials equals an unsigned Stirling number of the first kind,

$$B_{m,r}(0!\,, 1!\,, \dots, (m-r)!\,) = |S(m,r)| = \begin{bmatrix} m \\ r \end{bmatrix}.$$

Likewise, by a numerical substitution into the definition of partial Bell polynomials (Eq. (10)),

$$B_{m,r}(0!\,, 1!\,, \dots, (m-r)!\,) = m! \sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m-r+1} \frac{1}{i^{y_{k,i}}(y_{k,i})!}.$$

Hence, by equating, we get

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m-r+1} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = \frac{1}{m!}\begin{bmatrix} m \\ r \end{bmatrix}.$$

As we stated in a previous remark, the biggest integer that can appear in a partition of an integer m using r terms is $m-r+1$ (which means that $y_{k,m-r} = \cdots = y_{k,m} = 0$). Thus, we get

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = \sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m-r+1} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = \frac{1}{m!}\begin{bmatrix} m \\ r \end{bmatrix}.$$

The second expression follows from the definition of unsigned Stirling numbers of the first kind. □

By adding the arguments of the sum from Lemma 4.1 for all possible partition lengths, we obtain the following identity.

Lemma 4.2. *Let m be a non-negative integer, the following sum over all partitions of m can be shown to equal 1 independently of the value of m,*

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = 1.$$

Proof. From Lemma 4.1, we have

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}}(y_{k,i})!} = \frac{1}{m!}\begin{bmatrix} m \\ r \end{bmatrix}.$$

Hence,

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!}=\sum_{r=0}^{m}\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}}(y_{k,i})!}=\sum_{r=0}^{m}\frac{1}{m!}\begin{bmatrix}m\\ r\end{bmatrix}=\frac{1}{m!}\sum_{r=0}^{m}\begin{bmatrix}m\\ r\end{bmatrix}.$$

Using Eq. (9), we obtain the lemma. □

A more general form of Lemma 4.1 is illustrated in the following lemma.

Lemma 4.3. *Let* $(y_{k,1},\ldots,y_{k,m})=\{(y_{1,1},\ldots,y_{1,m}),(y_{2,1},\ldots,y_{2,m}),\ldots\}$ *be the set of all partitions of* m*. Let* $(\ell_1,\ldots,\ell_m)$ *be a partition of* $j\leq m$ *such that* $\sum\ell_i=\ell$*.*

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ \#(\pi)=r\\ y_{k,i}\geq\ell_i}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\frac{1}{(m-j)!}\begin{bmatrix}m-j\\ r-\ell\end{bmatrix}\prod_{i=1}^{m}\frac{1}{i^{\ell_i}(\ell_i)!}.$$

Remark 8. Knowing that the largest element of a partition of j is at most j, we can rewrite it as

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ \#(\pi)=r\\ y_{k,i}\geq\ell_i}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\frac{1}{(m-j)!}\begin{bmatrix}m-j\\ r-\ell\end{bmatrix}\prod_{i=1}^{j}\frac{1}{i^{\ell_i}(\ell_i)!}.$$

Proof. We split the sum as follows,

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ \#(\pi)=r\\ \exists i,y_{k,i}<\ell_i}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}+\sum_{\substack{\pi(m)\\ \#(\pi)=r\\ y_{k,i}\geq\ell_i}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}.$$

Knowing that $\binom{y_{k,i}}{\ell_i}=0$, if $y_{k,i}<\ell_i$, then the elements of this sum for which $\exists i,y_{k,i}<\ell_i$ are zero. Thus, the first term is zero and we obtain the first equality of the lemma.

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{1}{i^{\ell_i}\ell_i!}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}-\ell_i}(y_{k,i}-\ell_i)!}.$$

As $\ell_1,\ldots,\ell_m$ are all constants, then $\prod_{i=1}^{m}\frac{1}{i^{\ell_i}\ell_i!}$ is constant. This factor is constant and is common to all terms of the sum, therefore, we can factor it and take it outside the sum.

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\left(\prod_{i=1}^{m}\frac{1}{i^{\ell_i}\ell_i!}\right)\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{1}{i^{y_{k,i}-\ell_i}(y_{k,i}-\ell_i)!}.$$

Having that $(\ell_1,\ldots,\ell_m)$ is a partition of $j\leq m$, hence, $\sum i.\ell_i=j\leq m$. Additionally, $\pi(m)$ corresponds to $\sum i.y_{k,i}=m$. Thus, the condition $\pi(m)$ can be replaced by $\sum i.(y_{k,i}-\ell_i)=\sum i.y_{k,i}-\sum i.\ell_i=m-j$. Similarly, $\ell=\sum\ell_i$ and $\#(\pi)=r$ corresponds to $\sum y_{k,i}=r$, hence, the condition $\#(\pi)=r$ can be replaced by $\sum(y_{k,i}-\ell_i)=\sum y_{k,i}-\sum\ell_i=r-\ell$. Hence, letting $Y_{k,i}=y_{k,i}-\ell_i$, we define the partitions $\Pi\equiv(Y_{k,1},\ldots,Y_{k,m-j})$ of $m-j$ with lengths specified by $r-\ell$. Hence, we can write:

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \left(\prod_{i=1}^{m} \frac{1}{i^{\ell_i}\ell_i!}\right) \sum_{\substack{\sum i.Y_{k,i}=m-j\\ \sum Y_{k,i}=r-\ell}} \prod_{i=1}^{m} \frac{1}{i^{Y_{k,i}}Y_{k,i}!} = \left(\prod_{i=1}^{m} \frac{1}{i^{\ell_i}\ell_i!}\right) \sum_{\substack{\Pi(m-j)\\ \#(\Pi)=r-\ell}} \prod_{i=1}^{m} \frac{1}{i^{Y_{k,i}}Y_{k,i}!}.$$

Knowing that the largest element of a partition of $(m-j)$ is at most $(m-j)$, hence,

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \left(\prod_{i=1}^{m} \frac{1}{i^{\ell_i}\ell_i!}\right) \sum_{\substack{\Pi(m-j)\\ \#(\Pi)=r-\ell}} \prod_{i=1}^{m-j} \frac{1}{i^{Y_{k,i}}Y_{k,i}!}.$$

Applying Lemma 4.1, with $y_{k,i}$ substituted by $Y_{k,i}$, m substituted by $m-j$, and r substituted by $r-\ell$, we get the second equality of the lemma. $\square$

Remark 9. If $j > m$, then $\sum i.Y_{k,i} = m - j < 0$, which makes Lemma 4.1 invalid, which then makes this lemma invalid.

Similarly, a more general form of Lemma 4.2 is illustrated in the following lemma.

Lemma 4.4. *Let* $(y_{k,1}, \ldots, y_{k,m}) = \{(y_{1,1}, \ldots, y_{1,m}), (y_{2,1}, \ldots, y_{2,m}), \ldots\}$ *be the set of all partitions of* m. *Let* $(\ell_1, \ldots, \ell_m)$ *be a partition of* $j \leq m$.

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \sum_{\substack{\pi(m)\\ y_{k,i}\geq \ell_i}} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \prod_{i=1}^{m} \frac{1}{i^{\ell_i}(\ell_i)!}.$$

Remark 10. Knowing that the largest element of a partition of j is at most j, we can rewrite it as

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \sum_{\substack{\pi(m)\\ y_{k,i}\geq \ell_i}} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \prod_{i=1}^{j} \frac{1}{i^{\ell_i}(\ell_i)!}.$$

Proof. We split the sum as follows,

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \sum_{\substack{\pi(m)\\ \exists i, y_{k,i}<\ell_i}} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} + \sum_{\substack{\pi(m)\\ y_{k,i}\geq \ell_i}} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}.$$

Knowing that $\binom{y_{k,i}}{\ell_i} = 0$, if $y_{k,i} < \ell_i$, then the elements of this sum for which $\exists i, y_{k,i} < \ell_i$ are zero. Thus, the first term is zero and we obtain the first equality of the lemma.

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{i^{\ell_i}\ell_i!} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}-\ell_i}(y_{k,i}-\ell_i)!}.$$

As $\ell_1, \ldots, \ell_m$ are all constants, then $\prod_{i=1}^{m} \frac{1}{i^{\ell_i}\ell_i!}$ is constant. This factor is constant and is common to all terms of the sum, therefore, we can factor it and take it outside the sum.

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!} = \left(\prod_{i=1}^{m} \frac{1}{i^{\ell_i}\ell_i!}\right) \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{i^{y_{k,i}-\ell_i}(y_{k,i}-\ell_i)!}.$$

Having that $(\ell_1, \ldots, \ell_m)$ is a partition of $j \leq m$, hence, $\sum i.\ell_i = j \leq m$. Additionally, the condition $\pi(m)$ is equivalent to $\sum i.y_{k,i} = m$. Thus, the condition $\pi(m)$ can be replaced by $\sum i.(y_{k,i} - \ell_i) = \sum i.y_{k,i} - \sum i.\ell_i = m - j (\geq 0)$. Hence, we define the set of partitions Π of $m - j$ were the multiplicities of the parts are given by $Y_{k,i} = y_{k,i} - \ell_i$. Thus, we have:

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\left(\prod_{i=1}^{m}\frac{1}{i^{\ell_i}\ell_i!}\right)\sum_{\sum i.Y_{k,i}=m-j}\prod_{i=1}^{m}\frac{1}{i^{Y_{k,i}}Y_{k,i}!}=\left(\prod_{i=1}^{m}\frac{1}{i^{\ell_i}\ell_i!}\right)\sum_{\Pi(m-j)}\prod_{i=1}^{m}\frac{1}{i^{Y_{k,i}}Y_{k,i}!}.$$

Knowing that the largest element of a partition of $(m-j)$ is at most $(m-j)$, hence,

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{\binom{y_{k,i}}{\ell_i}}{i^{y_{k,i}}(y_{k,i})!}=\left(\prod_{i=1}^{m}\frac{1}{i^{\ell_i}\ell_i!}\right)\sum_{\Pi(m-j)}\prod_{i=1}^{m-j}\frac{1}{i^{Y_{k,i}}Y_{k,i}!}.$$

Applying Lemma 4.2, with $y_{k,i}$ substituted by $Y_{k,i}$ and m substituted by $m-j$, we get the second equality of the lemma. $\square$

Remark 11. If $j>m$, then $\sum i.Y_{k,i}=m-j<0$, which makes Lemma 4.2 invalid, which then makes this lemma invalid.

Proposition 4.1. *Let $B_{m,r}(x_1,\ldots,x_{m-r+1})$ be the partial Bell polynomial and $B_m(x_1,\ldots,x_m)$ be the complete Bell polynomial. Then*

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}=\frac{1}{m!}\sum_{r=0}^{m}B_{m,r}(x_1,\ldots,x_{m-r+1})=\frac{1}{m!}B_m(x_1,\ldots,x_m)$$

where $x_i=(i-1)!\,(\sum_{N=q}^{n}(a_N)^i)$.

Proof. Knowing that the largest integer that can appear in a partition of an integer m using r terms is $m-r+1$, we can write

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}=\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m-r+1}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

We can notice that the right-hand side term of the previous expression corresponds to a multiple of a special value of the partial Bell polynomial where $x_i=(i-1)!\,(\sum_{N=q}^{n}(a_N)^i),\forall i\in[1,m]$. Hence,

$$\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}=\frac{1}{m!}B_{m,r}(x_1,\ldots,x_{m-r+1}).$$

Additionally, the sum over the partitions of m is equivalent to the sum for r going from 0 to m of the sums over the partitions of m of length r. Thus,

$$\begin{aligned}\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}&=\sum_{r=0}^{m}\sum_{\substack{\pi(m)\\ \#(\pi)=r}}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}\\&=\frac{1}{m!}\sum_{r=0}^{m}B_{m,r}(x_1,\ldots,x_{m-r+1}).\end{aligned}$$

Applying the definition of a complete Bell polynomial (Eq. (11)), we get the second equality. $\square$

Finally, we introduce a lemma that expresses a product of sums as a sum of products.

Lemma 4.5. *A product of sums can be turned into a sum of products as follows,*

$$\prod_{j=1}^{m}\sum_{N_j=q_j}^{n_j} a_{(j);N_j} = \sum_{N_m=q_m}^{n_m}\cdots\sum_{N_1=q_1}^{n_1}\prod_{j=1}^{m} a_{(j);N_j}.$$

Proof. We begin by rewriting the right-hand side term as follows,

$$\sum_{N_m=q_m}^{n_m}\cdots\sum_{N_1=q_1}^{n_1}\prod_{j=1}^{m} a_{(j);N_j} = \sum_{N_m=q_m}^{n_m} a_{(m);N_m}\cdots\sum_{N_2=q_2}^{n_2} a_{(2);N_2}\sum_{N_1=q_1}^{n_1} a_{(1);N_1}.$$

Knowing that the index of each sum is independent of that of the other sums, we can split this structure as follows:

$$\sum_{N_m=q_m}^{n_m}\cdots\sum_{N_1=q_1}^{n_1}\prod_{j=1}^{m} a_{(j);N_j} = \left(\sum_{N_m=q_m}^{n_m} a_{(m);N_m}\right)\cdots\left(\sum_{N_1=q_1}^{n_1} a_{(1);N_1}\right) = \prod_{j=1}^{m}\sum_{N_j=q_j}^{n_j} a_{(j);N_j}.$$

This completes the proof. □

Example 4.1. $(a_1+a_2+a_3)(b_1+b_2) = \sum_{N_2=1}^{2}\sum_{N_1=1}^{3} a_{N_1} b_{N_2}$.

Now that all the required lemmas have been proven, we show the following theorem which allows the representation of a recurrent sum in terms of non-recurrent sums.

Theorem 4.6 (Reduction theorem)**.** *Let m be a non-negative integer, k be the index of the k-th partition of m $(1 \le k \le p(m))$, i be an integer between 1 and m, and $y_{k,i}$ be the multiplicity of i in the k-th partition of m. The reduction theorem for recurrent sums is stated as follows:*

$$\sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1} = \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

Proof. <u>Base case (for $n=q$, $\forall m \in \mathbb{N}$):</u>

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{q}(a_N)^i\right)^{y_{k,i}} = \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!\, i^{y_{k,i}}}(a_q)^{i.y_{k,i}} = (a_q)^m\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!\, i^{y_{k,i}}}.$$

By applying Lemma 4.2, we get

$$\sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{q}(a_N)^i\right)^{y_{k,i}} = (a_q)^m = \sum_{N_m=q}^{q}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1}.$$

<u>Induction hypothesis (for n, $\forall m \in \mathbb{N}$):</u>

$$\sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{N_m}\cdots a_{N_1} = \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n}(a_N)^i\right)^{y_{k,i}}.$$

<u>Induction step:</u> To be concise, we denote by I the right term of the equality to be proven, i.e.,

$$I = \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!}\left(\frac{1}{i}\sum_{N=q}^{n+1}(a_N)^i\right)^{y_{k,i}} = \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!\, i^{y_{k,i}}}\left(\sum_{N=q}^{n}(a_N)^i + (a_{n+1})^i\right)^{y_{k,i}}.$$

By applying the binomial theorem, we get:

$$I = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \sum_{v=0}^{y_{k,i}} \binom{y_{k,i}}{v} \left(\sum_{N=q}^{n} (a_N)^i \right)^v \left((a_{n+1})^i \right)^{y_{k,i}-v}$$

$$= \sum_{\pi(m)} \prod_{i=1}^{m} \sum_{v=0}^{y_{k,i}} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v} \left(\sum_{N=q}^{n} (a_N)^i \right)^v (a_{n+1})^{i.y_{k,i}-i.v}.$$

Let

$$A_{v,i,k} = \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v} \left(\sum_{N=q}^{n} (a_N)^i \right)^v (a_{n+1})^{i.y_{k,i}-i.v}.$$

By expanding then regrouping (or by using Lemma 4.5), it can be seen that

$$\prod_{i=1}^{m} \sum_{v=0}^{y_{k,i}} A_{v,i,k} = \sum_{v_m=0}^{y_{k,m}} \cdots \sum_{v_1=0}^{y_{k,1}} \prod_{i=1}^{m} A_{v_i,i,k}.$$

This is because, for any given k, by expanding the product of sums (the left-hand side term), we will get a sum of products of the form $A_{v_1,1}A_{v_2,2}\cdots A_{v_m,m}$ ($\prod_{i=1}^{m} A_{v_i,i}$) for all combinations of $v_1, v_2, \ldots, v_m$ such that $0 \le v_1 \le y_{k,1}, \ldots, 0 \le v_m \le y_{k,m}$, which is equivalent to the right-hand side term. Hence,

$$I = \sum_{\pi(m)} \sum_{v_m=0}^{y_{k,m}} \cdots \sum_{v_1=0}^{y_{k,1}} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} (a_{n+1})^{i.y_{k,i}-i.v_i}.$$

A more compact way of writing the repeated sum over the v_i's is by expressing it with one sum that combines all the conditions. The set of conditions $0 \le v_1 \le y_{k,1}, \ldots, 0 \le v_m \le y_{k,m}$ can be expressed as the condition $0 \le v_i \le y_{k,i}$ for $i \in [1, m]$.

$$I = \sum_{\pi(m)} \sum_{0 \le v_i \le y_{k,i}} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} (a_{n+1})^{i.y_{k,i}-i.v_i}.$$

Similarly, let j represent $\sum i.v_i$. Hence, we can add the trivial condition that is $j = \sum i.v_i$ to the sum over v_i. This condition is equivalent to the condition $\Pi(j)$ which refers to the sum being over all partitions Π of j. Additionally,

- $\sum i.v_i = j$ is minimal when $v_1 = 0, \ldots, v_m = 0$. Hence $j_{min} = 0$.
- $\sum i.v_i = j$ is maximal when $v_1 = y_{k,1}, \ldots, v_m = y_{k,m}$. Hence $j_{max} = \sum i.y_{k,i} = m$.

Therefore, we have that $0 \le j \le m$ or, equivalently, that j can go from 0 to m. Hence, knowing that adding a true statement to a condition does not change the condition, we can add this additional condition to get

$$I = \sum_{\pi(m)} \sum_{\substack{j=0 \\ \Pi(j) \\ 0 \le v_i \le y_{k,i}}}^{m} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} (a_{n+1})^{i.y_{k,i}-i.v_i}.$$

Knowing that $\binom{y_{k,i}}{v_i} = 0$ if $v_i > y_{k,i}$, hence, the terms produced for $v_i > y_{k,i}$ would be zero. Thus, we can remove the condition $0 \le v_i \le y_{k,i}$ because terms that do not satisfy this condition will be zeros and, therefore, would not change the value of the sum.

$$I = \sum_{\pi(m)} \sum_{\substack{j=0 \\ \Pi(j)}}^{m} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} (a_{n+1})^{i.y_{k,i} - i.v_i}.$$

We expand the expression then, from all values of k (from all partitions $(y_{k,1}, \ldots, y_{k,m})$ of m), we regroup together the terms having a combination of exponents $(v_1, \ldots, v_m)$ that forms a partition of the same integer j and we do so for all $j \in [0, m]$. Hence, performing this manipulation allows us to interchange the sum over $\pi(m)$ (over $\sum_i i.y_{k,i} = m$) with the sums over j. Thus, the expression becomes as follows,

$$\begin{aligned} I &= \sum_{\substack{j=0 \\ \Pi(j)}}^{m} \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} (a_{n+1})^{i.y_{k,i} - i.v_i} \\ &= \sum_{\substack{j=0 \\ \Pi(j)}}^{m} \sum_{\pi(m)} (a_{n+1})^{\sum i.y_{k,i} - \sum i.v_i} \left[\prod_{i=1}^{m} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right] \left[\prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \right] \\ &= \sum_{\substack{j=0 \\ \Pi(j)}}^{m} (a_{n+1})^{m-j} \left[\prod_{i=1}^{m} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right] \left(\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \binom{y_{k,i}}{v_i} \right). \end{aligned}$$

Applying Lemma 4.4, we get

$$\begin{aligned} I &= \sum_{\substack{j=0 \\ \Pi(j)}}^{m} (a_{n+1})^{m-j} \left[\prod_{i=1}^{m} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right] \left(\prod_{i=1}^{m} \frac{1}{i^{v_i} (v_i)!} \right) \\ &= \sum_{\substack{j=0 \\ \Pi(j)}}^{m} (a_{n+1})^{m-j} \left(\prod_{i=1}^{m} \frac{1}{i^{v_i} (v_i)!} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right). \end{aligned}$$

Knowing that for any given value of j there are multiple combinations of $v_1, \ldots, v_m$ that satisfy $\sum i.v_i = j$, hence, every value of j corresponds to a sum of the sum's argument for all partitions of j (for all combinations of $v_1, \ldots, v_m$ satisfying $\sum i.v_i = j$). Therefore, we can split the outer sum with two conditions into two sums each with one of the conditions as follows,

$$I = \sum_{j=0}^{m} (a_{n+1})^{m-j} \sum_{\Pi(j)} \left(\prod_{i=1}^{m} \frac{1}{i^{v_i} (v_i)!} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right).$$

Knowing that the largest element of a partition of j is at most j,

$$I = \sum_{j=0}^{m} (a_{n+1})^{m-j} \left(\sum_{\Pi(j)} \prod_{i=1}^{j} \frac{1}{i^{v_i} (v_i)!} \left(\sum_{N=q}^{n} (a_N)^i \right)^{v_i} \right).$$

By using the induction hypothesis, the expression becomes

$$I = \sum_{j=0}^{m} (a_{n+1})^{m-j} \left(\sum_{N_j=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{N_j} \cdots a_{N_1} \right).$$

Using Corollary 2.2.1, we get

$$I = \sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{N_m} \cdots a_{N_1}.$$

The theorem is proven by induction. □

Corollary 4.6.1. *If the recurrent sum starts at* 1*, Theorem 4.6 becomes*

$$\sum_{N_m=1}^{n} \cdots \sum_{N_1=1}^{N_2} a_{N_m} \cdots a_{N_1} = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=1}^{n} (a_N)^i \right)^{y_{k,i}}.$$

Example 4.2. *For* $m = 2$*, we have*

$$\sum_{N_2=1}^{n} \sum_{N_1=1}^{N_2} a_{N_2} a_{N_1} = \frac{1}{2} \left(\sum_{N=1}^{n} a_N \right)^2 + \frac{1}{2} \left(\sum_{N=1}^{n} (a_N)^2 \right).$$

Example 4.3. *For* $m = 3$*, we have*

$$\sum_{N_3=1}^{n} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} a_{N_3} a_{N_2} a_{N_1} = \frac{1}{6} \left(\sum_{N=1}^{n} a_N \right)^3 + \frac{1}{2} \left(\sum_{N=1}^{n} a_N \right) \left(\sum_{N=1}^{n} (a_N)^2 \right) + \frac{1}{3} \left(\sum_{N=1}^{n} (a_N)^3 \right).$$

Example 4.4. *For* $m = 4$*, we have*

$$\begin{aligned} \sum_{N_4=1}^{n} \sum_{N_3=1}^{N_4} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} a_{N_4} a_{N_3} a_{N_2} a_{N_1} &= \frac{1}{24} \left(\sum_{N=1}^{n} a_N \right)^4 + \frac{1}{4} \left(\sum_{N=1}^{n} a_N \right)^2 \left(\sum_{N=1}^{n} (a_N)^2 \right) \\ &+ \frac{1}{3} \left(\sum_{N=1}^{n} a_N \right) \left(\sum_{N=1}^{n} (a_N)^3 \right) \\ &+ \frac{1}{8} \left(\sum_{N=1}^{n} (a_N)^2 \right)^2 + \frac{1}{4} \left(\sum_{N=1}^{n} (a_N)^4 \right). \end{aligned}$$

An additional partition identity that can be deduced from Theorem 4.6 is as follows.

Corollary 4.6.2. *For any* $m, n \in \mathbb{N}$*, we have that*

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!} \left(\frac{n}{i} \right)^{y_{k,i}} = \binom{n+m-1}{m}.$$

Proof. From paper [19], we have the following relation,

$$\sum_{N_m=1}^{n} \cdots \sum_{N_1=1}^{N_2} 1 = \binom{n+m-1}{m}.$$

By applying Theorem 4.6, we get

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!} \left(\frac{1}{i} \sum_{N=1}^{n} 1 \right)^{y_{k,i}} = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!} \left(\frac{n}{i} \right)^{y_{k,i}} = \binom{n+m-1}{m}.$$ □

Example 4.5. For $n = 1$, Corollary 4.6.2 reduces to Lemma 4.1.

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} = \binom{m}{m} = 1.$$

Example 4.6. For $n = 3$, Corollary 4.6.2 gives

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{3^{y_{k,i}}}{(y_{k,i})!\, i^{y_{k,i}}} = \binom{m+2}{m} = \frac{(m+1)(m+2)}{2}.$$

4.3 General Reduction Theorem

Let $|A|$ represent the number of elements in the set A. Let m be a non-negative integer and let $\{(y_{1,1}, \ldots, y_{1,m}), (y_{2,1}, \ldots, y_{2,m}), \ldots\}$ be the set of all partitions of m. Let us consider the set $M = \{1, \ldots, m\}$. The permutation group S_m is the set of all permutations of the set $\{1, \ldots, m\}$. Let $\sigma \in S_m$ be a permutation of the set $\{1, \ldots, m\}$ and let $\sigma(i)$ represent the i-th element of this given permutation. The number of such permutations is given by:

$$|S_m| = m!\,. \tag{12}$$

The cycle-type of a permutation σ is the ordered set where the i-th element represents the number of cycles of size i in the cycle decomposition of σ. The number of ways of arranging i elements cyclically is $(i-1)!$. The number of possible combinations of $y_{k,i}$ cycles of size i is $[(i-1)!\,]^{y_{k,i}}$. Hence, the number of permutations having cycle-type $(y_{k,1}, \ldots, y_{k,m})$ is given by:

$$\prod_{i=1}^{m} [(i-1)!\,]^{y_{k,i}}. \tag{13}$$

A partition P of a set M is a set of non-empty disjoint subsets of M such that every element of M is present in exactly one of the subsets. Let

$$P = \{\underbrace{P_{1,1}, \ldots, P_{1,y_1}}_{y_1 \text{ sets}}, \;\ldots,\; \underbrace{P_{m,1}, \ldots, P_{m,y_m}}_{y_m \text{ sets}}\}$$

represent a partition of a set of m elements (for our purpose let it be the set $\{1, \ldots, m\}$). Let $P_{i,y}$ represent the y-th subset of order (size) i and let y_i represent the number of subsets of size i contained in this partition of the set. It is interesting to note that $(y_1, \ldots, y_m)$ will always form a partition of the non-negative integer m. However, the number of partitions of m is different from the number of partitions of a set of m elements because there are more than one partition of the set of m elements that can be associated with a given partition of m. In fact, we can easily determine that the number of partitions of a set of m elements associated with the partition $(y_1, \ldots, y_m)$ is given by:

$$|\Omega_k| = \frac{m!}{1!^{y_{k,1}} \cdots m!^{y_{k,m}}\, (y_{k,1})! \cdots (y_{k,m})!} = \frac{m!}{\prod_{i=1}^{m} i!^{y_{k,i}}\, y_{k,i}!}, \tag{14}$$

where Ω_k is the set of all partitions of the set of m elements associated to partition $(y_{k,1}, \dots, y_{k,m})$. This is because the number of ways to divide m objects into l_1 groups of 1 element, l_2 groups of 2 elements, etc., and l_m groups of m elements is given by:

$$\frac{m!}{1!^{l_1} \cdots m!^{l_m} \, l_1! \cdots l_m!} = \frac{m!}{\prod_{i=1}^{m} i!^{l_i} \, l_i!}. \tag{15}$$

We will denote by Ω the set of all partitions of the set of m elements.

Finally, a partition P of a set M is a refinement of a partition ρ of the same set M if every element in P is a subset of an element in ρ. We denote this as $P \succeq \rho$.

Using the notation introduced, we can formulate a generalization of Theorem 4.6 where all sequences are distinct.

Theorem 4.7. *Let $m, n, q \in \mathbb{N}$ such that $n \geq q$. Let $a_{(1);N}, \dots, a_{(m);N}$ be m sequences defined in the interval $[q, n]$. we have that*

$$\sum_{\sigma \in S_m} \left(\sum_{N_m=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(\sigma(m));N_m} \cdots a_{(\sigma(1));N_1} \right) = \sum_{P \in \Omega} \prod_{i=1}^{m} [(i-1)!\,]^{y_{k,i}} \left[\prod_{g=1}^{y_{k,i}} \left(\sum_{N=q}^{n} \prod_{h \in P_{i,g}} a_{(h);N} \right) \right].$$

Remark 12. The theorem can also be written as

$$\begin{aligned} &\sum_{\sigma \in S_m} \left(\sum_{N_m=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(\sigma(m));N_m} \cdots a_{(\sigma(1));N_1} \right) \\ &\quad = \sum_{\pi(m)} \sum_{\Omega_k} \prod_{i=1}^{m} [(i-1)!\,]^{y_{k,i}} \left[\prod_{g=1}^{y_{k,i}} \left(\sum_{N=q}^{n} \prod_{h \in P_{i,g}} a_{(h);N} \right) \right] \\ &\quad = |S_m| \sum_{\pi(m)} \frac{1}{|\Omega_k|} \sum_{\Omega_k} \prod_{i=1}^{m} \frac{1}{y_{k,i}! \, i^{y_{k,i}}} \left[\prod_{g=1}^{y_{k,i}} \left(\sum_{N=q}^{n} \prod_{h \in P_{i,g}} a_{(h);N} \right) \right]. \end{aligned}$$

As every partition of a set of m elements is associated with a given partition of m, hence, adding up all the partitions of the set for every given partition of m is equivalent to adding up all partitions of the set. The first form is obtained by regrouping together, from the set of all partitions of the set $\{1, \dots, m\}$, those who are associated with a given partition of m.

The second expression is obtained by noting that

$$\frac{|S_m|}{|\Omega_k|} \prod_{i=1}^{m} \frac{1}{y_{k,i}! \, i^{y_{k,i}}} = \prod_{i=1}^{m} [(i-1)!\,]^{y_{k,i}}.$$

These forms are shown as they can be more easily used to show that this theorem reduces to Theorem 4.6 if all sequences are the same.

Proof. Both sides of the equation produce all combinations of terms which are products of the m sequences. Hence, the strategy of this proof is to show that every combination appears with the same multiplicity on both sides.

Without loss of generality, we can assume that all sequences are distinct. We can write:

$$\sum_{\sigma\in S_m}\left(\sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{(\sigma(m));N_m}\cdots\, a_{(\sigma(1));N_1}\right)=\sum_{\sigma\in S_m}\left(\sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2} a_{(m);N_{\sigma(m)}}\cdots\, a_{(1);N_{\sigma(1)}}\right).$$

Hence, we can consider the symmetric group S_m as acting on $N=(N_1,\ldots,N_m)$ that has an isotropy group $S_m(N)$ and an associated partition ρ of the set of m elements. The partition ρ is the set of all equivalence classes of the relation given by $a\sim b$ if and only if $N_a=N_b$ and $S_m(N)=\{\sigma\in S_m \mid \sigma(i)\sim i\ \forall i\}$. Thus,

$$a_{(m);N_m}\cdots\, a_{(1);N_1} \tag{16}$$

appears $|S_m(N)|$ times in the expansion of the left hand side of the theorem.

Likewise, in the right-hand side, (16) can only appear in the terms corresponding to partitions P which are refinements of ρ. The expression (16) appears

$$\sum_{P\succeq\rho}\prod_{i=1}^{m}[(i-1)!\,]^{y_{k,i}} \tag{17}$$

times in the right-hand side of the theorem. Also let us notice that $[(i-1)!\,]^{y_{k,i}}$ corresponds to $(|P_{i,1}|-1)!\cdots(|P_{i,y_{k,i}}|-1)!$ because $|P_{i,1}|=\cdots=|P_{i,y_{k,i}}|=i$. Hence, $\prod_{i=1}^{m}[(i-1)!\,]^{y_{k,i}}$ corresponds to $\prod_{P_{h,g}\subset P}(|P_{h,g}|-1)!$, which is equal to the number of permutations having cycle-type specified by P.

Knowing that any element of $S_m(N)$ has a unique cycle-type specified by a partition that refines ρ, hence, we conclude that

$$\sum_{P\succeq\rho}\prod_{i=1}^{m}[(i-1)!\,]^{y_{k,i}}=|S_m(N)|. \tag{18}$$

As both sides of the theorem produce the same terms and with the same multiplicity, we can say that these sides are equal to each other. □

Example 4.7. For $m=2$, Theorem 4.7 gives

$$\sum_{N_2=q}^{n}\sum_{N_1=q}^{N_2} a_{N_2}b_{N_1}+\sum_{N_2=q}^{n}\sum_{N_1=q}^{N_2} b_{N_2}a_{N_1}=\left(\sum_{N=q}^{n}a_N\right)\left(\sum_{N=q}^{n}b_N\right)+\left(\sum_{N=q}^{n}a_Nb_N\right).$$

Example 4.8. For $m=3$, Theorem 4.7 gives

$$\begin{aligned}&\sum_{\sigma\in S_3}\left(\sum_{N_3=q}^{n}\sum_{N_2=q}^{N_3}\sum_{N_1=q}^{N_2} a_{(\sigma(3));N_3}a_{(\sigma(2));N_2}a_{(\sigma(1));N_1}\right)\\ &=\left(\sum_{N=q}^{n}a_{(1);N}\right)\left(\sum_{N=q}^{n}a_{(2);N}\right)\left(\sum_{N=q}^{n}a_{(3);N}\right)\\ &\quad+\left(\sum_{N=q}^{n}a_{(1);N}\right)\left(\sum_{N=q}^{n}a_{(2);N}a_{(3);N}\right)+\left(\sum_{N=q}^{n}a_{(2);N}\right)\left(\sum_{N=q}^{n}a_{(1);N}a_{(3);N}\right)\\ &\quad+\left(\sum_{N=q}^{n}a_{(3);N}\right)\left(\sum_{N=q}^{n}a_{(1);N}a_{(2);N}\right)+2\left(\sum_{N=q}^{n}a_{(1);N}a_{(2);N}a_{(3);N}\right).\end{aligned}$$

Remark 13. Although Theorem 4.7 is a generalization of Theorem 4.6, Theorem 4.7 is limited by the fact that we cannot isolate a recurrent sum of a specific order, instead we need to add up the recurrent sum for all different orderings of the sequences. However, the author thinks that it cannot be simplified further for the general case, i.e., without specifying the sequences. For a given choice of sequences, one should potentially be able to isolate the recurrent sum with the desired order using the properties of the given sequences.

4.4 Applications to special sums

In this section, we will apply the reduction formula presented in Theorem 4.6 to simplify certain special recurrent sums. The first special sum that we will simplify is a recurrent sum of N^p which will produce a recurrent form of the Faulhaber formula. The second special sum is the multiple zeta star sum for positive even arguments.

4.4.1 Recurrent sums of powers

The Faulhaber formula is a formula developed by Faulhaber in a 1631 edition of *Academia Algebrae* [17] to calculate sums of powers (N^p). The Faulhaber formula is as follows

$$\sum_{N=1}^{n} N^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j n^{p+1-j}, \tag{19}$$

where B_j are the Bernoulli numbers of the first kind [31].

In this section, we will use the reduction formula to generalize the Faulhaber formula to a formula for recurrent sums of powers.

Theorem 4.8. *For any $m, n, p \in \mathbb{N}$, we have that*

$$\begin{aligned}\sum_{N_m=1}^{n}\cdots\sum_{N_1=1}^{N_2} {N_m}^p\cdots {N_1}^p &= \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!\, i^{y_{k,i}}}\left(\sum_{N=1}^{n} N^{ip}\right)^{y_{k,i}}\\ &= \sum_{\pi(m)}\prod_{i=1}^{m}\frac{1}{(y_{k,i})!\, i^{y_{k,i}}}\left(\frac{n^{ip+1}}{ip+1}\sum_{j=0}^{ip}(-1)^j\binom{ip+1}{j}\frac{B_j}{n^j}\right)^{y_{k,i}}.\end{aligned}$$

Proof. Applying Theorem 4.6 and then Faulhaber's formula, we get this theorem. □

Let us now consider a couple of particular cases:

- Case 1: $m = 2$

$$\begin{aligned}\sum_{N_2=1}^{n}\sum_{N_1=1}^{N_2} {N_2}^p {N_1}^p &= \frac{1}{2}\left(\sum_{N=1}^{n} N^p\right)^2 + \frac{1}{2}\left(\sum_{N=1}^{n} N^{2p}\right)\\ &= \frac{1}{2}\left[\left(\frac{n^{p+1}}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}\frac{B_j}{n^j}\right)^2 + \left(\frac{n^{2p+1}}{2p+1}\sum_{j=0}^{2p}(-1)^j\binom{2p+1}{j}\frac{B_j}{n^j}\right)\right].\end{aligned}$$

Example 4.9. For $p = 2, 3$, by applying this theorem, we can get the following formulas:

$$\sum_{N_2=1}^{n} \sum_{N_1=1}^{N_2} {N_2}^2 {N_1}^2 = \frac{n(n+1)(n+2)(2n+1)(2n+3)(5n-1)}{360} = \binom{2n+4}{5} \frac{5n-1}{4!}.$$

$$\sum_{N_2=1}^{n} \sum_{N_1=1}^{N_2} {N_2}^3 {N_1}^3 = \frac{n(n+1)(n+2)(21n^5+69n^4+45n^3-21n^2-6n+4)}{672}.$$

- Case 2: $m = 3$

$$\sum_{N_3=1}^{n} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} {N_3}^p {N_2}^p {N_1}^p = \frac{1}{6} \left(\sum_{N=1}^{n} N^p \right)^3 + \frac{1}{2} \left(\sum_{N=1}^{n} N^p \right) \left(\sum_{N=1}^{n} N^{2p} \right) + \frac{1}{3} \left(\sum_{N=1}^{n} N^{3p} \right).$$

Example 4.10. For $p = 1, 2$, by applying this theorem and exploiting Faulhaber's formula,

$$\sum_{N_3=1}^{n} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} N_3 N_2 N_1 = \frac{n^2(n+1)^2(n+2)(n+3)}{48} = \left(\sum_{N=1}^{n} N \right) \left[\frac{n(n+1)(n+2)(n+3)}{4!} \right].$$

$$\sum_{N_3=1}^{n} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} {N_3}^2 {N_2}^2 {N_1}^2 = \binom{2n+6}{7} \frac{35n^2-21n+4}{144}.$$

Remark 14. Some of these sequences as well as additional ones were added by the author to the OEIS:

- A346642 (`https://oeis.org/A346642`),
- A351766 (`https://oeis.org/A351766`),
- A351770 (`https://oeis.org/A351770`),
- A351105 (`https://oeis.org/A351105`).

4.4.2 Multiple zeta star values

In this section, using the formula developed by Euler and the reduction theorem, we prove an expression which can be used to calculate multiple zeta star values for positive even arguments. Then we present additional identities concerning MZSVs as well as special sums over partitions. For MZSVs of a repeated argument, we use the following notation: $\zeta^\star(\{p\}_m)$ represents $\zeta^\star(p, \ldots, p)$ where the multiplicity of p is m.

We start by applying Theorem 4.6 and using the expression of the zeta function for positive even values to get an expression for the recurrent sum of $\frac{1}{N^{2p}}$, that is, $\zeta^\star(\{2p\}_m)$.

Theorem 4.9. *For any $m \in \mathbb{N}$, $p \in \mathbb{N}^*$, we have that*

$$\zeta^\star(\{2p\}_m) = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \left(\zeta(2ip)\right)^{y_{k,i}} = (2i\pi)^{2pm} \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{B_{2ip}}{(2i)(2ip)!} \right)^{y_{k,i}},$$

$$\zeta^\star(\{2p\}_m) = \frac{(2i\pi)^{2pm}}{(2pm)!} \sum_{\pi(m)} \binom{2pm}{\boldsymbol{\beta}} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}} (B_{2ip})^{y_{k,i}}}{(2i)^{y_{k,i}} y_{k,i}!}, \quad \boldsymbol{\beta} = (\{2p\}_{y_1}, \ldots, \{2pm\}_{y_m}).$$

Proof. By applying Theorem 4.6,

$$\zeta^\star(\{2p\}_m) = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \left(\sum_{N=1}^{\infty} \left(\frac{1}{N^{2p}} \right)^i \right)^{y_{k,i}} = \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \left(\zeta(2ip)\right)^{y_{k,i}}.$$

Euler proved that, for $m \geq 1$ (see [2] for a proof),

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!} B_{2m}. \tag{20}$$

Hence,

$$\begin{aligned}\zeta^\star(\{2p\}_m) &= \sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \left((-1)^{ip+1} \frac{B_{2ip}(2\pi)^{2ip}}{2(2ip)!} \right)^{y_{k,i}} \\ &= (-1)^{pm}(2\pi)^{2pm} \sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{B_{2ip}}{(2i)(2ip)!} \right)^{y_{k,i}}. \end{aligned}$$

□

Example 4.11. *For $m = 2, 3$, Theorem 4.9 gives*

$$\zeta^\star(2p, 2p) = \frac{(2\pi)^{4p}}{(4p)!} \left[\binom{4p}{2p} \frac{(B_{2p})^2}{2^2 2!} - \frac{B_{4p}}{4} \right].$$

$$\zeta^\star(\{2p\}_3) = \frac{(-1)^{p+1}(2\pi)^{6p}}{(6p)!} \left[\binom{6p}{\{2p\}_3} \frac{B_{2p}^3}{2^3 3!} - \binom{6p}{2p, 4p} \frac{B_{2p}}{2} \frac{B_{4p}}{4} + \frac{B_{6p}}{6} \right].$$

By using the values of the zeta function for even arguments as well as Theorem 4.9 and playing with different values, we can notice some identities. In what follows, we prove several identities related to MZSVs, one of which represents a generalization of the Basel Problem solved by Euler.

We start with the following theorem which represents an alternative statement of Schneider's theorem [35].

Theorem 4.10. *For any $m \in \mathbb{N}$, we have that*

$$\zeta^\star(\{2\}_m) = \sum_{N_m=1}^{\infty} \cdots \sum_{N_1=1}^{N_2} \frac{1}{N_m^2 \cdots N_1^2} = \frac{(-1)^{m+1}\left(2^{2m}-2\right) B_{2m}\pi^{2m}}{(2m)!} = \left(2 - \frac{1}{2^{2(m-1)}}\right)\zeta(2m)$$

or, identically (from Theorem 4.6),

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{1}{(y_{k,i})!\, i^{y_{k,i}}} \left(\zeta(2i)\right)^{y_{k,i}} = \frac{(-1)^{m+1}\left(2^{2m}-2\right) B_{2m}\pi^{2m}}{(2m)!} = \left(2 - \frac{1}{2^{2(m-1)}}\right)\zeta(2m).$$

Proof. In [35], the following relation was proven but in another notation,

$$\sum_{N_m=1}^{\infty} \cdots \sum_{N_1=1}^{N_2} \frac{1}{N_m^2 \cdots N_1^2} = \left(\frac{2^{2m-1}-1}{2^{2m-2}} \right) \zeta(2m) = \left(2 - \frac{1}{2^{2(m-1)}}\right)\zeta(2m).$$

Using the expression of $\zeta(2m)$ from Eq. (20), we get

$$\sum_{N_m=1}^{\infty} \cdots \sum_{N_1=1}^{N_2} \frac{1}{N_m^2 \cdots N_1^2} = \frac{(-1)^{m+1}\left(2^{2m}-2\right) B_{2m}\pi^{2m}}{(2m)!}.$$

The first equation is proven. Applying Theorem 4.6, we get the second equation. □

Remark 15. From Theorem 4.10, we can deduce that $(\pi x)csc(\pi x)$ is a generating function for $\zeta^\star(\{2\}_m)$. In Part 2 of this study [21], we will generalize this result.

Corollary 4.10.1. *For any $m \in \mathbb{N}$, we have that*

$$\sum_{\pi(m)} \prod_{i=1}^{m} \frac{(-1)^{y_{k,i}}}{(y_{k,i})!} \left(\frac{B_{2ip}}{(2i)(2i)!} \right)^{y_{k,i}} = \left(\frac{1}{2^{2m-1}} - 1 \right) \frac{B_{2m}}{(2m)!}$$

Proof. By applying Theorem 4.9 with $p = 1$ to Theorem 4.10, we obtain the corollary. $\square$

We will use this to prove that this recurrent harmonic series (or multiple zeta star values) with $2p = 2$ will converge to 2 as the number of summations m goes to infinity.

Theorem 4.11. *For any $m \in \mathbb{N}$, we have that*

$$\lim_{m\to\infty} \zeta^\star(\{2\}_m) = \lim_{m\to\infty} \left(\sum_{N_m=1}^{\infty} \cdots \sum_{N_1=1}^{N_2} \frac{1}{N_m^2 \cdots N_1^2} \right) = 2.$$

Proof. It is known that $\lim_{m\to\infty} \zeta(2m) = 1$. By applying Theorem 4.10,

$$\lim_{m\to\infty} \left(\sum_{N_m=1}^{\infty} \cdots \sum_{N_1=1}^{N_2} \frac{1}{N_m^2 \cdots N_1^2} \right) = \lim_{m\to\infty} \left(2 - \frac{1}{2^{2(m-1)}} \right) \times \lim_{m\to\infty} \zeta(2m) = 2.$$

$\square$

Example 4.12. For $m = 4$, we have

$$\sum_{N_4=1}^{\infty} \sum_{N_3=1}^{N_4} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} \frac{1}{N_4^2 N_3^2 N_2^2 N_1^2} = \frac{1}{24} (\zeta(2))^4 + \frac{1}{4} (\zeta(2))^2 \zeta(4) + \frac{1}{3}\zeta(2)\zeta(6) + \frac{1}{8} (\zeta(4))^2 + \frac{1}{4}\zeta(8)$$
$$= \frac{127\pi^8}{604800} = \left(2 - \frac{1}{2^{2(3)}} \right) \zeta(8) \approx 1.992466004.$$

Similarly, we will use this to show that the sum (over all non-negative values of m) of the recurrent harmonic series with $2p = 2$ will diverge.

Remark 16. From Theorem 4.11, we can easily see that

$$\sum_{m=0}^{\infty} \zeta^\star(\{2\}_m) = \sum_{m=0}^{\infty} \left(\sum_{N_m=1}^{\infty} \cdots \sum_{N_1=1}^{N_2} \frac{1}{N_m^2 \cdots N_1^2} \right) \to \infty.$$

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