Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 1, Pages 81–91
DOI: 10.7546/nntdm.2022.28.1.81-91

Set partitions with isolated successions

Toufik Mansour[1] and Augustine O. Munagi[2]

[1] Department of Mathematics, University of Haifa
3498838 Haifa, Israel
e-mail: tmansour@univ.haifa.ac.il

[2] School of Mathematics, University of the Witwatersrand
Johannesburg 2050, South Africa
e-mail: augustine.munagi@wits.ac.za

Received: 24 March 2021 **Revised:** 14 February 2022
Accepted: 15 February 2022 **Online First:** 15 February 2022

Abstract: We enumerate partitions of the set $\{1, \ldots, n\}$ according to occurrences of isolated successions, that is, integer strings $a, a+1, \ldots, b$ in a block when neither $a-1$ nor $b+1$ lies in the same block. Our results include explicit formulas and generating functions for the number of partitions containing isolated successions of a given length. We also consider a corresponding analog of the associated Stirling numbers of the second kind.
Keywords: Partition, Isolated succession, Recurrence, Generating function.
2020 Mathematics Subject Classification: 05A18, 05A15, 05A19.

1 Introduction

A partition of $[n] = \{1, 2, \ldots, n\}$ is a decomposition of $[n]$ into nonempty subsets called *blocks*. A partition into k-blocks is also called a k-partition and denoted by $B_1/B_2/\ldots/B_k$, where the blocks are arranged in standard order: $\min(B_1) < \cdots < \min(B_k)$ (see [4]).

The number of k-partitions of $[n]$ is the Stirling number of the second kind $S(n,k)$ which satisfies the recurrence relation:

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k), \tag{1}$$

where $S(0,0) = 1,\ S(n,0) = S(0,n) = 0$ for $n > 0$.

The classical associated Stirling number of the second kind $S_2(n, k)$ enumerates k-partitions of $[n]$ into non-singleton blocks (see [2, 8]). A further refinement of $S(n, k)$ is the t-associated Stirling number of the second kind $S_t(n, k)$ which is the number of k-partitions of $[n]$ into blocks of size $\geq t$. In particular $S_1(n, k) = S(n, k)$. These numbers are defined by the triangular recurrence relation

$$S_t(n, k) = k\, S_t(n-1, k) + \binom{n-1}{t-1} S_t(n-t, k-1). \tag{2}$$

Inspired by these results, Munagi [7] recently studied the enumeration of partitions with respect to occurrences of 'isolated singletons', where an isolated singleton refers to an element a in a block B_i such that $a-1,\ a+1 \notin B_i$. For example, the partition $1, 2, 4, 6/3/5, 7, 8$ contains four isolated singletons, namely $4, 6, 3, 5$. The number $g_0(n, k)$ of k-partitions of $[n]$ containing no isolated singletons is given by [7, Theorem 2]:

$$g_0(n, k) = \sum_{j\geq 1} \binom{n-j-1}{j-1} S(j-1, k-1). \tag{3}$$

In this paper we generalize the singletons case and consider the enumeration of partitions of $[n]$ by strings of consecutive integers. A maximal string of $t > 0$ consecutive integers will also be called a *succession of length* t or a t*-succession*.

Notation. Let $[a, b]$ denote $\{a, a+1, \ldots, b\} \subseteq [n]$ and let $[b]$ denote $[1, b]$. So $[a, b]$ represents a succession of length $b - a + 1$.

Definition. *Let $B_1/B_2/\cdots/B_k$ be a partition of $[n]$ and let $[a, a+t-1] \subseteq B_i$, $1 \leq i \leq k$. Then we say that $[a, a+t-1]$ is an* isolated succession *of length t (or an* isolated t-succession*) if $|B_i| = t$ or $a-1 \notin B_i$ and $a+t \notin B_i$.*

For example, the partition $1, 3, 4, 6, 7, 8/2, 5/9, 10/11$ contains two isolated 2-successions, namely $[3, 4]$ and $[9, 10]$. The partition $1, 3, 6, 7, 8/2, 5/4, 9, 11/10$ contains none.

A related subsisting idea in the literature is concerned with the enumeration of partitions according to the number of occurrences of general unrestricted successions (see [5] and [6]). Here the partition $1, 3, 4, 6, 7, 8/2, 5/9, 10/11$ is deemed to contain four unrestricted 2-successions, namely $[3, 4], [6, 7], [7, 8]$ and $[9, 10]$. The notion of enumeration of partitions according to the number of isolated successions of length > 1 appears to be new.

In this paper we obtain enumeration results for the number of partitions of $[n]$ according to the number of occurrences of isolated t-successions for any positive integer $t \leq n$.

Let $g_r(n, k, t)$ denote the number of k-partitions of $[n]$ containing r isolated t-successions, and let $g_0(n, k, t) = g(n, k, t)$.

In Section 2 we first consider the isolated succession analog of the t-associated Stirling numbers of the second. Then in Section 3 we obtain a recursive formula and an explicit formula for the function $g(n, k, t)$. These results will lead to the derivation of corresponding formulas for $g_r(n, k, t)$ in Section 4.

2 Isolated Stirling numbers

Define $q_t(n,k)$ to be the number of k-partitions of $[n]$ containing only isolated successions of length $\geq t$. The numbers $q_t(n,k)$ satisfy the recurrence:

Proposition 2.1. *Given integers n, k, t with $1 < k < n,\ 1 < t < n$, we have*

$$q_t(n,k) = q_t(n-1,k) + q_t(n-t,k-1) + (k-1)q_t(n-t,k)$$
$$q_t(0,0) = 1,\ q_t(n,1) = 1,\ q_t(n,n) = \delta_{1t},\ q_n(n,k) = \delta_{1k},\ q_1(n,k) = S(n,k), \tag{4}$$

where δ_{ij} is the Kronecker delta ($\delta_{ii} = 1, \delta_{ij} = 0, i \neq j$).

Proof. We construct an enumerated partition $p = B_1/\cdots/B_k$ by considering the length of the maximal string of consecutive integers containing n.

The number of partitions p in which n belongs to a succession of length $\geq t+1$ is $q_t(n-1,k)$ (obtained by putting n into the block containing $n-1$).

The number of partitions p containing the block $[n-t+1,n]$ is $q_t(n-t,k-1)$ (obtained by inserting $[n-t+1,n]$ into a partition enumerated by $q_t(n-t,k-1)$).

The number of partitions p in which $[n-t+1,n] \subsetneq B_i$ and $n-t \notin B_i$, $i \in [k]$ is $(k-1)q_t(n-t,k)$ (obtained by putting the elements $n-t+1,\ldots,n$ into any block except the block containing $n-t$ in a partition enumerated by $q_t(n-t,k)$).

Addition of the three classes of partitions gives the main result. The initial values are evident and may be verified separately. □

Remark 1. *Observe that $q_2(n,k) = g_0(n,k)$, where the explicit formula is stated in* (3).

We next obtain exact computational formulas for $q_t(n,k)$.

Define $Q_t(x;k) = \sum_{n\geq k} q_t(n,k)x^n$. Then by Proposition 2.1, we have, for $t \geq 2$,

$$Q_t(x;k) = xQ_t(x;k) + x^tQ_t(x;k-1) + x^t(k-1)Q_t(x;k),$$

which leads to

$$Q_t(x;k) = \frac{x^t}{1-x-(k-1)x^t}Q_t(x;k-1).$$

Note that $Q_t(x;1) = x^t/(1-x)$. Thus, by induction on k, we have

$$Q_t(x;k) = \frac{x^{kt}}{(1-x)\prod_{j=2}^{k}(1-x-(j-1)x^t)}.$$

Hence, using the fact that $\frac{x^k}{\prod_{j=1}^{k}(1-jx)} = \sum_{n\geq k} S(n,k)x^n$, we obtain

$$\begin{aligned} Q_t(x;k) &= \frac{x^{kt}}{(1-x)^k\prod_{j=1}^{k-1}\left(1-j\frac{x^t}{1-x}\right)} \\ &= x^t\sum_{n\geq k} S(n,k-1)\frac{x^{nt}}{(1-x)^{n+1}} \\ &= \sum_{i\geq k}\sum_{j\geq 0}\binom{i+j}{j}S(n,k-1)x^{it+t+j}. \end{aligned}$$

Hence we have proved the following result.

Proposition 2.2. *The generating function and the exact formula for the t-isolated Stirling number $q_t(n,k)$ are given by*

$$\sum_{n\geq 0} q_t(n,k)x^n = \frac{x^{kt}}{(1-x)\prod_{j=1}^{k-1}(1-x-jx^t)},\ t\geq 2$$

and

$$q_t(n,k) = \sum_{i\geq k-1} \binom{n+i-(i+1)t}{i} S(i,k-1). \tag{5}$$

Some values of $q_t(n,k)$ are illustrated for $t = 2,3$ in Table 1. Note that in order to have $q_t(n,k) > 0$ it is necessary that $1 \leq k \leq \lfloor \frac{n}{t} \rfloor$. When k is maximal, then $n = kt + r,\ 0 \leq r < t$, and Equation (5) reduces to

$$q_t(n,k) = \binom{k+r-1}{r},\ n = kt+r,\ 0 \leq r < t.$$

$q_2(n,k) = g(n,k,1)$

$n\backslash k$	1	2	3	4	5	$q_2(n)$
2	1	0	0	0	0	1
3	1	0	0	0	0	1
4	1	1	0	0	0	2
5	1	2	0	0	0	3
6	1	4	1	0	0	6
7	1	7	3	0	0	11
8	1	12	9	1	0	23
9	1	20	22	4	0	47
10	1	33	52	16	1	103
11	1	54	116	50	5	226

$q_3(n,k)$

$n\backslash k$	1	2	3	4	$q_3(n)$
3	1	0	0	0	1
4	1	0	0	0	1
5	1	3	0	0	1
6	1	1	0	0	2
7	1	2	0	0	3
8	1	3	0	0	4
9	1	5	1	0	7
10	1	8	3	0	12
11	1	12	6	0	19
12	1	18	13	1	33

Table 1. Tables of $q_t(n,k),\ q_t(n) = \sum_k q_t(n,k)$ for $t = 2,3$

3 Partitions avoiding isolated t-successions

The number $g(n,k,t)$ of k-partitions of $[n]$ containing no isolated t-successions satisfies the following recurrence.

Theorem 3.1. *Given integers n,k,t with $0 < k < n,\ 0 < t < n$, we have*

$$\begin{aligned} g(n,k,t) &= g(n-1,k-1,t) + kg(n-1,k,t) - g(n-t,k-1,t) - (k-1)g(n-t,k,t) \\ &\quad + g(n-t-1,k-1,t) + (k-1)g(n-t-1,k,t). \end{aligned} \tag{6}$$

Alternatively, we have

$$g(n,k,t) = \sum_{i\geq 1}(g(n-i,k-1,t)+(k-1)g(n-i,k,t)) - g(n-t,k-1,t) - (k-1)g(n-t,k,t). \tag{7}$$

$$g(0,0,t) = 1,\ g(n,1,t) = 1-\delta_{nt},\ g(n,n,t) = 1-\delta_{1t},\ g(n,k,n) = S(n,k)(1-\delta_{1k}).$$

Proof. For each $j \in [t-1]$, if $[n-j+1, n] = B_i,\ i \in [k]$, we obtain $g(n-j, k-1, t)$ partitions.

We construct an enumerated partition $p = B_1/\cdots/B_k$ in three ways, as follows.

If $[n-j+1, n] = B_i,\ i \in [k]$ for each $j \in [t-1]$, we obtain $g(n-j, k-1, t)$ partitions. But when $[n-j+1, n] \subsetneq B_i$ and $n-j \notin B_i$, we obtain $(k-1)g_t(n-j, k, t)$ partitions.

So the total number of partitions in which n belongs to an isolated succession of length $< t$ is

$$\sum_{i=1}^{t-1}(g(n-i, k-1, t) + (k-1)g(n-i, k, t)). \tag{8}$$

To obtain a partition in which n belongs to an isolated succession of length $\geq t+2$, we put n into the block containing $n-1$ in a partition enumerated by $g(n-1, k, t)$ provided that $n-1$ belongs to an isolated succession of length $\geq t+1$. So the number of partitions in which n belongs to an isolated succession of length $\geq t+2$ is (using (8) with $n-1$ in place of n):

$$g(n-1, k, t) - \sum_{i=1}^{t-1}(g((n-1)-i, k-1, t) + (k-1)g((n-1)-i, k, t)). \tag{9}$$

It remains to account for partitions in which n belongs to an isolated succession of length $t+1$. Their number is clearly

$$g(n-t-1, k-1, t) + (k-1)g(n-t-1, k, t). \tag{10}$$

Adding Equations (8) to (10) we obtain

$$\begin{aligned} g(n,k,t) = &\sum_{i=1}^{t-1}(g(n-i, k-1, t) + (k-1)g(n-i, k, t)) \\ &+ g(n-1, k, t) - \sum_{i=1}^{t-1}(g(n-i-1, k-1, t) + (k-1)g(n-i-1, k, t)) \\ &+ g(n-t-1, k-1, t) + (k-1)g(n-t-1, k, t). \end{aligned} \tag{11}$$

Then, on shifting limits in the second summation and canceling terms between the two summations, we obtain the recurrence (6):

$$\begin{aligned} g(n,k,t) = g(n-1, k-1, t) &+ (k-1)g(n-1, k, t) \\ &+ g(n-1, k, t) - g(n-t, k-1, t) - (k-1)g(n-t, k, t) \\ &+ g(n-t-1, k-1, t) + (k-1)g(n-t-1, k, t). \end{aligned} \tag{12}$$

A different approach is obtained by noting that the number of partitions in which n belongs to an isolated succession of length $\geq t+1$ is given directly by

$$\sum_{i=t+1}^{n-k+1}(g(n-i, k-1, t) + (k-1)g(n-i, k, t)). \tag{13}$$

Thus addition of Equations (8) and (13) gives the second recurrence (7).

The initial values are intuitive. For example, the trivial 1-block partition $[n]$ implies that $g_n(n, 1) = 0$ and $g_t(n, 1) = 1$ when $t \neq n$, but if $k > 1$, then $g_n(n, k) = S(n, k)$ since no block can contain an n-succession. □

The solution of the recurrence (6) or (7) is given by the following explicit formula.

Theorem 3.2. *We have*

$$g(n,k,t)=\sum_{j\geq k}\sum_{i\geq 0}(-1)^i\binom{j}{i}\binom{n-1-it}{j-1-i}S(j-1,k-1).$$

In order to prove this theorem we first obtain a result on the number of integer compositions of n into k summands without t, to be denoted by $w_t(n,k)$. (A recurrence relation for $w_t(n,k)$ is derived in [1]). The generating function is

$$w_t(x;k)=\sum_{n\geq 0}w_t(n,k)x^n=((x+x^2+x^3+\cdots)-x^t)=\left(\frac{x}{1-x}-x^t\right)^k$$

which leads to

$$\begin{aligned}w_t(x;k)&=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{x^{k-j+jt}}{(1-x)^{k-j}}\\&=\sum_{j=0}^{k}\sum_{i\geq 0}(-1)^j\binom{k}{j}\binom{k-1-j+i}{k-1-j}x^{k-j+jt+i}.\end{aligned}$$

Hence,

$$w_t(n,k)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{n-1-jt}{k-1-j},$$

where we define $\binom{n}{-1}=\delta_{-1,n}$.

Thus, we can state the following result.

Lemma 3.3. *We have*

$$w_t(n,k)=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{n-1-jt}{k-1-j}. \tag{14}$$

Proof of Theorem 3.2. The function $g(n,k,t)$ enumerates k-partitions of $[n]$ in which every block consists of distinct successions of lengths $\neq t$. Any pair of successive distinct successions, $a,a+1,\dots,a+u-1$ and $b,b+1,\dots,b+v-1$ in a block, satisfy $0<u,v\neq t$ and $a+u<b$. A partition with this property may be constructed in two steps. First obtain a j-partition of $[n]$, say $\{H_1,\dots,H_j\}$, in which every block consists of one isolated succession of length $\neq t$, where $j\geq k$. To obtain such partitions simply divide the sequence $1,2,\dots,n$ into j segments by inserting $j-1$ separators between the terms such that no segment has length t. Second, obtain a partition of $\{H_1,\dots,H_j\}$ (regarded as just a set of j distinct objects) into k blocks of nonconsecutive label numbers, $\{B_1,\dots,B_k\}$, that is, if $H_q,H_s\in B_i$, then $|q-s|>1$. Last, a desired partition $P=\{S_1,\dots,S_k\}$ is obtained by setting $S_i=\bigcup_{H\in B_i}B$, $1\leq i\leq k$.

The construction of a partition in the first step corresponds to the process of putting $j-1$ stars between n bars on a line such that each bar separates m stars, $m\neq t$. This procedure is known to generate the compositions of n into j summands different from t (see [3]). For example, with $t = 2$, the segments $123/4/5678/9$ correspond to $*\,*\,*\,|\,*\,|\,*\,*\,*\,*\,|\,*$ which identifies the composition $3+1+4+1$. The corresponding number of partitions is the number of compositions of n into j summands without t, that is, $w_t(n, j)$. It is clear that the subsequent construction in the second step generates as many partitions as the number of k-partitions of $[j]$ into blocks of nonconsecutive elements. This number is known to be $S(j-1, k-1)$ (see [4]). Thus for each j the number of partitions P is $w_t(n, j)S(j-1, k-1)$. Hence $g(n, k, t) = \sum_j w_t(n, j)S(j-1, k-1)$ which proves the theorem in view of (14). □

Remark 2. *Since $g(n, k, 1) = g_0(n, k)$, one may use Theorem 3.2 and Equation* (3) *to obtain the following identity which is reminiscent of other identities in Shattuck's collection [9]:*

$$\sum_{m=k}^{n-1}\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}\binom{n-1-i}{n-m}S(m-1, k-1) = \sum_{j\geq 1}\binom{n-j-1}{j-1}S(j-1, k-1),\ n > k > 1.$$

Corollary 3.3.1. *The number of partitions of $[n]$ that contain no isolated t-successions is given by $f(n, t) = \sum_k g(n, k, t)$:*

$$f(n, t) = \sum_{j\geq k}\sum_{i\geq 0}^{j}(-1)^i\binom{j}{i}\binom{n-1-it}{j-1-i}B(j-1),$$

where $B(n)$ denotes the n-th Bell number, defined by $B(n) = \sum_k S(n, k)$.

4 Partitions containing isolated t-successions

We now consider the general enumeration of k-partitions of $[n]$ containing $r \geq 0$ isolated t-successions.

Theorem 4.1. *Given positive integers n, k, t, r with $1 < k < n$, $0 < rt < n$, $1 < r$, we have either of the following relations:*

$$\begin{aligned} g_r(n, k, t) = g_r(n-1, k-1, t) &+ kg_r(n-1, k, t) \\ &- g_r(n-t, k-1, t) - (k-1)g_r(n-t, k, t) \\ &- g_{r-1}(n-t-1, k-1, t) - (k-1)g_{r-1}(n-t-1, k, t) \\ &+ g_r(n-t-1, k-1, t) + (k-1)g_r(n-t-1, k, t) \\ &+ g_{r-1}(n-t, k-1, t) + (k-1)g_{r-1}(n-t, k, t), \end{aligned} \tag{15}$$

$$\begin{aligned} g_r(n, k, t) = \sum_{j\geq 1}(g_r(n-j, k-1, t) &+ (k-1)g_r(n-j, k, t)) + g_{r-1}(n-t, k-1, t) \\ &+(k-1)g_{r-1}(n-t, k, t) - g_r(n-t, k-1, t) - (k-1)g_r(n-t, k, t). \end{aligned} \tag{16}$$

$$g_t(0, 0, t) = \delta_{0r},\ g_0(n, k, t) = g(n, k, t),\ g_r(rt, k, t) = S(r-1, k-1),\ g_r(n, 1, t) = \delta_{nt}\delta_{r1},$$
$$g_r(n, n, t) = \delta_{nr}\delta_{1t},\ g_r(t+1, 1, t) = \delta_{0r},\ g_r(t+1, 2, t) = \delta_{1t}\delta_{2r} + 2(1-\delta_{1t})\delta_{1r}.$$

Proof. Let $G_r(n,k,t)$ denote the set of partitions enumerated by $g_r(n,k,t)$. The proof is obtained in each case by constructing a partition $P = B_1/\cdots/B_k \in G_r(n,k,t)$.

Proof of (15). (i) The number of partitions in P containing the isolated succession $[n-j+1,n]$, $j \in [t-1]$ is

$$\sum_{j=1}^{t-1}(g_r(n-j,k-1,t)+(k-1)g_r(n-j,k,t)). \tag{17}$$

(ii) The number of partitions in $G_r(n-1,k,t)$ that do not contain the isolated successions $[(n-1)-j+1,(n-1)]$, $j \in [t]$, i.e., partitions in which $n-1$ belongs to a succession of length $\geq t+1$, is

$$\begin{aligned} g_r(n-1,k,t) - \sum_{j=1}^{t-1}&(g_r(n-1-j,k-1,t)+(k-1)g_r(n-1-j,k,t)) \\ &- (g_{r-1}(n-t-1,k-1,t)+(k-1)g_{r-1}(n-t-1,k,t)). \end{aligned} \tag{18}$$

So we put n into the block containing $n-1$ to obtain a partition P containing $[n-\ell+1,n]$, $\ell \geq t+2$.

(iii) The number of partitions in P containing the isolated succession $[n-t,n]$ is

$$g_r(n-t-1,k-1,t)+(k-1)g_r(n-t-1,k,t). \tag{19}$$

(iv) Lastly, the number of partitions in P containing the t-succession $[n-t+1,n]$ is

$$g_{r-1}(n-t,k-1,t)+(k-1)g_{r-1}(n-t,k,t). \tag{20}$$

Addition of the expressions in (17) to (20) gives

$$\begin{aligned} g_r(n,k,t) =& \sum_{j=1}^{t-1}(g_r(n-j,k-1,t)+(k-1)g_r(n-j,k,t)) \\ &+ g_r(n-1,k,t) - \sum_{j=1}^{t-1}(g_r(n-1-j,k-1,t)+(k-1)g_r(n-1-j,k,t)) \\ &- (g_{r-1}(n-t-1,k-1,t)+(k-1)g_{r-1}(n-t-1,k,t)) \\ &+ g_r(n-t-1,k-1,t)+(k-1)g_r(n-t-1,k,t) \\ &+ g_{r-1}(n-t,k-1,t)+(k-1)g_{r-1}(n-t,k,t), \end{aligned} \tag{21}$$

which simplifies to the first recurrence (15).

Proof of (16). The summation accounts for the number of partitions P in which n belongs to an isolated succession of length $j \neq t$ provided that the t-th summand is excluded, that is, the subtracted quantity $g_r(n-t,k-1,t)+(k-1)g_r(n-t,k,t)$. When added to the number of partitions in which n belongs to an isolated t-succession, given by (20), we obtain (16).

The initial values may be verified independently except possibly the following two. First, $g_r(rt,k,t) = S(r-1,k-1)$ because the $r>1$ instances of t-successions, namely, $(1,\ldots,t)$, $(t+1,\ldots,2t),\ldots,(t(r-1)+1,\ldots,rt)$, appear in valid partitions of $[rt]$ provided that each pair in a block are isolated or non-consecutive. So the number of cases is equal to the number of partitions of $[r]$ into blocks of non-consecutive integers, that is $S(r-1, t-1)$. Second, $g_r(t+1, 2, t) = 1$ or 2. Indeed $g_1(t+1, 2, t) = 2$ because of the partitions $1/2, \ldots, t+1$ and $1, \ldots, t/t+1$, while $g_2(t+1, 2, t) = 1$ provided that $t = 1$ giving the trivial partition $1/2$. Other values of t, r give $g_r(t+1, 2, t) = 0$. Combining the three cases we obtain $g_r(t+1, 2, t) = \delta_{1t}\delta_{2r} + 2(1-\delta_{1t})\delta_{1r}$. □

In order to obtain an explicit formula for $g_r(n, k, t)$, we need the formula for the number of compositions of n into j summands that contain r occurrences of t. Any such composition may be obtained by designating r of the j positions to hold t's, in $\binom{j}{r}$ ways; and then obtaining a composition of $n - rt$ without t's into the other $j - r$ positions, in $w_t(n - rt, j - r)$ ways. The number of compositions generated is $\binom{j}{r} w_t(n - rt, j - r)$.

Therefore, using a reasoning similar to the proof of Theorem 3.2 we obtain the solution of (15) and (16) in the form

$$g_r(n, k, t) = \sum_{j \geq 1} \binom{j}{r} w_t(n - rt, j - r) S(j - 1, k - 1), \quad n > rt. \tag{22}$$

Then applying Equation (14), the next theorem follows.

Theorem 4.2. *We have*

$$g_r(n, k, t) = \sum_{j \geq 1} \binom{j}{r} \sum_{i=0}^{j-r} (-1)^i \binom{j-r}{i} \binom{n - rt - 1 - it}{j - r - 1 - i} S(j - 1, k - 1), \quad n > rt,$$

where $g_r(rt, k, t) = S(r - 1, t - 1)$.

For example, the 2-partitions of $[5]$ are distributed according to containment of isolated 2-successions as follows:

$g_0(5, 2, 2) = 6 : 1, 2, 3, 4/5; 1, 2, 3, 5/4; 1, 3, 4, 5/2; 1, 3, 5/2, 4; 1, 5/2, 3, 4; 1/2, 3, 4, 5.$

$g_1(5, 2, 2) = 6 : 1, 2, 3/4, 5; 1, 2, 4/3, 5; 1, 3, 4/2, 5; 1, 4/2, 3, 5; 1, 2/3, 4, 5; 1, 3/2, 4, 5.$

$g_2(5, 2, 2) = 3 : 1, 2, 4, 5/3; 1, 4, 5/2, 3; 1, 2, 5/3, 4.$

$g_t(5, 2, 2) = 0,\ t > 2.$

The values of $g_r(n, k, t)$ are illustrated in Table 2 with $t = 2$ and $r = 1$ for $1 \leq n \leq 10$.

Corollary 4.2.1. *The number of k-partitions of $[n]$ containing r isolated t-successions is given by $f_r(n, t) = \sum_k g_r(n, k, t)$:*

$$f_r(n, t) = \sum_{j \geq 1} \binom{j}{r} \sum_{i=0}^{j-r} (-1)^i \binom{j-r}{i} \binom{n - rt - 1 - it}{j - r - 1 - i} B(j - 1), \quad n > rt,$$

where $f_r(rt, t) = B(r - 1)$.

$g_1(n,k,2)$

$n\backslash k$	1	2	3	4	5	6	7	8	9	$f_1(n,t)$
2	1	0	0	0	0	0	0	0	0	1
3	0	2	0	0	0	0	0	0	0	2
4	0	3	3	0	0	0	0	0	0	6
5	0	6	12	4	0	0	0	0	0	22
6	0	13	41	30	5	0	0	0	0	89
7	0	26	132	162	60	6	0	0	0	386
8	0	50	402	762	475	105	7	0	0	1801
9	0	96	1178	3302	3120	1150	168	8	0	9022
10	0	184	3368	13560	18389	10110	2436	252	9	48308

Table 2. Values of $g_r(n,k,t),\ f_r(n,t)=\sum_k g_r(n,k,t)$ for $t=2,\ r=1$.

4.1 Generating functions

In this section we obtain generating functions for $g(n,k,t)$ and $g_r(n,k,t)$.

Let $G(x;k,t)=\sum_{n\geq k} g(n,k,t)x^n$. Since $g(n,k,t)=\sum_{j\geq 1} w_t(n,j)S(j-1,k-1)$, we have

$$G(x;k,t)=\sum_{j\geq k} w_t(x;j)S(j-1;k-1)=\sum_{j\geq k}\left(\frac{x}{1-x}-x^t\right)^j S(j-1;k-1),$$

which gives

$$G(x;k,t)=\left(\frac{x}{1-x}-x^t\right)\sum_{j\geq k-1}\left(\frac{x}{1-x}-x^t\right)^j S(j;k-1).$$

Using the fact that $\sum_{j\geq k} S(j,k)x^j=\dfrac{x^k}{\prod_{j=1}^{k}(1-jx)}$, we have

$$G(x;k,t)=\frac{\left(\frac{x}{1-x}-x^t\right)^k}{\prod_{j=1}^{k-1}\left(1-j\left(\frac{x}{1-x}-x^t\right)\right)}.$$

Let $w_t(n,k,r)$ denote the number of compositions of n into k summands which contains r summands equal to t. Then from the proof of Theorem 4.2 we know that

$$w_t(n,k,r)=\binom{k}{r}w_t(n-rt,k-r).$$

Let $G_r(x;k,t)=\sum_{n\geq k} g_r(n,k,t)x^n$ be the generating function for $g_r(n,k,t)$. Then Equation (22) implies

$$G_r(x;k,t)=\sum_{j\geq k} w_t(x;j,r)S(j-1,k-1).$$

On other hand,

$$w_t(x;k,r)=\binom{k}{r}x^{rt}\left(\frac{x}{1-x}-x^t\right)^{k-r}.$$

Thus,

$$G_r(x;k,t) = \frac{x^{rt}}{\left(\frac{x}{1-x} - x^t\right)^r} \sum_{j\geq k} \binom{j}{r} \left(\frac{x}{1-x} - x^t\right)^j S(j-1,k-1),$$

which leads to

$$G_r(x;k,t) = \frac{x^{rt}}{r!\left(\frac{x}{1-x} - x^t\right)^r} \frac{d^r}{dy^r} \frac{y^k}{\prod_{i=1}^{k-1}(1-ix)}\Bigg|_{y=x/(1-x)-x^t}.$$

5 Conclusion

This paper has undertaken a thorough enumeration of set partitions according to distinct strings of consecutive integers, i.e., successions, lying in a block. We have provided complete formulas—recursive, generating function and explicit—for the new function $q_t(n,k)$ that enumerates the k-partitions of $[n]$ containing only isolated successions of length $\geq t$, in analogy with the t-associated Stirling numbers of the second kind. We have also considered the generalized enumeration function $g_r(n,k,t)$, $r \geq 0$ using the same agenda, and stated the corresponding results.

References

[1] Chinn, P., & Heubach, S. (2003). Compositions of n with no occurrence of k. *Congressus Numerantium*, 164, 33–51.

[2] Comtet, L. (1974). *Advanced Combinatorics. The Art of Finite and Infinite Expansions*. D. Reidel Publishing Co., Dodrecht.

[3] Heubach, S., & Mansour, T. (2010). *Combinatorics of Compositions and Words*. CRC, Taylor & Francis Group, Boca Raton.

[4] Mansour, T. (2013). *Combinatorics of Set Partitions*. CRC, Taylor & Francis Group, Boca Raton.

[5] Mansour, T., & Shattuck, M. (2013). Parity successions in set partitions. *Linear Algebra and Its Applications*, 439(9), 2642–2650.

[6] Munagi, A. O. (2008). Extended set partitions with successions. *European Journal of Combinatorics*, 29 (5), 1298–1308.

[7] Munagi, A. O. (2018). Set partitions with isolated singletons.*The American Mathematical Monthly*, 125(5), 447–452.

[8] Riordan, J. (1958). *An Introduction to Combinatorial Analysis*, John Wiley, New York.

[9] Shattuck, M. (2015). Combinatorial proofs of some Stirling number formulas. *Pure Mathematics and Applications*, 25(1), 107–113.