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Linear recurrence sequence associated to rays of negatively extended Pascal triangle

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Abstract: We consider the extension of generalized arithmetic triangle to negative values of rows and we describe the recurrence relation associated to the sum of diagonal elements laying along finite rays. We also give the corresponding generating function. We conclude by an application to Fibonacci numbers and Morgan-Voyce polynomials with negative subscripts.
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1 Introduction

It is well known that for x and y two real numbers and n a nonnegative integer, we have the identity

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \tag{1}$$

where the number $\binom{n}{k}$ is the binomial coefficient.

Pascal's triangle is defined by the recursive description on nonnegative integers n and k,

$$\binom{n}{k} = \begin{cases} \binom{n-1}{k} + \binom{n-1}{k-1}, & \text{for } n > k > 0, \\ 1, & \text{for } k = 0, \\ 1, & \text{for } k = n, \end{cases}$$

with the convention $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$.

The definition of the factorial of x of degree k, see [8], allows the extension $\binom{x}{k}$, the binomial coefficient of degree k defined for every real number x as follows.

$$\binom{x}{k} = \frac{(x)_k}{k!},\ k = 0, 1, 2, \ldots$$

where $(x)_k = x\,(x-1)\,(x-2)\cdots(x-k+1)\,,\ k = 0, 1, 2, \ldots,$ with $(x)_0 = 1$.

We have

$$(x+k-1)_k = (-1)^k\,(-x)_k\,. \tag{2}$$

For x a nonnegative integer, we recover the classical binomial coefficient. For x a negative integer, we get the negative vertical binomial coefficient which has the sign of $(-1)^k$.

The nonnegative integer $(-1)^k\binom{-n}{k}$ gives the number of k combinations of n with repetition, as follows from the relation (2),

$$\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}. \tag{3}$$

The number $\binom{-n}{k}$ represents the coefficient of $x^k y^{-n-k}$ in the expansion of $(x+y)^{-n}$.

A conventional extension of binomial coefficients to arbitrary integers n and k is given in the paper of Sprugnoli, see [13]. For more details, see also [11]. One can also see the book of Atanassov [4].

Formula (3) provides an extension of Pascal's triangle to negative rows. With given nonnegative integers n and k, $\binom{-n}{k}$ denotes the k-th entry of the $(-n)$-th row.

The entry $\binom{-n}{k}$ in the negative Pascal's triangle is determined for $n \geq 1$, as

$$\binom{-n}{k} = \begin{cases} \binom{-n+1}{k} - \binom{-n}{k-1}, & \text{for } k > 0, \\ 1, & \text{for } k = 0, \\ 0, & \text{for } k < 0. \end{cases}$$

The first values of the negative Pascal's triangle are given in Table 1 below.

	$\boldsymbol{k=0}$	$\boldsymbol{k=1}$	$\boldsymbol{k=2}$	$\boldsymbol{k=3}$	$\boldsymbol{k=4}$	$\boldsymbol{k=5}$	$\boldsymbol{\cdots}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$n=-5$	1	-5	15	-35	70	-126	$\cdots$
$n=-4$	1	-4	10	-20	35	-56	$\cdots$
$n=-3$	1	-3	6	-10	15	-21	$\cdots$
$n=-2$	1	-2	3	-4	5	-6	$\cdots$
$n=-1$	1	-1	1	-1	1	-1	$\cdots$

Table 1. First values of negative Pascal's triangle.

The Arithmetic triangle is the original triangle defined by Pascal himself. Since then, several generalizations have been constructed and studied in many ways.

Inspired by the Arithmetic triangle, Enseley [9] defined what he called GAT, the generalized arithmetic triangle, which is a generalization of the arithmetic triangle, by considering as the edges of this triangle the numerical sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. He described a few particular GATs as, for example, $a_n = b_n = F_n$, with F_n being the Fibonacci number. Also, we can find in [7], a detailed description of what the authors called Fibonacci and Lucas triangles.

In [5] and [6], the authors consider the generalized arithmetic triangle with $a_n = x^n$, $b_n = y^n$ where they change the rule of addition. More precisely, the triangle considered is defined as follows.

Let x and y be two real numbers. The generalized arithmetic triangle contains elements $\left\langle {n \atop k} \right\rangle$ in the n-th row and k-th column defined for $n \geq 2$ by:

$$\left\langle {n \atop k} \right\rangle = \begin{cases} x\left\langle {n-1 \atop k} \right\rangle + y\left\langle {n-1 \atop k-1} \right\rangle, & \text{for } 1 \leq k \leq n-1, \\ x^n, & \text{for } k = 0, \\ y^n, & \text{for } k = n, \end{cases}$$

with the convention $\left\langle {n \atop k} \right\rangle = 0$ whenever $k < 0$ or $k > n$

In the present paper, we consider the problem of the extension of generalized arithmetic triangle as defined by Belbachir and Szalay in [5], to negative rows.

With given nonnegative integers n and k, $\left\langle {-n \atop k} \right\rangle$ denotes the k-th entry of the $(-n)$-th row.

The entry $\left\langle {-n \atop k} \right\rangle$ in the extension of the generalized arithmetic triangle to negative rows is determined for $x \neq 0$ and $n \geq 2$, by:

$$\left\langle {-n \atop k} \right\rangle = \begin{cases} x^{-1}\left\langle {-n+1 \atop k} \right\rangle - x^{-1}y\left\langle {-n \atop k-1} \right\rangle, & \text{for } 1 \leq k \leq n-1, \\ x^{-n}, & \text{for } k = 0, \\ 0, & \text{for } k < 0. \end{cases}$$

We call this extension to negative rows, the generalized negative arithmetic triangle.

The first rows of the generalized negative arithmetic triangle are given in Table 2 below.

	$\boldsymbol{k=0}$	$\boldsymbol{k=1}$	$\boldsymbol{k=2}$	$\boldsymbol{k=3}$	$\boldsymbol{k=4}$	$\boldsymbol{k=5}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$n=-8$	$1x^{-8}$	$-8x^{-9}y^1$	$36x^{-10}y^2$	$-120x^{-11}y^3$	$330x^{-12}y^4$	$-792x^{-13}y^5$	$\cdots$
$n=-7$	$1x^{-7}$	$-7x^{-8}y^1$	$28x^{-9}y^2$	$-84x^{-10}y^3$	$210x^{-11}y^4$	$-462x^{-12}y^5$	$\cdots$
$n=-6$	$1x^{-6}$	$-6x^{-7}y^1$	$21x^{-8}y^2$	$-56x^{-9}y^3$	$126x^{-10}y^4$	$-252x^{-11}y^5$	$\cdots$
$n=-5$	$1x^{-5}$	$-5x^{-6}y^1$	$15x^{-7}y^2$	$-35x^{-8}y^3$	$70x^{-9}y^4$	$-126x^{-10}y^5$	$\cdots$
$n=-4$	$1x^{-4}$	$-4x^{-5}y^1$	$10x^{-6}y^2$	$-20x^{-7}y^3$	$35x^{-8}y^4$	$-56x^{-9}y^5$	$\cdots$
$n=-3$	$1x^{-3}$	$-3x^{-4}y^1$	$6x^{-5}y^2$	$-10x^{-6}y^3$	$15x^{-7}y^4$	$-21x^{-8}y^5$	$\cdots$
$n=-2$	$1x^{-2}$	$-2x^{-3}y^1$	$3x^{-4}y^2$	$-4x^{-5}y^3$	$5x^{-6}y^4$	$-6x^{-7}y^5$	$\cdots$
$n=-1$	$1x^{-1}$	$-1x^{-2}y^1$	$1x^{-3}y^2$	$-1x^{-4}y^3$	$1x^{-5}y^4$	$-1x^{-6}y^5$	$\cdots$

Table 2. Some values of the generalized negative arithmetic triangle.

In this triangle, the elements lying on a finite ray defined by a fixed direction (r, q) and a fixed value of p, form the finite sequence

$$\binom{-n-qk}{p+rk} x^{-n-p-(r+q)k} y^{p+rk},\ k = 0, 1, \ldots, \lfloor (n-1)/(-q) \rfloor,$$

where, $r \in \mathbb{N}$, $q \in \mathbb{Z}^-$, $0 \leq p < r$. For the concept of direction, one may consult [5].

Our aim is to calculate the sum of these elements along a fixed direction (r, q). For doing so, we consider the sequence $(V_{-n}^{(r,q,p)})_{n \geq 1}$ defined by:

$$V_{-n}^{(r,q,p)} = \sum_{k=0}^{\lfloor (n-1)/(-q) \rfloor} \binom{-n-qk}{p+rk} x^{-n-p-(r+q)k} y^{p+rk}, \tag{4}$$

with $V_0 = 0$.

Observe that for $1 \leq n < -q+1$, we have for each fixed p,

$$V_{-n}^{(r,q,p)} = \binom{-n}{p} x^{-n-p} y^p. \tag{5}$$

2 Examples

First, we present some examples to illustrate such sequences with particular directions.

- The direction $(r, q) = (1, -1)$. It deals with the sequence $W_{-n} = V_{-n}^{(1,-1,0)}$ given by

$$W_{-n} = \sum_{k=0}^{n-1} \binom{-n+k}{k} x^{-n} y^k,$$

satisfying

$$\begin{cases} W_{-1} = x^{-1}, \\ xW_{-n-1} = (1-y)W_{-n},\ n \geq 1. \end{cases}$$

- The direction $(r, q) = (1, -2)$. It concerns the sequence $\widetilde{U}_{-n} = V_{-n}^{(1,-2,0)}$ given by

$$\widetilde{U}_{-n} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{-n+2k}{k} x^{-n+k} y^k.$$

For $(x, y) = (-1, 1)$, we obtain the sequence $(\widetilde{F}_{-n})_{n \geq 1}$:

$$\widetilde{F}_{-n} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{-n+2k}{k} (-1)^{-n+k},$$

satisfying

$$\begin{cases} \widetilde{F}_0 = 0, \widetilde{F}_{-1} = -1, \\ \widetilde{F}_{-n} = -\widetilde{F}_{-n+1} + \widetilde{F}_{-n+2},\ n \geq 2. \end{cases}$$

$(\widetilde{F}_{-n})_{n \geq 1} = (-1, 1, -2, 3, -5, 8, -13, \ldots)$.

One has $(\widetilde{F}_{-n})_{n\geq 1} = (-F_{-n})_{n\geq 1}$, where $(F_{-n})_{n\geq 1} = (1, -1, 2, -3, 5, -8, 13, \ldots)$ is the negatively subscripted Fibonacci sequence defined by:

$$\begin{cases} F_0 = 0, F_1 = 1, \\ F_{-n} = -F_{-n+1} + F_{-n+2},\ n \geq 1. \end{cases}$$

For $(x, y) = (-\frac{1}{2}, \frac{1}{2})$, we obtain the sequence $(\widetilde{P}_{-n})_{n\geq 1}$:

$$\widetilde{P}_{-n} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{-n+2k}{k} (-\frac{1}{2})^{-n+k} (\frac{1}{2})^k,$$

satisfying

$$\begin{cases} \widetilde{P}_0 = 0, \widetilde{P}_{-1} = -2, \\ \widetilde{P}_{-n} = -2\widetilde{P}_{-n+1} + \widetilde{P}_{-n+2},\ n \geq 2. \end{cases}$$

$(\widetilde{P}_{-n})_{n\geq 1} = (-2, 4, -10, 24, -58, \ldots)$.
Note that $(\widetilde{P}_{-n})_{n\geq 1} = (-2P_{-n})_{n\geq 1}$ and $(P_{-n})_{n\geq 1} = (1, -2, 5, -12, 29, \ldots)$ is the negatively subscripted Pell sequence defined by:

$$\begin{cases} P_0 = 0, P_1 = 1, \\ P_{-n} = -2P_{-n+1} + P_{-n+2},\ n \geq 1. \end{cases}$$

For $(x, y) = (-2, 1)$, we obtain the sequence $(\widetilde{J}_{-n})_{n\geq 1}$:

$$\widetilde{J}_{-n} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{-n+2k}{k} (-2)^{-n+k},$$

satisfying

$$\begin{cases} \widetilde{J}_0 = 0, \widetilde{J}_{-1} = -\frac{1}{2}, \\ \widetilde{J}_{-n} = -\frac{1}{2}\widetilde{J}_{-n+1} + \frac{1}{2}\widetilde{J}_{-n+2},\ n \geq 2. \end{cases}$$

$(\widetilde{J}_{-n})_{n\geq 1} = \left(-\frac{1}{2}, \frac{1}{4}, -\frac{3}{8}, \frac{5}{16}, -\frac{11}{32}, \ldots\right)$.
Observe that $(\widetilde{J}_{-n})_{n\geq 1} = (-J_{-n})_{n\geq 1}$, where $(J_{-n})_{n\geq 1} = \left(\frac{1}{2}, -\frac{1}{4}, \frac{3}{8}, -\frac{5}{16}, \frac{11}{32}, \ldots\right)$ is the negatively subscripted Jacobsthal sequence defined by:

$$\begin{cases} J_0 = 0, J_1 = 1, \\ J_{-n} = -\frac{1}{2}J_{-n+1} + \frac{1}{2}J_{-n+2},\ n \geq 1. \end{cases}$$

For $(x, y) = (-\frac{2}{3}, \frac{1}{3})$, we obtain the sequence $(\widetilde{\phi}_{-n})_{n\geq 1}$:

$$\widetilde{\phi}_{-n} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{-n+2k}{k} (-\frac{2}{3})^{-n+k} (\frac{1}{3})^k,$$

satisfying

$$\begin{cases} \widetilde{\phi}_0 = 0, \widetilde{\phi}_{-1} = -\frac{3}{2}, \\ \widetilde{\phi}_{-n} = -\frac{3}{2}\widetilde{\phi}_{-n+1} + \frac{1}{2}\widetilde{\phi}_{-n+2},\ n \geq 2. \end{cases}$$

$(\widetilde{\phi}_{-n})_{n\geq 1} = \left(-\frac{3}{2}, \frac{9}{4}, -\frac{33}{8}, \frac{117}{16}, -\frac{417}{32}, \ldots\right)$.

One has $(\widetilde{\phi}_{-n})_{n\geq 1} = (-3\phi_{-n})_{n\geq 1}$, with $(\phi_{-n})_{n\geq 1} = \left(\frac{1}{2}, -\frac{3}{4}, \frac{11}{8}, -\frac{39}{16}, \frac{139}{32}, \ldots\right)$ is the negatively subscripted Fermat sequence defined by:

$$\begin{cases} \phi_0 = 0, \phi_1 = 1 \\ \phi_{-n} = -\frac{3}{2}\phi_{-n+1} + \frac{1}{2}\phi_{-n+2},\ n \geq 1. \end{cases}$$

Note that for x and y real numbers such that $x \neq 0$, we have

$$\begin{cases} \widetilde{U}_0 = 0, \widetilde{U}_{-1} = x^{-1}, \\ \widetilde{U}_{-n} = x^{-1}\widetilde{U}_{-n+1} - yx^{-1}\widetilde{U}_{-n+2},\ n \geq 2. \end{cases}$$

$(\widetilde{U}_{-n})_{n\geq 1}$ is the extension of the sequence $(\widetilde{U}_n)_{n\geq 0}$ defined by

$$\begin{cases} \widetilde{U}_0 = 0, \widetilde{U}_1 = -y^{-1}, \\ \widetilde{U}_n = y^{-1}\widetilde{U}_{n-1} - y^{-1}x\widetilde{U}_{n-2},\ n \geq 2. \end{cases}$$

Also, $(\widetilde{U}_{-n})_{n\geq 1} = (-y^{-1}U_{-n})_{n\geq 1}$, with $(U_{-n})_{n\geq 1}$ is the negatively subscripted $(U_n)_{n\geq 0}$ sequence defined by

$$\begin{cases} U_0 = 0, U_1 = 1, \\ U_n = y^{-1}U_{n-1} - y^{-1}xU_{n-2},\ n \geq 2. \end{cases}$$

3 Main results: Sum of the elements lying along a finite ray

Our purpose in this section is to establish the recurrence relation associated to $\left(V_{-n}^{(r,q,p)}\right)_n$ formed by the sum of the elements lying along a finite ray in the generalized negative arithmetic triangle. First, we will present a well-known lemma (Lemma 3.2) playing a key role in determining the main result of this paper. We prove it for convenience. Let's start with

Lemma 3.1. *Let x, y, z, be nonnegative integers satisfying $z \leq y \leq x$. Then*

$$\sum_{j=0}^{z} (-1)^j \binom{z}{j}\binom{x+j}{y} = (-1)^z \binom{x}{y-z}.$$

Proof. Using the Vandermonde identity, $\binom{x+z}{y} = \sum_{i=0}^{n} \binom{x}{i}\binom{z}{y-i}$ for all x, y and z nonnegative integers, we obtain:

$$\begin{aligned} \sum_{j=0}^{z} (-1)^j \binom{z}{j}\binom{x+j}{y} &= \sum_{j=0}^{z} (-1)^j \binom{z}{j} \sum_{i=0}^{j} \binom{x}{y-i}\binom{j}{i} \\ &= \sum_{i=0}^{z} \binom{x}{y-i}\binom{z}{i} \sum_{j=i}^{z} (-1)^j \binom{z-i}{j-i} \\ &= \sum_{i=0}^{z} (-1)^i \binom{x}{y-i}\binom{z}{i} \sum_{k=0}^{z-i} (-1)^k \binom{z-i}{k}. \end{aligned}$$

Since for all $i = 0, \ldots, z-1$, $\sum_{k=0}^{z-i} (-1)^k \binom{z-i}{k} = 0$, we get the result. □

Lemma 3.2. *Let x be a negative integer and y, z be nonnegative integers satisfying $y \geq z$. Then*

$$\sum_{j=0}^{z} (-1)^j \binom{z}{j}\binom{x-j}{y} = \binom{x-z}{y-z}.$$

Proof. Since x is a negative integer and y a nonnegative integer, then from (3), we have:

$$\sum_{j=0}^{z} (-1)^j \binom{z}{j}\binom{x-j}{y} = \sum_{j=0}^{z} (-1)^j \binom{z}{j} (-1)^y \binom{-x+y-1+j}{y}.$$

From Lemma 3.1, we obtain

$$\sum_{j=0}^{z} (-1)^j \binom{z}{j}\binom{x-j}{y} = (-1)^{y+z} \binom{-x+y-1}{y-z} = \binom{x-z}{y-z}.$$

This completes the proof. □

Now, we give our main theorem.

Theorem 3.3. *Let n, $r \in \mathbb{N}$, $q \in \mathbb{Z}^-$, $0 \leq p < r$. The terms of the sequence*

$$V_{-n} = V_{-n}^{(r,q,p)} = \sum_{k=0}^{\lfloor (n-1)/(-q) \rfloor} \binom{-n-qk}{p+rk} x^{-n-p-(r+q)k} y^{p+rk}, \tag{6}$$

satisfy for $n > -r-q$ the linear recurrence relation

$$V_{-n} - x\binom{r}{1}V_{-n-1} + x^2\binom{r}{2}V_{-n-2} + \cdots + (-x)^r \binom{r}{r} V_{-n-r} = y^r V_{-n-q-r}. \tag{7}$$

Proof. Note first, that we can extend the summation in formula (6), up to $\lfloor (n-1)/(-q) \rfloor$ without changing the total sum since for $n \geq 1$, $k > \lfloor (n-1)/(-q) \rfloor$ one gets $p + rk > -n - qk > 0$ and hence $\binom{-n-qk}{p+rk} = 0$. Now, for $n > -r-q$, we have

$$\begin{aligned}\sum_{j=0}^{r} (-x)^j \binom{r}{j} V_{-n-j} &= \sum_{j=0}^{r}\sum_{k\geq 0} (-x)^j \binom{r}{j}\binom{-n-j-qk}{p+rk} x^{-n-j-p-(r+q)k} y^{p+rk} \\ &= \sum_{k\geq 0} x^{-n-p-(r+q)k} y^{p+rk} \sum_{j=0}^{r} (-1)^j \binom{r}{j}\binom{-n-j-qk}{p+rk}.\end{aligned}$$

While for $n \geq 1$,

$$\begin{aligned}\sum_{j=0}^{r} (-1)^j \binom{r}{j}\binom{-n-j-qk}{p+rk} &= \sum_{j=0}^{r} (-1)^j \sum_{m=0}^{p} \binom{r-p}{j-m}\binom{p}{m}\binom{-n-j-qk}{p+rk} \\ &= \sum_{l=0}^{r-m} (-1)^{l+m} \sum_{m=0}^{p} \binom{r-p}{l}\binom{p}{m}\binom{-n-l-m-qk}{p+rk} \\ &= \sum_{l=0}^{r-p} (-1)^l \binom{r-p}{l} \sum_{m=0}^{p} (-1)^m \binom{p}{m}\binom{-n-l-m-qk}{p+rk}.\end{aligned}$$

By Lemma 3.2,

$$\sum_{m=0}^{p}(-1)^m\binom{p}{m}\binom{-n-l-m-qk}{p+rk}=\binom{-n-l-p-qk}{rk}.$$

So,

$$\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{-n-j-qk}{p+rk}=\sum_{l=0}^{r-p}(-1)^l\binom{r-p}{l}\binom{-n-l-p-qk}{rk}=\binom{-n-qk-r}{rk-r+p}.$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{r}(-x)^j\binom{r}{j}V_{-n-j} &= \sum_{k\geq 1}x^{-n-p-(r+q)k}y^{p+rk}\binom{-n-qk-r}{p+r(k-1)}\\ &= \sum_{k\geq 0}x^{-n-p-(r+q)(k+1)}y^{p+r(k+1)}\binom{-n-r-q(k+1)}{p+rk}\\ &= y^rV_{-n-q-r}.\end{aligned}$$

This completes the proof. □

The order of the recurrence sequence given in Theorem 3.3 is equal to $(-q)$ for $r+q<0$ and is equal to r for $r+q>0$. In the last situation, one deals with the Morgan-Voyce phenomenon. For more details, see [1,2].

Indeed, since $q<0$ and $r+q>0$, we can write (7) as

$$V_{-n}-x\binom{r}{1}V_{-n-1}+\cdots+\left((-x)^{r+q}\binom{r}{r+q}-y^r\right)V_{-n-r-q}+\cdots+(-x)^r\binom{r}{r}V_{-n-r}=0.$$

Note that the coefficient y^r of V_{-n-r-q} is subtracted from one of the terms $V_{-n-1},\ldots,V_{-n-r}$.

Example 3.4. *The sequence* $\left(V_{-n}^{(2,-1,0)}\right)_n$ *satisfies the following recurrence relation of order* $r=2$,

$$V_{-n}=(2x+y^2)V_{-n-1}-x^2V_{-n-2},$$

with $V_{-1}=x^{-1}$, $V_{-2}=x^{-2}+x^{-3}y^2$. *However, the sequence* $\left(V_{-n}^{(1,-3,0)}\right)_n$ *satisfies the recurrence relation of order* $(-q)=3$,

$$V_{-n+2}=y^{-1}V_{-n}-xy^{-1}V_{-n-1},$$

with $V_0=0$, $V_{-1}=x^{-1}$, $V_{-2}=x^{-2}$.

At this level, a comparison with Theorem 1 in [5] is interesting. The recurrence sequence given here by Theorem 3.3, is not an extension to negative subscripts of the recurrence given by Theorem 1 in [5] . Indeed, for example, for $r=2$, $q=-1$, $p=0$, $(T_n)_{n\geq 0}$ in [5] satisfies for $n\geq 2$ the recurrence

$$T_n=(2x+y^2)T_{n-1}-x^2T_{n-2},$$

with $T_0 = 0$, $T_1 = 1$. So, by extension to negative subscripts, we deduce that $T_{-1} = -x^{-2}$ while for $r = 2$, $q = -1$, $p = 0$, we have $V_{-1} = x^{-1}$ and for $n \geq 1$,

$$V_{-n} = (2x + y^2)V_{-n-1} - x^2 V_{-n-2}.$$

So, $(V_{-n})_n$ is not the extension to negative subscripts of the sequence $(T_n)_{n\geq 0}$ given in [5]. It is not sufficient to replace n by $(-n)$.

4 Application to the negatively subscripted Fibonacci numbers

As an application of Theorem 3.3, we obtain a new formula for the Fibonacci numbers with negative subscript.

Theorem 4.1. *For* $m \geq 1$,

$$\begin{aligned} F_{-2m+1} &= \sum_{k=0}^{2m-1} \sum_{s=0}^{\lfloor k/2 \rfloor} (-1)^{m+k+s} \, 2^{k-2s} \binom{k}{2s} \binom{-2m+k}{2k}, \\ F_{-2m} &= \sum_{k=0}^{2m-1} \sum_{s=0}^{\lfloor (k-1)/2 \rfloor} (-1)^{m+k+s+1} \, 2^{k-2s-1} \binom{k}{2s+1} \binom{-2m+k}{2k}. \end{aligned}$$

Proof. From (9) , for $r = 2$, $q = -1$, $x = i$ and $y^2 = 1 - 2i$, we get

$$V_{-n} = i^{-n} \sum_{k=0}^{n-1} \binom{-n+k}{2k} (-i-2)^k.$$

While, if we put in Theorem 3.3, $r = 2$, $q = -1$, we obtain

$$-x^2 V_{-n} = -(2x + y^2)V_{-n+1} + V_{-n+2}.$$

So, for $x = i$ and $y^2 = 1 - 2i$, we get $V_{-n} = -V_{-n+1} + V_{-n+2}$. Hence, $V_{-n} = F_{-n+1} - iF_{-n}$, as $V_{-1} = -i$, $V_{-2} = 1 + i$. It follows that

$$F_{-n+1} = Re \left(i^{-n} \sum_{k=0}^{n-1} \binom{-n+k}{2k} (-i-2)^k \right)$$

and

$$F_{-n} = Im \left(i^{-n} \sum_{k=0}^{n-1} \binom{-n+k}{2k} (-i-2)^k \right).$$

Then, by putting $n = 2m$, and after some combinatorial computations, we deduce F_{-2m} and F_{-2m+1}. □

5 Generating function of sum of elements lying along a finite ray

In this section we use the following well known lemma, see for example [6], in order to prove our Theorem 5.2.

Lemma 5.1. *For $r \geq k+p$, we have*

$$\sum_{j=0}^{k} (-1)^j \binom{r}{k-j}\binom{j+p}{p} = \binom{r-p-1}{k}.$$

Now, for a fixed direction (r, q) with $q < 0$, $r+q < 0$, and a fixed value of p, we consider the sequence $\left(V_{-n}^{(r,q,p)}\right)_{n\in\mathbb{N}}$ of sums of elements lying on the corresponding ray in the generalized negative arithmetic triangle. The generating function $T(z) = \sum_{n\geq 0} V_{-n-1} z^n$ is given in the following theorem.

Theorem 5.2. *Let x, y and z be real numbers such that $x \neq 0$. Then, the generating function associated to $\left(V_{-n}^{(r,q,p)}\right)_{n\in\mathbb{N}}$ is given by*

$$T(z) = \frac{y^p z^q (z-x)^{r-p-1}}{y^r - z^q (z-x)^r}.$$

Proof. In Theorem 3.3, for $r+q<0$, we obtain with $n > -q$

$$V_{-n+r} - x\binom{r}{1}V_{-n+r-1} + x^2\binom{r}{2}V_{-n+r-2} + \cdots + (-x)^r\binom{r}{r}V_{-n} = y^r V_{-n-q}.$$

Hence, for $n > -q$, we can write

$$V_{-n} = \sum_{j=1}^{r} (-1)^{j+1} x^{-j}\binom{r}{r-j}V_{-n+j} + \left(\frac{y}{-x}\right)^r V_{-n-q}.$$

Setting

$$a_j = \begin{cases} x^{-j}\binom{r}{j} & \text{if } 1 \leq j \leq r, \\ 0 & \text{if } r+1 \leq j \leq -q-1, \\ (-1)^{-q+1} y^r (-x)^{-r} & \text{for } j = -q, \end{cases}$$

and $U_{-n} = V_{-n-1}$, we deduce for $n > -q-1$,

$$U_{-n} = \sum_{j=1}^{-q} (-1)^{j+1} a_j U_{-n+j},$$

and from relation (5), we obtain for $n \leq -q-1$,

$$U_{-n} = \binom{-n-1}{p} x^{-n-1-p} y^p.$$

On the one hand, we have

$$\begin{aligned}T(z) &= \sum_{n\geq 0} U_{-n}z^n \\ &= \sum_{n=0}^{-q-1} U_{-n}z^n + \sum_{n\geq -q} U_{-n}z^n \\ &= \sum_{n=0}^{-q-1} U_{-n}z^n + \sum_{n\geq -q}\sum_{j=1}^{-q} \left((-1)^{j+1} a_j U_{-n+j}\right) z^n \\ &= \sum_{n=0}^{-q-1} U_{-n}z^n + \sum_{j=1}^{-q} (-1)^{j+1} a_j z^j \left(T(z) - \sum_{k=0}^{-q-1-j} U_{-k}z^k\right).\end{aligned}$$

Considering $a_0 = 1$, we obtain

$$\left(\sum_{j=0}^{-q} (-1)^j a_j z^j\right) T(z) = \sum_{n=0}^{-q-1} U_{-n}z^n + \sum_{k=0}^{-q-1} \left(\sum_{j=1}^{k} (-1)^j a_j U_{-k+j}\right) z^k.$$

Hence,

$$T(z) = \frac{\sum_{k=0}^{-q-1}\left(\sum_{j=0}^{k} (-1)^j a_j U_{-k+j}\right) z^k}{\sum_{j=0}^{-q} (-1)^j a_j z^j}.$$

On the other hand,

$$\begin{aligned}\sum_{j=0}^{-q} (-1)^j a_j z^j &= \sum_{j=0}^{r} (-1)^j a_j z^j - \left(-\frac{y}{x}\right)^r z^{-q} \\ &= \sum_{j=0}^{r} (-1)^j x^{-j} \binom{r}{j} z^j - \left(-\frac{y}{x}\right)^r z^{-q} \\ &= \left(1-\frac{z}{x}\right)^r - \left(-\frac{y}{x}\right)^r z^{-q},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{-q-1}\left(\sum_{j=0}^{k} (-1)^j a_j U_{-k+j}\right) z^k &= \sum_{k=0}^{r}\left(\sum_{j=0}^{k} (-x)^{-j} \binom{r}{j} U_{-k+j}\right) z^k \\ &= \sum_{k=0}^{r}\left(\sum_{j=0}^{k} (-x)^{-k+j} \binom{r}{k-j} U_{-j}\right) z^k \\ &= \sum_{k=0}^{r}\left(\sum_{j=0}^{k} (-x)^{-k+j} \tbinom{r}{k-j}\tbinom{-j-1}{p} x^{-j-1-p} y^p\right) z^k.\end{aligned}$$

From (3), $\binom{-j-1}{p} = (-1)^p \binom{j+p}{p}$. So

$$\sum_{k=0}^{-q-1}\left(\sum_{j=0}^{k} (-1)^j a_j U_{-k+j}\right) z^k = (-y)^p x^{-1-p} \sum_{k=0}^{r}\sum_{j=0}^{k} (-1)^j \binom{r}{k-j} \tbinom{j+p}{p} \left(-\frac{z}{x}\right)^k.$$

From Lemma 5.1, we finish the proof by

$$\sum_{k=0}^{-q-1}\left(\sum_{j=0}^{k}(-1)^j a_j U_{-k+j}\right)z^k = (-y)^p x^{-1-p}\sum_{k=0}^{r}\binom{r-p-1}{k}\left(-\frac{z}{x}\right)^k$$
$$= (-y)^p x^{-1-p}\sum_{k=0}^{r-p-1}\binom{r-p-1}{k}\left(-\frac{z}{x}\right)^k$$
$$= (-y)^p x^{-1-p}\left(1-\frac{z}{x}\right)^{r-p-1}.$$

This completes the proof. □

For $r = 1$, $q = -2$, $x = -1$, $y = 1$ and from the second example in Section 2, we have $V_{-n} = -F_{-n}$. Using Theorem 5.2, we get the well-known generating function of the negatively subscripted of Fibonacci numbers,

$$\sum_{n\geq 0} F_{-n-1}z^n = \frac{1}{1+z-z^2}.$$

5.1 Application to Morgan-Voyce polynomials

Let s be an integer and consider the sequence $(M_n(t))_n$ defined by

$$M_0(t) = 1,\ M_1(t) = 1+s+t,\ M_n(t) = (2+t)M_{n-1}(t) - M_{n-2}(t),\ n\geq 2.$$

For $s = 0, 1, 2$, we obtain respectively the Morgan-Voyce polynomials $b_n(t)$, $B_n(t)$ and $c_n(t)$ (see [3, 10, 12, 14]). Many interesting results have been established regarding these polynomials. In [14], the author gives the closed form expressions

$$B_n(t) = \sum_{k=0}^{n}\binom{n+k+1}{1+2k}t^k, \tag{8}$$
$$b_n(t) = \sum_{k=0}^{n}\binom{n+k}{2k}t^k, \tag{9}$$
$$c_n(t) = \sum_{k=0}^{n}\frac{2n+1}{2k+1}\binom{n+k}{2k}t^k, \tag{10}$$

and their generating functions

$$\sum_{n\geq 0} B_n(t)z^n = \frac{1}{1-(t+2)z-z^2},$$
$$\sum_{n\geq 0} b_n(t)z^n = \frac{1-z}{1-(t+2)z-z^2},$$
$$\sum_{n\geq 0} c_n(t)z^n = \frac{1+z}{1-(t+2)z-z^2}.$$

It is easy to establish the following equalities involving negative subscripts.

$$B_{-n}(t) = -B_{n-2}(t),\ n\geq 0, \tag{11}$$
$$b_{-n}(t) = b_{n-1}(t),\ n\geq 0,$$
$$c_{-n}(t) = -c_{n-1}(t),\ n\geq 0.$$

So, it follows from $(8), (9)$ and (10) that

$$B_{-n}(t) = \sum_{k=0}^{n-2} \binom{-n+k+1}{1+2k} t^k, \tag{12}$$

$$b_{-n}(t) = \sum_{k=0}^{n-1} \binom{-n+k}{2k} t^k, \tag{13}$$

$$c_{-n}(t) = \sum_{k=0}^{n-1} \frac{1-2n}{1+2k} \binom{-n+k}{2k} t^k. \tag{14}$$

In this section, as an application of Theorem 5.2, we establish a new formula for $B_{-n}(t)$, $b_{-n}(t)$ and $c_{-n}(t)$.

Proposition 5.3. *We have,*

$$B_{-n}(t) = -\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{-n+2k+1}{k} (t+2)^{n-2-2k}, \ n \geq 2$$

$$b_{-n}(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{-n+2k}{k} (t+2)^{n-1-2k} - \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{-n+2k+1}{k} (t+2)^{n-2-2k}, \ n \geq 1$$

$$c_{-n}(t) = -\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{-n+2k}{k} (t+2)^{n-1-2k} - \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{-n+2k+1}{k} (t+2)^{n-2-2k}, \ n \geq 1$$

Proof. By putting in Theorem 5.2, $r = 1, q = -2, p = 0, x = y = (t+2)^{-1}$, we get

$$\sum_{n \geq 0} V_{-n-1} z^n = \frac{t+2}{1-(t+2)z+z^2},$$

$$\sum_{n \geq 0} V_{-n-1} z^n = (t+2) \sum_{n \geq 0} B_n(t) z^n.$$

So, for $n \geq 0$, $V_{-n-1} = (t+2) B_n(t)$ and from (11),

$$B_{-n}(t) = -(t+2)^{-1} V_{-n+1} = -\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{-n+2k+1}{k} (t+2)^{n-2-2k}$$

for $n \geq 2$.

In the same way, one gets (13) and (14). □

6 Conclusion

In this paper, we considered an extension of generalized arithmetic triangle to negative values of rows and we established the recurrence relation associated to the sum of diagonal elements laying along finite rays, this last one is of order r or $-q$ according to the sign of $r+q$, where $r \in \mathbb{N}$ and $q \in \mathbb{Z}^-$. We also wrote down the corresponding generating function. We conclude by an application to Fibonacci numbers and Morgan-Voyce polynomials with negative subscripts.

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