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Remark on the transcendence of real numbers generated by Thue–Morse along squares

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Abstract: In 1929, Mahler proved that the real number generated by Thue–Morse sequence is transcendental. Later, Adamczewski and Bugeaud gave a different proof of the transcendence of this number using a combinatorial transcendence criterion. Moreover, Kumar and Meher gave the generalization of the combinatorial transcendence criterion under the subspace Lang conjecture. In this paper, we prove under the subspace Lang conjecture that the real number generated by Thue–Morse along squares is transcendental by using the combinatorial transcendence criterion of Kumar and Meher.
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1 Introduction

Let $\sigma(n)$ be the sum of digits in the binary expansion of a non-negative integer n. We introduce the Thue–Morse sequence $(t(n))_{n\geq 0}$ as follows:

$$t(n) := \begin{cases} 0 & \sigma(n) \text{ is even,} \\ 1 & \sigma(n) \text{ is odd.} \end{cases}$$

Let b be an integer with $b \geq 2$. In 1929, Mahler [8] proved that the series $\sum_{n=0}^{\infty} \frac{t(n)}{b^{n+1}}$ is transcendental by using the certain functional equation, which is called Mahler function (see also [12]).

Later, Adamczewski and Bugeaud [1] gave a different proof of the transcendence of $\sum_{n=0}^{\infty} \frac{t(n)}{b^{n+1}}$ using a combinatorial transcendence criterion, whose proof relies on the subspace theorem (see [2, Theorem 1] or [1, Theorem 5], see Section 2). More precisely, for any sequence $\mathbf{a} = (a(n))_{n\geq 0}$ whose values lie in $\{0, 1, \ldots, b-1\}$ and a positive integer m, we define the complexity function $p_{\mathbf{a}}(m)$ as follows

$$p_{\mathbf{a}}(m) = \#\{a(j)a(j+1)\cdots a(j+m-1) \mid j \geq 0\}.$$

Adamczewski and Bugeaud proved the following result (see [1, Theorem 1]).

Theorem 1.1 ([1]). *Let b be an integer with $b \geq 2$ and $\mathbf{a} = (a(n))_{n\geq 0}$ be a non-periodic sequence whose values lie in $\{0, 1, \ldots, b-1\}$. If, for any non-negative integer m, there exist a positive constant C such that $p_{\mathbf{a}}(m) \leq Cm$, then the series $\sum_{n=0}^{\infty} \frac{a(n)}{b^{n+1}}$ is transcendental.*

By [3, Theorem 10.3.1] and the non-periodicity of Thue–Morse sequence, Thue–Morse sequence satisfies the assumption of Theorem 1.1. Therefore, $\sum_{n=0}^{\infty} \frac{t(n)}{b^{n+1}}$ is transcendental. Moreover, Kumar and Meher [7] gave the generalization of the combinatorial transcendence criterion under the subspace Lang conjecture, which was conjectured by [4] (see [4, Section 2] or [7, Conjecture 1]).

On the other hand, Thue–Morse along squares $(t(n^2))_{n\geq 0}$ was investigated by several authors (see [5, 6, 9, 11], etc.). Moshe [11] proved the following result (see [11, Corollary 3]).

Theorem 1.2 ([11]). *We define the sequence $\mathbf{s}$ by $\mathbf{s} = (s(n))_{n\geq 0} = (t(n^2))_{n\geq 0}$. Then, for any non-negative integer m,*

$$p_{\mathbf{s}}(m) = 2^m. \tag{1}$$

Later, Drmota, Mauduit and Rivat [5] improved Theorem 1.2 as follows:

Theorem 1.3 ([5]). *The sequence $\mathbf{s}$ is normal.*

Remark 1.1 ([1, 5, 10]). A sequence $\mathbf{a} = (a(n))_{n\geq 0}$ whose values lie in $\{0, 1, \ldots, b-1\}$ is normal if, for any positive integer m, each one of the b^m blocks of length m on the alphabet $\{0, 1, \ldots, b-1\}$ occurs in $\mathbf{a} = (a(n))_{n\geq 0}$ with the same frequency $\frac{1}{b^m}$.

By (1), one can easily deduce that the series $\sum_{n=0}^{\infty} \frac{s(n)}{b^{n+1}}$ is irrational. On the other hand, the Borel conjecture (see Section 1 in [1]) predicts that digital expansions of real algebraic irrational numbers are normal. Therefore, we propose the following problem.

Problem 1.1. *Let b be an integer with $b \geq 2$. Is the series $\sum_{n=0}^{\infty} \frac{s(n)}{b^{n+1}}$ transcendental?*

Remark 1.2. In view of Theorem 1.2, Theorem 1.1 cannot be applied to $\sum_{n=0}^{\infty} \frac{s(n)}{b^{n+1}}$.

This paper gives the proof of the transcendence of the series $\sum_{n=0}^{\infty} \frac{s(n)}{b^{n+1}}$ by using the combinatorial transcendence criterion of Kumar and Meher.

Now we state the main result as follows.

Theorem 1.4. *Suppose that the subspace Lang conjecture is true. The series* $\sum_{n=0}^{\infty} \frac{s(n)}{b^{n+1}}$ *is transcendental.*

Moreover, we give extensions of Theorem 1.4 as follows. Let $\tau(n)$ be the number of pattern 11 in the binary expansion of a non-negative integer n. We introduce the Rudin–Shapiro sequence $(r(n))_{n\geq 0}$ as follows:

$$r(n) := \begin{cases} 0 & \tau(n) \text{ is even,} \\ 1 & \tau(n) \text{ is odd.} \end{cases}$$

In [10], Müllner proved that the Rudin–Shapiro sequence $(r(n))_{n\geq 0}$ is also normal along squares. From the normality of the sequence $(r(n^2))_{n\geq 0}$, one can easily deduce that the series $\sum_{n=0}^{\infty} \frac{r(n^2)}{b^{n+1}}$ is irrational. We also give the similar theorem of Theorem 1.4 for Rudin–Shapiro sequence along squares as follows.

Theorem 1.5. *Suppose that the subspace Lang conjecture is true. The series* $\sum_{n=0}^{\infty} \frac{r(n^2)}{b^{n+1}}$ *is transcendental.*

On the other hand, Moshe [11] proved the following result (see [11, Corollary 3]).

Theorem 1.6 ([11])**.** *We define the sequence* $\mathbf{v}$ *by* $\mathbf{v} = (v(n))_{n\geq 0} = (t(n^3))_{n\geq 0}$*. Then, for any non-negative integer* m*,*

$$p_{\mathbf{v}}(m) \geq 2^{\frac{m}{2}}. \tag{2}$$

The normality of the sequence $\mathbf{v}$ is known as a difficult problem. By (2), one can easily deduce that the series $\sum_{n=0}^{\infty} \frac{v(n)}{b^{n+1}}$ is irrational. We also give the similar theorem of Theorem 1.4 for $\mathbf{v}$ as follows.

Theorem 1.7. *Suppose that the subspace Lang conjecture is true. The series* $\sum_{n=0}^{\infty} \frac{v(n)}{b^{n+1}}$ *is transcendental.*

This paper is organized as follows. In Section 2, we introduce the two combinatorial transcendence criteria of Adamczewski, Bugeaud and Luca [2] and Kumar and Meher [7] (Theorem 2.2). In Section 3, we prove Theorems 1.4, 1.5 and 1.7 by using Theorem 2.2. In Section 4, we comment on future works.

2 The two combinatorial transcendence criteria

In this section, we introduce the two combinatorial transcendence criteria of Adamczewski, Bugeaud and Luca [2] (see also [1]) and Kumar and Meher [7]. Adamczewski, Bugeaud and Luca [2] proved the following combinatorial transcendence criterion by using the subspace theorem.

Theorem 2.1 ([1, 2])**.** *Let* b *be integer with* $b \geq 2$*. Let* $(a(n))_{n\geq 0}$ *be a non-periodic sequence whose values lie in* $\{0, 1, \ldots, b-1\}$*. Suppose there exist a positive constant* ϵ *and infinitely many 3-tuples* (j_n, k_n, l_n) *of natural numbers satisfying*

$$a(j_n + j) = a(j_n + k_n + j) \text{ for all } 0 \leq j \leq l_n - 1 \text{ and for all non-negative integer } n \tag{3}$$

and

$$\epsilon(j_n + k_n) \le l_n \le k_n. \tag{4}$$

Then the series $\sum_{n=0}^{\infty} \frac{a(n)}{b^{n+1}}$ *is transcendental.*

Remark 2.1. We tried to apply Theorem 2.1 to $\sum_{n=0}^{\infty} \frac{s(n)}{b^{n+1}}$, but it seems to be difficult. In Section 4, we propose this problem.

Kumar and Meher [7] proved the following combinatorial transcendence criterion under the subspace Lang conjecture.

Theorem 2.2 ([7]). *Let* b *be integer with* $b \ge 2$*. Let* $(a(n))_{n\ge 0}$ *be a non-periodic sequence whose values lie in* $\{0, 1, \ldots, b-1\}$*. Suppose there exist a positive constant* ϵ *and infinitely many* 3*-tuples* (j_n, k_n, l_n) *of natural numbers satisfying* (3) *and*

$$\frac{(2+\epsilon)(\log(j_n + k_n) + \log\log b)}{\log b} \le l_n \le k_n. \tag{5}$$

If the subspace Lang Conjecture is true, the series $\sum_{n=0}^{\infty} \frac{a(n)}{b^{n+1}}$ *is transcendental.*

3 The proof of Theorems

Now we prove Theorem 1.4.

Proof. Let j, m, and n be non-negative integers with $j < 2^n$. By the definition of $t(n)$, we see

$$t(2^n m) = t(m) \text{ and } t(j + 2^n m) \equiv t(j) + t(2^n m) \pmod 2. \tag{6}$$

By (6), for any integers n and j with $n > 1$ and $0 \le j \le 2^n - 1$, we get

$$s(j + 2^{2n} + 2^{4n}) \equiv t(j^2) + 3 + t(2^{2n+1}j) + t(2^{4n+1}j) \equiv t(j^2) + 1 \equiv s(j) + 1 \pmod 2, \tag{7}$$
$$s(j + 2^{3n} + 2^{6n}) \equiv t(j^2) + 3 + t(2^{3n+1}j) + t(2^{6n+1}j) \equiv t(j^2) + 1 \equiv s(j) + 1 \pmod 2. \tag{8}$$

By (7), (8) and applying Theorem 2.2 with $\epsilon = 1$ and $(j_n, k_n, l_n) = (2^{2n}+2^{4n}, 2^{3n}+2^{6n}-2^{2n}-2^{4n}, 2^n)$ $(n > 1)$ to the series $\sum_{n=0}^{\infty} \frac{s(n)}{b^{n+1}}$, we complete the proof of Theorem 1.4. □

Now we prove Theorem 1.5.

Proof. Let j, m, and n be non-negative integers with $j < 2^n$. By the definition of $r(n)$, we see

$$r(2^n m) = r(m) \text{ and } r(j + 2^{n+2} m) \equiv r(j) + r(2^{n+2} m) \pmod 2. \tag{9}$$

We have

$$(j + 2^{3n} + 2^{6n})^2 = j^2 + 2^{3n+1}j + 2^{6n}(1 + 2j) + 2^{9n+1} + 2^{12n}, \tag{10}$$
$$(j + 2^{4n} + 2^{8n})^2 = j^2 + 2^{4n+1}j + 2^{8n}(1 + 2j) + 2^{12n+1} + 2^{16n}. \tag{11}$$

By the definition of $r(n)$ and (9)–(11), for any integers n and j with $n > 2$ and $0 \le j \le 2^n - 1$, we get

$$r((j + 2^{3n} + 2^{6n})^2) = r((j + 2^{4n} + 2^{8n})^2). \tag{12}$$

By (12) and applying Theorem 2.2 with $\epsilon = 1$ and $(j_n, k_n, l_n) = (2^{3n} + 2^{6n}, 2^{4n} + 2^{8n} - 2^{3n} - 2^{6n}, 2^n)$ $(n > 2)$ to the series $\sum_{n=0}^{\infty} \frac{r(n^2)}{b^{n+1}}$, we complete the proof of Theorem 1.5. □

Now we prove Theorem 1.7.

Proof. Let n and j be non-negative integers with $n > 2$ and $0 \le j \le 2^n - 1$. We have

$$\begin{aligned}(j + 2^{4^2 n} + 2^{4^4 n} + 2^{4^6 n} + 2^{4^8 n})^3 = j^3 &+ \sum_{k=1}^{4} 3j 2^{2\times 4^{2k} n} + \sum_{k=1}^{4} 3j^2 2^{4^{2k} n} + \sum_{1\le k<l\le 4} 3j 2^{4^{2k} n} 2^{4^{2l} n+1} \\ &+ \sum_{1\le k<l<m\le 4} 3 \times 2^{4^{2k} n} 2^{4^{2l} n} 2^{4^{2m} n+1} + \sum_{1\le k<l\le 4} 3 \times 2^{4^{2k} n} 2^{2\times 4^{2l} n} \\ &+ \sum_{1\le k<l\le 4} 3 \times 2^{2\times 4^{2k} n} 2^{4^{2l} n} + \sum_{k=1}^{4} 2^{3\times 4^{2k} n}.\end{aligned} \tag{13}$$

For any integers j and k with $1 \le j \le 2^n - 1$ and $1 \le k \le 4$, we get

$$j^3 = \sum_{i=0}^{3n-1} m_{i,1} 2^i, \tag{14}$$

$$3j^2 2^{4^{2k} n} = \sum_{i=4^{2k} n}^{(4^{2k}+3)n} m_{i,2} 2^i, \tag{15}$$

$$3j 2^{2\times 4^{2k} n} = \sum_{i=2\times 4^{2k} n}^{(2\times 4^{2k}+2)n} m_{i,3} 2^i, \tag{16}$$

$$2^{3\times 4^{2k} n} = \sum_{i=3\times 4^{2k} n}^{3\times 4^{2k} n} 2^i, \tag{17}$$

where $m_{i,1}, m_{i,2}, m_{i,3} \in \{0, 1\}$ (for all integer i in the sets of the indexes of the sum of the right-hand sides). For any integers j and k, l with $1 \le j \le 2^n - 1$ and $1 \le k < l \le 4$, we get

$$3j 2^{4^{2k} n} 2^{4^{2l} n+1} = \sum_{i=(4^{2k}+4^{2l})n}^{(4^{2k}+4^{2l}+3)n} m_{i,4} 2^i, \tag{18}$$

$$3 \times 2^{4^{2k} n} 2^{2\times 4^{2l} n} = \sum_{i=(4^{2k}+2\times 4^{2l})n}^{(4^{2k}+2\times 4^{2l}+1)n} m_{i,5} 2^i, \tag{19}$$

$$3 \times 2^{2\times 4^{2k} n} 2^{4^{2l} n} = \sum_{i=(2\times 4^{2k}+4^{2l})n}^{(2\times 4^{2k}+4^{2l}+1)n} m_{i,6} 2^i, \tag{20}$$

where $m_{i,4}, m_{i,5}, m_{i,6} \in \{0,1\}$ (for all integer i in the sets of the indexes of the sum of the right-hand sides). For any integers k, l, m with $1 \le k < l < m \le 4$, we get

$$3 \times 2^{4^{2k}n} 2^{4^{2l}n} 2^{4^{2m}n+1} = \sum_{i=(4^{2k}+4^{2l}+4^{2m})n}^{(4^{2k}+4^{2l}+4^{2m}+2)n} m_{i,7} 2^i \tag{21}$$

where $m_{i,7} \in \{0,1\}$ (for all integer i in the sets of the indexes of the sum of the right-hand sides). By the uniqueness of the base 4-representation of non-negative integers, the indexes of the sum of the right-hand sides of equations (14)–(21) are all different. Therefore, by (6), for any integers n and j with $n > 2$ and $0 \le j \le 2^n - 1$, we have

$$\begin{aligned} v(j + 2^{4^2 n} + 2^{4^4 n} + 2^{4^6 n} + 2^{4^8 n}) &\equiv t(j^3) + \sum_{k=1}^{4} t(3j 2^{2\times 4^{2k} n}) + \sum_{k=1}^{4} t(3j^2 2^{4^{2k} n}) \\ &+ \sum_{1\le k<l\le 4} t(3j 2^{4^{2k}n} 2^{4^{2l}n+1}) + \sum_{1\le k<l<m\le 4} t(3 \times 2^{4^{2k}n} 2^{4^{2l}n} 2^{4^{2m}n+1}) \\ &+ \sum_{1\le k<l\le 4} t(3 \times 2^{4^{2k}n} 2^{2\times 4^{2l} n}) + \sum_{1\le k<l\le 4} t(3 \times 2^{2\times 4^{2k} n} 2^{4^{2l}n}) \\ &+ \sum_{k=1}^{4} t(2^{3\times 4^{2k} n}) \\ &\equiv t(j^3) \equiv v(j) \pmod 2. \end{aligned} \tag{22}$$

By (22) and applying Theorem 2.2 with $\epsilon = 1$ and $(j_n, k_n, l_n) = (0, 2^{4^2 n} + 2^{4^4 n} + 2^{4^6 n} + 2^{4^8 n}, 2^n)$ ($n > 2$) to the series $\sum_{n=0}^{\infty} \frac{v(n)}{b^{n+1}}$, we complete the proof of Theorem 1.7. □

4 Comments on future works

In this section, we comment on future works. The infinitely many 3-tuples (j_n, k_n, l_n) of natural numbers in the proof of Theorem 1.4 satisfy

$$\frac{j_n + k_n}{2 \times 2^{5n}} \le l_n \le k_n. \tag{23}$$

We think the subspace Lang Conjecture to be difficult because it involves the Lang Conjecture, which is known as a longstanding conjecture (see [4]). Hence, we are considering refinement of (23) to prove Problem 1.1. Now we propose the following problems.

Problem 4.1. *Does there exist a positive constant ϵ and infinitely many 3-tuples (j_n, k_n, l_n) of natural numbers satisfying*

$$s(j_n + j) = s(j_n + k_n + j) \quad \textit{for all} \;\; 0 \le j \le l_n - 1 \;\; \textit{and for all non-negative integer}\; n \tag{24}$$

and

$$\epsilon(j_n + k_n) \le l_n \le k_n? \tag{25}$$

If this problem is resolved, the positive answer of Problem 1.1 follows from Theorem 2.1.

Problem 4.2. *Does there exist a positive constant* ϵ *and infinitely many* 3*-tuples* (j_n, k_n, l_n) *of natural numbers satisfying*

$$s(j_n + j) = s(j_n + k_n + j) \ \textit{for all } 0 \le j \le l_n - 1 \textit{ and for all non-negative integer } n \quad (26)$$

and

$$\frac{\epsilon(j_n + k_n)}{n} \le l_n \le k_n? \quad (27)$$

If this problem is resolved, then Problem 1.1 may be resolved by a generalization of subspace theorem that is much weaker than the subspace Lang Conjecture.

On the other hand, we propose other problems. Thue–Morse sequence and Rudin–Shapiro sequence are known as non-periodic automatic sequences (see [3]). Now we propose the following problem as an extension of Theorems 1.4 and 1.5.

Problem 4.3. *Let* b *be an integer with* $b \ge 2$ *and* $\mathbf{a} = (a(n))_{n\ge0}$ *be a non-periodic automatic sequence whose values lie in* $\{0, 1, \ldots, b-1\}$*. Is the series* $\sum_{n=0}^{\infty} \frac{a(n^2)}{b^{n+1}}$ *transcendental under the subspace Lang Conjecture?*

Finally, we also propose the following problem as an extension of Theorems 1.4 and 1.7.

Problem 4.4. *Let* b *and* p *be integers with* $b, p \ge 2$*. Is the series* $\sum_{n=0}^{\infty} \frac{t(n^p)}{b^{n+1}}$ *transcendental under the subspace Lang Conjecture?*

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