Sums involving generalized harmonic and Daehee numbers

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Abstract: In this paper, we establish some sums involving generalized harmonic and Daehee numbers which are derived from the generating functions. For example, for \( n, r \geq 1 \),

\[
\sum_{i=0}^{n} H(i, r-1, \alpha) H_{n-i}^r(\alpha) = \sum_{l_1+l_2+\cdots+l_{r+1}=n} H_{l_1}(\alpha) H_{l_2}(\alpha) \cdots H_{l_{r+1}}(\alpha).
\]

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1 Introduction

The harmonic numbers are defined by

\[
H_0 = 0 \quad \text{and} \quad H_n = \sum_{i=1}^{n} \frac{1}{i} \quad \text{for} \quad n \geq 1.
\]

It is well known that

\[
H_n = \int_0^1 \frac{1 - t^n}{1 - t} \, dt = \gamma + \psi(n + 1),
\]
where \( \gamma \) denotes the Euler–Mascheroni constant, defined by

\[
\gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \log n \right) = -\psi'(1) \approx 0.57721566490153286066512 \ldots
\]

Harmonic numbers are closely related to the Riemann \( \xi \)-function defined by

\[
\xi(s) = \sum_{i=1}^{\infty} \frac{1}{i^s} = \prod_p \left(1 - p^{-s}\right)^{-1},
\]

where the product is over all primes \( p \).

These numbers have been generalized by some authors [1, 2, 4, 9, 16, 18]. In [9], for any \( \alpha \in \mathbb{R}^+ \), the generalized harmonic numbers \( H_n(\alpha) \) are defined by

\[
H_0(\alpha) = 0 \quad \text{and} \quad H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i^\alpha} \quad \text{for } n \geq 1.
\]

For \( \alpha = 1 \), \( H_n(1) = H_n \) are the usual harmonic numbers and the generating function of the generalized harmonic numbers is

\[
\sum_{n=1}^{\infty} H_n(\alpha)x^n = -\frac{\ln \left(1 - \frac{x}{\alpha}\right)}{1 - x}.
\]

In [15], for the generalized harmonic numbers \( H_n(\alpha) \), the authors defined the generalized hyperharmonic numbers of order \( r, H_n^r(\alpha) \) as follows:

**Definition 1.** For \( r < 0 \) or \( n \leq 0 \), \( H_n^r(\alpha) = 0 \) and for \( n \geq 1 \), the generalized hyperharmonic numbers of order \( r, H_n^r(\alpha) \), are defined by

\[
H_n^r(\alpha) = \sum_{i=1}^{n} H_{i-1}^r(\alpha), \quad r \geq 1,
\]

where \( H_0^r(\alpha) = \frac{1}{n^\alpha} \).

For \( \alpha = 1 \), \( H_n^r(1) = H_n^r \) are the hyperharmonic numbers of order \( r \). The generating function of the generalized hyperharmonic numbers of order \( r \) is

\[
\sum_{n=1}^{\infty} H_n^r(\alpha)x^n = -\frac{\ln \left(1 - \frac{x}{\alpha}\right)}{(1 - x)^r}.
\]

In [4, 18], the generalized harmonic numbers \( H(n, r) \) of rank \( r \) are defined as for \( n \geq 1 \) and \( r \geq 0 \),

\[
H(n, r) = \sum_{1 \leq n_0 + n_1 + \cdots + n_r \leq n} \frac{1}{n_0n_1\cdots n_r}.
\]

It is clear that \( H(n, 0) = H_n \). The generating function of the generalized harmonic numbers \( H(n, r) \) of rank \( r \) is defined by

\[
\sum_{n=0}^{\infty} H(n, r)x^n = \frac{(-\ln (1-x))^{r+1}}{1-x}.
\]

In [8], inspired from works [4, 15, 18], \( H(n, r, \alpha) \) are defined as for \( n \geq 1 \) and \( r \geq 0 \),

\[
H(n, r, \alpha) = \sum_{1 \leq n_0 + n_1 + \cdots + n_r \leq n} \frac{1}{n_0n_1\cdots n_r\alpha^{n_0+n_1+\cdots+n_r}}.
\]
For $\alpha = 1$, $H(n, r, 1) = H(n, r)$. The generating function of the generalized harmonic numbers of rank $r$, $H(n, r, \alpha)$, is given by
\[
\sum_{n=0}^{\infty} H(n, r, \alpha) x^n = \frac{(-\ln(1 - \frac{x}{\alpha}))^{r+1}}{1 - x}.
\]  
(2)

The Cauchy numbers of order $r$, $C'_r$, are defined by the generating functions to be
\[
\left( \frac{x}{\ln (1 + x)} \right)^r = \sum_{n=0}^{\infty} C'_r x^n n!.
\]  
(3)

The Daehee numbers of order $r$, $D'_r$, are defined by the generating functions to be
\[
\left( \frac{\ln (1 + x)}{x} \right)^r = \sum_{n=0}^{\infty} D'_r x^n n!.
\]  
(4)

For $r = 1$, $D'_1 = D_n$ are Daehee numbers. It is clear that $D_0 = 1$, $D_1 = -\frac{1}{2}$, ..., $D_n = (-1)^n \frac{n!}{n+1}$.

The derangement numbers $d_n$ are defined by the generating functions to be
\[
e^{-x} \frac{1}{1 - x} = \sum_{n=0}^{\infty} d_n x^n n!.
\]  
(5)

and $d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$ [5].

The generalized geometric series are given by for $a, b \in \mathbb{Z}^+$,
\[
\sum_{n=b}^{\infty} \binom{a + n - b}{n - b} x^n = \frac{x^b}{(1 - x)^{a+1}},
\]  
(6)

and the exponential generating function is
\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.
\]

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be two generating functions. The product of these functions is given as follows:
\[
F(x)G(x) = \left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n,
\]  
(7)

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Recently, there are many studies including generalized harmonic and special numbers which are obtained by generating functions [6–8, 10–14, 20].

In [17], Rim et al. investigated some identities with hyperharmonic, Daehee and derangement numbers. For example, for any positive integer $n$,
\[
\sum_{i=0}^{n} H_i \frac{(-1)^n}{(n-i)!} \frac{d_{n-i}}{(n-i)!} = \sum_{i=0}^{n} H_i^{r-1} \frac{d_{n-i}}{(n-i)!}.
\]
In [8], Duran et al. obtained sums including generalized harmonic numbers and special numbers. For example, for any positive integers \( n, r \) and \( m \),

\[
H(n, r, \alpha) = \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{n-j-r} \binom{m-1}{n-i} H^m_j(\alpha) s(i-j, r) r!
\]

where the Stirling numbers of the first kind \( s(n, i) \) are given by

\[
x^n = \sum_{i=0}^{n} s(n, i)x^i,
\]

where for \( n \geq 0 \), \( s(n, 0) = \delta_{n0} \), \( \delta_{ni} \) is the Kronecker delta [3, 19]. \( x^n \) stands for the falling factorial defined by

\[
x^n = x (x-1) \ldots (x-n+1).
\]

In [11], Kim et al. gave some new identities involving harmonic and hyperharmonic numbers which are derived from the transfer formula for the associated sequences. For example, for \( n, r \geq 1 \) and \( 1 \leq k \leq n \),

\[
\begin{align*}
\left( \frac{(r+3)}{n-k-1} \right) \left( n-k \right) \left( n-1 \right)^{n-k} \\
= \sum_{a=k}^{n} \sum_{l=0}^{n-a} \sum_{i=0}^{n} \left( \sum_{j_l+1}^{j_n+1} \cdots \sum_{m_n=1}^{m_1} \right) \left( m_1 \cdots m_n H_{r_{m_1}} \cdots H_{r_{m_n}} \right) l! r^l \\
\times \binom{n+l-1}{r+1} \binom{n-1}{a-1} s(n-a, l)(a-1)^{a-k}.
\end{align*}
\]

2 Sums with the generalized harmonic numbers of rank \( r \) and special numbers

This section, we will give some sums involving these numbers, using the generating functions of the generalized harmonic numbers of rank \( r \) and special numbers.

**Theorem 2.1.** Let \( n \) be a positive integer. For \( r \geq 1 \),

\[
\sum_{i=0}^{n} H(i, r-1, \alpha) H_{n-i}^r(\alpha) = \sum_{l_1+l_2+\cdots+l_{r+1}=n} H_{n}^1(\alpha) H_{n}^2(\alpha) \cdots H_{n}^{r+1}(\alpha) = \sum_{i=0}^{n} (-1)^i \binom{n-i-1}{r} D_{i+1}^{r+1} \alpha^{i+r+1}.
\]

**Proof.** By (1) and (2), we consider that

\[
\frac{-\ln \left( 1 - \frac{x}{\alpha} \right)}{1-x}, \frac{-\ln \left( 1 - \frac{x}{\alpha} \right)}{(1-x)^r} = \left( \sum_{n=0}^{\infty} H(n, r-1, \alpha) x^n \right) \left( \sum_{n=0}^{\infty} H_r^r(\alpha) x^n \right),
\]

and using (7), equals

\[
\sum_{n=0}^{\infty} \sum_{i=0}^{n} H(i, r-1, \alpha) H_{n-i}^r(\alpha) x^n,
\]
and
\[
\frac{(-\ln \left(1 - \frac{x}{\alpha}\right))^{r+1}}{(1-x)^{r+1}} = \left(\frac{-\ln \left(1 - \frac{x}{\alpha}\right)}{1-x}\right) \times \left(\frac{-\ln \left(1 - \frac{x}{\alpha}\right)}{1-x}\right) \times \ldots \times \left(\frac{-\ln \left(1 - \frac{x}{\alpha}\right)}{1-x}\right)
\]
\[
= \left(\sum_{l_1=0}^{\infty} H_{l_1}(\alpha)x^{l_1}\right) \left(\sum_{l_2=0}^{\infty} H_{l_2}(\alpha)x^{l_2}\right) \ldots \left(\sum_{l_{r+1}=0}^{\infty} H_{l_{r+1}}(\alpha)x^{l_{r+1}}\right)
\]
\[
= \sum_{n=0}^{\infty} \sum_{l_1+l_2+\ldots+l_{r+1}=n} H_{l_1}(\alpha)H_{l_2}(\alpha) \cdots H_{l_{r+1}}(\alpha)x^n.
\] (9)

Also, from (4) and (6), we have
\[
\frac{(-\ln \left(1 - \frac{x}{\alpha}\right))^{r+1}}{(1-x)^{r+1}} = \frac{\ln \left(1 - \frac{x}{\alpha}\right)}{-x} \frac{x^{r+1}}{(1-x)^{r+1}}
\]
\[
= \frac{1}{\alpha^{r+1}} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!\alpha^n} \sum_{n=r}^{\infty} \frac{n}{r} x^{n-1}
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!\alpha^{n+r+1}} \sum_{n=r+1}^{\infty} \frac{n}{r} x^{n}
\]
\[
= \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^i \left(\frac{n-i}{r}\right) \frac{x^{n-i}}{\alpha^{i+r+1}}
\] (10)

Hence from (8), (9) and (10), the desired results are obtained.

**Theorem 2.2.** Let \(n\) and \(r\) be positive integers. For \(m \geq 2\),

\[
\sum_{i=0}^{n} H(i, m-2, \alpha) H_{n-i}^{m-1}(\alpha) = \sum_{l_1+l_2+\ldots+l_m=n} H_{l_1}(\alpha)H_{l_2}(\alpha) \cdots H_{l_m}(\alpha),
\]

and

\[
\sum_{i=0}^{n} H(i, rm-2, \alpha) H_{n-i}^{m-1}(\alpha) = \sum_{l_1+l_2+\ldots+l_m=n} H(l_1, r-1, \alpha)H(l_2, r-1, \alpha) \cdots H(l_m, r-1, \alpha).
\]

**Proof.** The proof is similar to the proof of Theorem 2.1.

**Theorem 2.3.** Let \(n\) be a positive integer. For \(r \geq 1\),

\[
\sum_{i=0}^{n} (-1)^i \frac{C_i}{\alpha^{i-1}i!} H(n-i+1, r+1, \alpha) = H(n, r, \alpha).
\]

**Proof.** From (2) and (3), we write

\[
\sum_{n=0}^{\infty} H(n, r, \alpha)x^n = \frac{(-\ln \left(1 - \frac{x}{\alpha}\right))^{r+2}}{(1-x)^{r+2}} \frac{-x/\alpha}{\ln \left(1 - \frac{x}{\alpha}\right)} x^n
\]
\[
= \sum_{n=0}^{\infty} H(n, r+1, \alpha)x^n \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{\alpha^{n-1}n!}
\]
\[
= \sum_{n=0}^{\infty} H(n+1, r+1, \alpha)x^n \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{\alpha^{n-1}n!},
\]

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and by (7)
\[ \sum_{n=0}^{\infty} H(n, r, \alpha) x^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^i \frac{C_i}{\alpha^i i!} H(n - i + 1, r + 1, \alpha) x^n \]
as claimed. So, the proof is complete.

**Theorem 2.4.** Let \( n \) be a positive integer. For \( r \geq 1 \),
\[ \sum_{j=0}^{n} \sum_{i=0}^{j} H(i, r - 1, \alpha) H_{j-i}(\alpha) \frac{(-1)^{n-j}}{(n-j)!} = \sum_{i=0}^{n} H(i, r, \alpha) \frac{d_{n-i}}{(n-i)!} \]
and
\[ \sum_{i=0}^{n} \frac{(-1)^i d_{n-i-r-1} D_{i+1}^{r+1}}{\alpha^{i+r+1} i! (n-i-r-1)!} = \sum_{i=0}^{n} H(i, r, \alpha) \frac{(-1)^{n-i}}{(n-i)!} \]

**Proof.** By (7), we observe that
\[ \left( -\ln \left( 1 - \frac{x}{\alpha} \right) \right)^{r+1} e^{-x} = \sum_{n=0}^{\infty} H(n, r, \alpha) x^n \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n \]
\[ = \sum_{n=0}^{\infty} \sum_{i=0}^{n} H(i, r, \alpha) \frac{d_{n-i}}{(n-i)!} x^n \] (11)
and
\[ \left( -\ln \left( 1 - \frac{x}{\alpha} \right) \right)^{r+1} e^{-x} = \left( -\ln \left( 1 - \frac{z}{\alpha} \right) \right) - \frac{\ln\left( 1 - \frac{x}{\alpha} \right)}{1-x} e^{-x} \]
\[ = \sum_{n=0}^{\infty} H(n, r - 1, \alpha) x^n \sum_{n=0}^{\infty} H_n(\alpha) x^n \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \]
\[ = \sum_{n=0}^{\infty} \sum_{i=0}^{n} H(i, r - 1, \alpha) H_{n-i}(\alpha) x^n \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \]
\[ = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{i=0}^{j} H(i, r - 1, \alpha) H_{j-i}(\alpha) \frac{(-1)^{n-j}}{(n-j)!} x^n. \] (12)

From here, (11) and (12) yield the desired result.

From (7), we write
\[ \left( -\ln \left( 1 - \frac{z}{\alpha} \right) \right)^{r+1} e^{-x} = \sum_{n=0}^{\infty} H(n, r, \alpha) x^n \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \]
\[ = \sum_{n=0}^{\infty} \sum_{i=0}^{n} H(i, r, \alpha) \frac{(-1)^{n-i}}{(n-i)!} x^n, \] (13)
and, by (4) and (5),

\[
\frac{\left( -\ln\left( 1 - \frac{x}{\alpha} \right) \right)^{r+1}}{1 - x} e^{-x} = \left( \frac{\ln(1 - x/\alpha)}{-x} \right)^{r+1} \frac{e^{-x}}{1 - x} x^{r+1}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \frac{D_{n+r+1}^1}{\alpha^{n+r+1} n!} x^n \sum_{m=0}^{\infty} \frac{d_n}{n!} x^{n+r+1}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \frac{D_{n+r+1}^1}{\alpha^{n+r+1} n!} x^n \sum_{n=0}^{\infty} \frac{d_{n-r-1}}{(n-r-1)!} x^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^i d_{n-i-r-1}}{\alpha^{i+r+1} i!} \frac{D_{i}^{r+1}}{(n-i-r-1)!} x^n.
\]

(14)

With the help of (13) and (14), we have a relation between the generalized harmonic numbers of rank \( r \), \( H(n,r,\alpha) \), and Daehee numbers of order \( r \).

\( \square \)

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**References**


