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On two new combined 3-Fibonacci sequences. Part 3

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Abstract: Two new combined 3-Fibonacci sequences are introduced and the explicit formulae for their n-th members are given.
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1 Introduction and main result

In [1, 2, 4], five different combined 3-Fibonacci sequences have been introduced so far.

Here, we continue this direction of research, introducing two new 3-Fibonacci sequences that are different from the previous ones, thus further elaborating the series of extensions of the nature of the Fibonacci sequence (see, e.g., [3]).

Let everywhere below, a, b, c, d, e be arbitrary real numbers.

The first newly introduced sequence has the form:

$$\alpha_0 = 2a, \quad \beta_0 = 2b, \quad \gamma_0 = c, \quad \alpha_1 = 2d, \quad \beta_1 = 2e$$

and for each natural number $n \geq 1$:

$$\alpha_{n+1} = \alpha_n + \alpha_{n-1},$$
$$\beta_{n+1} = \beta_n + \beta_{n-1},$$
$$\gamma_{n+1} = \frac{\alpha_n + \beta_n}{2} + \gamma_n.$$

The first values of sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are given in the following Table 1.

n	α_n	γ_n	β_n
0	$2a$		$2b$
0		c	
1	$2d$		$2e$
1		$a+b+c$	
2	$2a+2d$		$2b+2e$
2		$a+b+c+d+e$	
3	$2a+4d$		$2b+4e$
3		$2a+2b+c+2d+2e$	
4	$4a+6d$		$4b+6e$
4		$3a+3b+c+4d+4e$	
5	$6a+10d$		$6b+10e$
5		$5a+5b+c+7d+7e$	
6	$10a+16d$		$10b+16e$
6		$8a+8b+c+12d+12e$	
	$\dots$	$\dots$	$\dots$

Table 1. The first values of sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$

Let $\{F_n\}_{n=0}^{\infty}$ be the standard Fibonacci sequence, where $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for each natural number $n \geq 0$.

Theorem 1. *For each natural number $n \geq 1$:*

$$\alpha_n = 2F_{n-1}a + 2F_n d,$$
$$\beta_n = 2F_{n-1}b + 2F_n e,$$
$$\gamma_n = F_n a + F_n b + c + (F_{n+1} - 1)d + (F_{n+1} - 1)e.$$

Proof. We can prove the Theorem, for example, by induction. For $n = 1$ and $n = 2$, the validity of the Theorem is checked directly from the above table. Let us assume that the Theorem is valid for some natural number $n \geq 2$. Then:

$$\begin{aligned}\alpha_{n+1} &= \alpha_n + \alpha_{n-1}\\ &= 2F_{n-1}a + 2F_n d + 2F_{n-2}a + 2F_{n-1}d\\ &= 2F_n a + 2F_{n+1}d.\end{aligned}$$

$$
\begin{aligned}
\beta_{n+1} &= \beta_n + \beta_n \\
&= 2F_{n-1}b + 2F_n e + 2F_{n-2}b + 2F_{n-1}e \\
&= 2F_n b + 2F_{n+1}e. \\
\gamma_{n+1} &= \frac{\alpha_n + \beta_n}{2} + \gamma_n \\
&= \frac{1}{2}((2F_{n-1}a + 2F_n d) + (2F_{n-1}b + 2F_n e)) + F_n a + F_n b + c + (F_{n+1} - 1)d \\
&\quad + (F_{n+1} - 1)e \\
&= F_{n-1}a + F_n d + F_{n-1}b + F_n e + F_n a + F_n b + c + (F_{n+1} - 1)d + (F_{n+1} - 1)e \\
&= F_{n+1}a + F_{n+1}b + c + (F_{n+2} - 1)d + (F_{n+2} - 1)e.
\end{aligned}
$$

The remaining formulas are checked by analogy. □

The second sequence introduced herewith has the form:

$$\alpha_0 = a, \quad \beta_0 = b, \quad \gamma_0 = c, \quad \alpha_1 = 2d, \quad \beta_1 = 2e$$

and for each natural number $n \geq 1$:

$$
\begin{aligned}
\alpha_{n+1} &= \alpha_n + \alpha_{n-1}, \\
\beta_{n+1} &= \beta_n + \beta_{n-1}, \\
\gamma_{n+1} &= \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \gamma_n.
\end{aligned}
$$

The first values of sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are given in the following Table 2.

n	α_n	γ_n	β_n
0	$2a$		$2b$
0		c	
1	$2d$		$2e$
1		$c+d+e$	
2	$2a+2d$		$2b+2e$
2		$a+b+c+2d+2e$	
3	$2a+4d$		$2b+4e$
3		$2a+2b+c+4d+4e$	
4	$4a+6d$		$4b+6e$
4		$4a+4b+c+7d+7e$	
5	$6a+10d$		$6b+10e$
5		$7a+7b+c+12d+12e$	
6	$10a+16d$		$10b+16e$
6		$12a+12b+c+20d+20e$	
	...	...	...

Table 2. The first values of sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$

Theorem 2. *For each natural number $n \geq 1$:*

$$\alpha_n = 2F_{n-1}a + 2F_n d,$$
$$\beta_n = 2F_{n-1}b + 2F_n e,$$
$$\gamma_n = F_n a + F_n b + c + (F_{n+1} - 1)d + (F_{n+1} - 1)e.$$

2 Conclusion

Here, two new combined 3-Fibonacci sequences from a new type were introduced and explicit formulas for their members are given.

Other new schemes, modifying the standard form of the 2- and 3-Fibonacci sequences and the above two sequences, will be discussed in future.

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