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On certain inequalities for the prime counting function – Part II

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Abstract: As a continuation of [6], we deduce some inequalities of a new type for the prime counting function $\pi(x)$.
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1 Introduction

Let $\pi(x)$ denote the number of primes $\leq x$, where $x \geq 1$ denotes a positive integer. There exist many inequalities as well as conjectures for this function. For a survey of results, see the monograph [4] and the recent papers [1, 6]. The famous Hardy–Littlewood conjecture states that for any $x, y \geq 1$ one has

$$\pi(x+y) \leq \pi(x) + \pi(y). \tag{1}$$

There exist many inequalities related to (1), which remains unproved up to now. In [6] we established the following counterpart of (1):

$$\pi(x+y) \geq \frac{2}{3} \cdot [\pi(x) + \pi(y)] \ \ (x, y \geq 1), \tag{2}$$

where the constant $\frac{2}{3}$ is the best possible one. This inequality offers a new proof, as well as a refinement of one of the main results from [1].

The aim of this paper is to offer certain new inequalities for $\pi(x)$. Particularly, inequalities involving the square root of $\pi(x)$ will be considered.

2 Main results

Theorem 1. *One has*

$$\sqrt{3\pi(x+y)} \geq \sqrt{\pi(x)} + \sqrt{\pi(y)} \quad \textit{for all } x, y \geq 1 \tag{3}$$

and

$$\sqrt{x\pi(x)} + \sqrt{y\pi(y)} \leq \frac{\sqrt{6}}{2} \cdot \sqrt{(x+y)\pi(x+y)} \quad \textit{for all } x, y \geq 1. \tag{4}$$

Proof. The following basic algebraic inequality will be used:

$$\sqrt{(a+b)(c+d)} \geq \sqrt{a \cdot c} + \sqrt{b \cdot d}, \tag{5}$$

where $a, b, c, d > 0$. Particularly, for $c = d = 1$ we get:

$$\sqrt{2 \cdot (a+b)} \geq \sqrt{a} + \sqrt{b}. \tag{6}$$

Applying inequality (2), one has

$$\sqrt{3(\pi(x+y))} \geq \sqrt{2 \cdot (\pi(x) + \pi(y)} \geq \sqrt{\pi(x)} + \sqrt{\pi(y)},$$

by (6). Thus, inequality (3) follows, in an improved form.

Applying again relation (2), we get by (5):

$$\sqrt{(x+y)\pi(x+y)} \geq \sqrt{\frac{2}{3}} \cdot \sqrt{(x+y)(\pi(x)+\pi(y))} \geq \sqrt{\frac{2}{3}} \cdot \left(\sqrt{x\pi(x)} + \sqrt{y\pi(y)}\right),$$

where $a = x$, $b = y$, $c = \pi(x)$, $d = \pi(y)$ in (5).

As $\sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$, inequality (4) follows, too. □

Theorem 2. *For any* $x, y \geq 2$ *one has*

$$\sqrt{\pi(x+y)} < \sqrt{\pi(x)} + \sqrt{\pi(y)}. \tag{7}$$

For infinitely many (x, y) *one has*

$$\sqrt{2\pi(x+y)} \geq \sqrt{\pi(x)} + \sqrt{\pi(y)} \tag{8}$$

and for infinitely many (x, y) *one has*

$$\sqrt{2\pi(x+y)} \leq \sqrt{\pi(x)} + \sqrt{\pi(y)}. \tag{9}$$

Proof. Let $2 \leq y \leq x$. For the proof of (7) we will use an inequality by Montgomery–Vaughan (see e.g. [4]):

$$\pi(x+y) \leq \pi(x) + 2\pi(y). \tag{10}$$

Thus $\sqrt{\pi(x+y)} \leq \sqrt{\pi(x) + 2\pi(y)}$. Let $a = \pi(x)$, $b = \pi(y)$. We will prove that $\sqrt{a+2b} < \sqrt{a} + \sqrt{b}$. Indeed, this is equivalent to $2b < 2\sqrt{ab}$, i.e., $b < 4a$. As $b = \pi(y)$ and $y \leq x$, one has $\pi(y) \leq \pi(x) < 4\pi(x)$, so $b < 4a$ is trivially true. This proves inequality (7).

For the proof of (8), set e.g. $x = n!$, $y = 2$, where $n \geq 4$. As $\pi(n!+3) = \pi(n!+1) = m$, the inequality $\sqrt{2m} > \sqrt{m} + 1$ becomes $\sqrt{m} \cdot (\sqrt{m} - 2) > 1$ which is true for $\sqrt{m} > 3$; i.e., $m > 9$, true for $n \geq 9$.

To prove (9), let $x = p_k$, $y = p_k+1$, where p_k is the k^{th} prime number. Then $\pi(x) = \pi(y) = k$. By Landau's inequality ([1], [4]) we get $\pi(2n) \leq 2\pi(n)$, without equality for $n \geq 10$, thus $\pi(2p_k) \leq 2p_k - 1$, if $p_k \geq 10$ (i.e., $k \geq 5$), so inequality (9) follows. $\square$

Theorem 3. *For any $x, y \geq 2$ one has*

$$(x+y)\sqrt{\pi(x+y)} \leq x \cdot \sqrt{2\pi(x)} + y\sqrt{2\pi(y)}. \tag{11}$$

Proof. We will use the famous Rosser–Schoenfeld inequality (see e.g. [3,4,6]) and L. Panaitopol (see [2])

$$\frac{x}{\log x - \frac{1}{2}} < \pi(x) < \frac{x}{\log x - 1.12} \quad \text{for} \quad x \geq 67. \tag{12}$$

Let us introduce the function

$$f(x) = \frac{x^{3/2}}{\left(\log x - \frac{1}{2}\right)^{1/2}} \quad (x > 0).$$

After elementary computations, we get for the second derivative of third function:

$$8\sqrt{x} \cdot f''(x) \cdot y^{5/2} = 6y^2 - 7y + 6, \tag{13}$$

where $y = \log x - 1/2$. As $6y^2 - 7y + 6 > 0$, we get first the function $f(x)$ is strictly convex.

By using the left-hand side of (12) and the convexity of $f(x)$, we can write that

$$x\sqrt{\pi(x)} + y\sqrt{\pi(y)} > f(x) + f(y) \geq 2f\left(\frac{x+y}{2}\right).$$

On the other hand, by the right-hand side of (12) we get

$$(x+y)\sqrt{\pi(x+y)} < \frac{(x+y)^{3/2}}{\Big(\log(x+y) - 1.12\Big)^{1/2}}.$$

Thus, in order to have (11), we have to consider the inequality

$$\frac{(x+y)^{3/2}}{\left(\log(x+y) - 1.12\right)^{1/2}} < \frac{\sqrt{2} \cdot 2 \cdot \left(\frac{x+y}{2}\right)^{\frac{3}{2}}}{\Big(\log\left(\frac{x+y}{2}\right) - \frac{1}{2}\Big)^{1/2}},$$

i.e.,

$$\log(x+y) - \frac{1}{2} - \log 2 < \log(x+y) - 1.12,$$

i.e., $\frac{1}{2} + \log 2 > 1.12$ which is true as $0.5 + 0.69\ldots = 1.19\ldots > 1.12$.

Therefore, we have proved inequality (11) for any $x, y \geq 67$.

It remains to consider $x \geq y$ and $y \leq 67$. Then $\pi(y) \leq \pi(67) = 18$. As $y\sqrt{\pi(y)} \geq 2\sqrt{2}$, we have to consider the inequality

$$(x+66)\sqrt{\pi(x)+18} < x\sqrt{2\pi(x)} + 2\sqrt{2}. \tag{14}$$

This can be rewritten as

$$\pi(x)\cdot(x^2-132\cdot x-66^2)+8x\sqrt{\pi(x)}>18x^2+18\cdot 132x+18\cdot 66^2. \tag{15}$$

We will prove that this is true if, e.g., $\pi(x)\geq 60$, i.e., $x\geq 281$. Indeed, after elementary computations, (15) can be written as $x\cdot(42x-78\cdot 132+56)>78\cdot 66^2$ or as $x\geq 281$, as $438922>339768$, which is true.

Now, a computer search shows that inequality (11) is true for $2\leq y\leq x\leq 280$.

The computer search shows also that there is equality in (11) only for $(x,y)=(2,2);(4,3);(4,4);(10,10)$, if $2\leq y\leq x$.

This finishes the proof of Theorem 3. □

Theorem 4. *The inequality*

$$\pi(x+y)\cdot\sqrt{x+y}\leq\pi(x)\cdot\sqrt{2x}+\pi(y)\cdot\sqrt{2y} \tag{16}$$

is true for any $2\leq y\leq x$, *except for the following values:* $(x,y)=(4,3);(10,9)$.

Proof. The proof is similar to the proof of Theorem 3, by considering the application

$$g(x)=\frac{x^{3/2}}{\log x-\frac{1}{2}},$$

which is convex for $x>e^{1/2}$.

By using again the double inequality (12), it follows that (16) holds true for any $x,y\geq 67$.

If $x\geq y$, $y\leq 66$, it is sufficient to prove the relation

$$\sqrt{x+66}\cdot(\pi(x)+18)\leq\pi(x)\cdot\sqrt{2x}+2. \tag{17}$$

It is easy to show that this inequality holds true for $\pi(x)\geq 70$, i.e., $x\geq 349$.

Finally, for $2\leq y\leq x\leq 349$, a computer search can be done, which shows that the inequality is correct, except for $(x,y)=(4,3);(10,9)$. There is equality only if $(x,y)=(2,2);(4,4);(10,10)$. □

Remark 1. Though inequality (9) of Theorem 2 is not generally true, for the values (x,y) for which it is true, the inequality is stronger than the Hardy–Littlewood inequality (1). Indeed, on the basis of (9) we can write.

$$2\pi(x+y)\leq\pi(x)+\pi(y)+2\sqrt{\pi(x)+\pi(y)}\leq 2(\pi(x)+\pi(y))$$

by $2\sqrt{ab}\leq a+b$ for $a=\pi(x)$, $b=\pi(y)$. Thus (1) follows.

In an analogous manner, it can be shown that the following inequality:

$$\sqrt{(x+y)\pi(x+y)}\leq\sqrt{x\pi(x)}+\sqrt{y\pi(y)}, \tag{18}$$

which is a counterpart to (4) is not generally true. However, for the values (x,y) for which (18) is true, it is stronger than the Hardy–Littlewood inequality.

Indeed, on the basis of (5) one can write

$$\sqrt{(x+y)\pi(x+y)}\leq\sqrt{x\pi(x)}+\sqrt{y\pi(y)}\leq\sqrt{(x+y)(\pi(x)+\pi(y))},$$

so (1) follows from (18).

Finally we mention two inequalities which can be proved by the methods shown above:

Theorem 5. *For any* $x, y \geq 2$ *one has*

$$\pi(x^2 + y^2) \geq \frac{5}{6} \cdot [\pi(x^2) + \pi(y^2)], \tag{19}$$

and

$$\sqrt{\frac{12}{5} \cdot \pi(x^2 + y^2)} \geq \sqrt{\pi(x^2)} + \sqrt{\pi(y^2)}. \tag{20}$$

Remark 2. Connections of $\pi(x)$ with other arithmetic functions can be found in [5].

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