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A note on generalized and extended Leonardo sequences

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Abstract: This note considers some real and complex extensions and generalizations of the Leonardo sequence, which is embedded within each of these two types of intriguing sequences, intriguing because there are still some unanswered questions. The connections between inhomogeneous and homogeneous forms are used as examples of a possible reason that the Leonardo sequences have been, in a sense, historically neglected.

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1 Introduction

Leonardo sequences satisfy a second order inhomogeneous linear recurrence relation of the form

$$a_n = a_{n-1} + a_{n-2} + n,\ n \geq 2, \tag{1.1}$$

with $a_0 = a_1 = 1$, [10]. A revival of interest in these Leonard Fibonacci sequences seems to have been initiated by the paper from Paula Catarino and Anabela Borges [2]. There was also some passing attention in the early days of the Fibonacci Association [7] in order to emphasize the genius of Leonard Fibonacci, but for the most part it was a case of converting nonhomogeneous second order forms into higher order homogeneous forms, such as when $n = 1$, the second order

Equation (1.1) can be converted to the third order linear homogeneous form $b_n = 2b_{n-1} - b_{n-3}$, as in the corresponding row of Table 1 below. This possibly accounts for the relative dearth of number theory specifically about Leonardo sequences *per se.*

When $a_0 = a_1 = 0$, we get the last row in the array in Table 2 below. Deveci [3] has considered links to the variation

$$a_n = a_{n-1} + a_{n-2} + (-1)^n, \ n \geq 2. \tag{1.2}$$

The ultimate aim of this particular note is to consider further generalizations and extensions, including

$$a_n = a_{n-1} + a_{n-2} + (-1)^n j, \ \ n \geq 2, \ \ j \geq 0, \tag{1.3}$$

and

$$a_n = a_{n-1} + a_{n-2} + (n + j), \ n \geq 2. \tag{1.4}$$

2 Examples of Equation (1.3)

A collection of generalized Leonardo sequences assembled from Equation (1.3) are set out in Table 1. These actually include the well-known standard Fibonacci and Leonardo sequences.

$n \rightarrow$ / $j \downarrow$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Comments**
0	1	1	2	**3**	**5**	**8**	**13**	**21**	**34**	Fibonacci
1	1	1	3	**5**	**9**	**15**	**25**	**41**	**67**	Leonardo
2	1	1	4	3	9	**10**	**21**	**29**	**52**	Red = no systematic difference patterns
3	1	1	5	3	11	**11**	**25**	**33**	**61**	
4	1	1	6	3	13	7	**24**	**27**	**55**	
5	1	1	7	3	15	13	33	41	79	
6	1	1	8	3	17	14	37	45	88	
7	1	1	9	3	19	15	41	49	97	
8	1	1	10	3	21	16	45	53	106	
9	1	1	11	3	23	17	49	57	115	
10	1	1	12	3	25	18	53	61	124	
Differences	0	0	1	0	2	1	4	4	9	

Table 1. Generalized Leonardo sequences

Do the differences in the last row of Table 1 have two sequences embedded within them, namely, {0, 1, 1, 4, 9, …} and {0, 0, 1, 4, …}? If so, do the other rows, and why?

3 Examples of Equation (1.4)

Another collection of extended Leonardo sequences can also then be built from Equation (1.4)

$$a_n = a_{n-1} + a_{n-2} + (n + j),\ n \geq 2,$$

and the first few can be represented in the array in Table 2.

$j\downarrow$ \ $n \rightarrow$	0	1	2	3	4	5	6	7	8	OEIS
–2	1	1	1	2	4	8	15	27	47	A000126
–1	1	1	2	4	8	15	27	47	80	A000071*
0	1	1	3	6	12	22	39	67	113	A066982
1	1	1	4	8	16	29	51	87	146	A030119
2	1	1	5	10	20	36	63	107	179	A210677
3	1	1	6	12	24	43	74	127	212	A210678
Differences	0	0	1	2	4	7	12	20	33	A000071

Table 2. Extended Leonardo sequences

In the row * where $j = -1$, the differences between the elements within the row follow the pattern $\{F_n - 1\} \equiv \{0, 1, 2, 4, 7, 12, 20, 33, \ldots\}$. It is trivial, but tedious, to establish that the elements of Table 2 also satisfy the fourth order linear homogeneous recurrence relation

$$a_n = 3a_{n-1} - 2a_{n-2} - a_{n-3} + a_{n-4},\ n \geq 4, \tag{3.1}$$

which is another example of the nonhomogeneous case being turned into a homogeneous equivalent.

Next we define the complex-type Leonardo sequence by the following homogeneous linear recurrence relation as another type of extension

$$a_n^{(i)} = ia_{n-1}^{(i)} - a_{n-2}^{(i)} + i^n,\ n \geq 2, \tag{3.2}$$

with $\sqrt{i} = -1, a_0^{(i)} = 0, a_1^{(i)} = 1.$

By mathematical induction on n, we obtain the following related results:

- if n is even, then

$$\operatorname{Re}(a_n^{(i)}) = -(\operatorname{Re}(a_{n-1}^{(i)}) + \operatorname{Im}(a_{n-1}^{(i)}))$$

  and

$$\operatorname{Im}(a_n^{(i)}) = (\operatorname{Re}(a_{n-1}^{(i)}) - \operatorname{Im}(a_{n-2}^{(i)}))\,;$$

- if n is odd, then

$$\operatorname{Re}(a_n^{(i)}) = -(\operatorname{Im}(a_{n-1}^{(i)}) + \operatorname{Re}(a_{n-2}^{(i)}))$$

  and

$$\operatorname{Im}(a_n^{(i)}) = \operatorname{Re}(a_{n-1}^{(i)}) - \operatorname{Im} a_{n-1}^{(i)}\,.$$

We now use the particular Leonardo sequence $\{b_n\}$ (Equations (3.3) and (4.2) below), which satisfies

$$b_n = b_{n-1} + b_{n-2} + 1,$$

with initial terms $b_0 = 0$, $b_1 = 1$, so that the first few terms are $\{0, 1, 2, 4, 7, 12, 20, 33, \ldots\}$. We will return to $\{b_n\}$ later. We can now obtain by induction that

$$a_n^{(i)} = \begin{cases} -iF_{n-1} + b_{n-1} & \text{for } n \equiv 0 \pmod 4, \\ +ib_{n-1} + F_{n-1} & \text{for } n \equiv 1 \pmod 4, \\ +iF_{n-1} - b_{n-1} & \text{for } n \equiv 2 \pmod 4, \\ -ib_{n-1} - F_{n-1} & \text{for } n \equiv 3 \pmod 4, \end{cases}$$

for $n \geq 4$, and

$$F_n + L_{n+1} = \begin{cases} -\operatorname{Im}(a_{n+3}^{(i)}) & \text{for } n \equiv 0 \pmod 4, \\ +\operatorname{Re}(a_{n+3}^{(i)}) & \text{for } n \equiv 1 \pmod 4, \\ +\operatorname{Im}(a_{n+3}^{(i)}) & \text{for } n \equiv 2 \pmod 4, \\ -\operatorname{Re}(a_{n+3}^{(i)}) & \text{for } n \equiv 3 \pmod 4, \end{cases}$$

for $n \geq 0$, where F_n and L_n (second and third rows of Table 1) are the n-th Fibonacci and Leonardo numbers, respectively. Now we have

$$b_n = \left|\operatorname{Re}(a_n^{(i)})\right| + \left|\operatorname{Im}(a_n^{(i)})\right|$$

for $n \geq 0$, and by using the sequence $\{b_n\}$, we find the following relationships among the elements of the sequences F_n and L_n:

- $b_n = F_{n+1} - 1$ for $n \geq 0$,
- $L_n - b_{n-1} = F_n$ for $n \geq 1$

and so

$$L_{n+4} = b_{n+4} + b_{n+2} - b_n, n \geq 0. \tag{3.3}$$

4 Other connections

Not surprisingly, there are also some other connections with and among various combinations of the Fibonacci and Lucas numbers; see, for example, A081659, A000071 and A066982 [11]. Riordan has dealt with some of these sequence connections from a combinatorial position [9], and Jarden [6] has also considered them from the point of view of the following variation of the Leonardo equation related to Equation (1.2) above:

$$a_n = a_{n-1} + a_{n-2} \mp 1, \; n \geq 2, \tag{4.1}$$

and the associated third order linear recurrence

$$b_n = 2b_{n-1} - b_{n-3}, \; n \geq 3, \tag{4.2}$$

to which both the A000071 sequence and the Leonardo sequence conform, the difference between these two sequences being the Fibonacci sequence itself. In fact, Jarden considers the sequences in Table 3 which can bring out the corresponding analogies with the Fibonacci and Lucas sequences.

(+1)	0	1	2	3	4	5	6	7	8
u_n	–1	0	0	1	2	4	7	12	20
v_n	1	0	2	3	6	10	17	28	46

(–1)	0	1	2	3	4	5	6	7	8
U_n	1	2	2	3	4	6	9	14	22
V_n	3	2	4	5	8	12	19	30	48

Table 3. Jarden's examples of Equation (4.1)

Among other properties, we then have respectively

$$v_n = u_{n-1} + u_{n+1} + 1 \text{ and } V_n = U_{n-1} + U_{n+1} - 1.$$

Other properties can be built with combinations of these sequences by analogy with those in Lucas [8] from

$$U_n = u_n + 2 \text{ and } V_n = v_n + 2.$$

5 Concluding comments

Dijkstra [4], a recipient of the Turing Award in 1972 from the Association for Computing Machinery, extended and generalized the Leonardo numbers with his two sequences $\{H_n\}$ and $\{K_n\}$ and the Leonardo-like format of their recurrence relations

$$H_n = H_{n-1} + H_{n-2} + (F_n),\ n \geq 1, \tag{5.1}$$

and

$$K_n = K_{n-1} + K_{n-2} + (L_n - 1),\ n \geq 1, \tag{5.2}$$

with initial conditions $H_0 = -1$, $H_1 = -1$, $K_0 = 0$, $K_1 = 0$, and where F_n and L_n represent the Fibonacci and Leonardo numbers respectively. For comparisons, the first few terms of these 'Leonardo' sequences are set out in Table 4.

n	0	1	2	3	4	5	6	7	8
H_n	–1	–1	0	2	7	14	29	56	106
K_n	0	0	2	6	16	30	60	114	214
L_n	1	1	3	5	9	15	25	41	67

Table 4. Leonardo-type sequences

Further developments could also include generalizing and expressing Equations (1.3) and (1.4) in terms of their Horadam extensions [5] by applying Asveld's techniques [1] to the sequence $\{w_n\}$ defined by

$$w_n = pw_{n-1} - qw_{n-2} + (p - q - 1)\sum_{j=0}^{k} \alpha_j n^j \,, \tag{5.3}$$

and appropriate initial conditions, and in which p and q are non-zero arbitrary integers.

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