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Cycles of higher-order Collatz sequences

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Abstract: Consider a sequence of numbers $x_n \in \mathbb{Z}_+$ defined by $x_{n+1} = \frac{x_n}{2}$ if x_n is even, and $x_{n+1} = \frac{x_n + 2x_{n-1} + q}{2}$ if x_n is odd. A 1-cycle is a periodic sequence with one transition from odd to even numbers. We prove theoretical and computational results for the existence of 1-cycles, and discuss a generalization to more complex cycles.
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1 Introduction

1.1 Definitions and terminology

Consider the higher-order Collatz sequence, defined by the function

$$x_{n+1} = \frac{x_n}{2} \text{ if } x_n \equiv 0 \pmod 2 \text{ and } x_{n+1} = \frac{x_n + 2x_{n-1} + 1}{2} \text{ if } x_n \equiv 1 \pmod 2 \tag{1}$$

starting with $x_0 = 7, x_1 = 11$, i.e. $(7, 11, 13, 18, 9, 23, 21, 34, \ldots)$. By definition, this higher-order sequence differs from the original Collatz sequence. If n is odd, instead of $x_{n+1} = \frac{3x_n + 1}{2}$ we now have $x_{n+1} = \frac{x_n + 2x_{n-1} + 1}{2}$.

We define an m-cycle in a different way compared to the definition of Simons and de Weger [14] for the original Collatz function. See the Section 6, Remark (1). An m-cycle has K odd and L even numbers and consists of m pairs of a subsequence of odd numbers, followed by a subsequence of even numbers. If $x_0, x_1 = \frac{x_0}{2}$ are the first numbers, then $x_{K+L} = x_0$. The reason for our definition is the crucial role of m, the number of transitions in the sequence from odd to even numbers, in our analysis. As usual, we denote the number of odd and even numbers by K, L for m-cycles and by k, ℓ for 1-cycles.

For the original Collatz sequence, Steiner and also Davison [3, 15] call a subsequence of odd numbers, followed by a subsequence of even numbers, a circuit. Brox [2] considers cycles of odd numbers only and calls a number a descendent if the next number is smaller. A descendent is the (odd) predecessor of a local maximum in an m-cycle.

1.2 Motivation

Cycle existence for the original Collatz sequence had been researched extensively [7]. Roughly speaking, if a cycle exists then K, L must satisfy $2^{K+L} \simeq 3^K$ while transcendental number theory shows that $|(K + L)\log 2 - K \log 3|$ cannot be arbitrary small. This leads to an upper bound for K. Steiner proves that the only 1-cycle is $(1, 2)$. He uses a lemma on linear log forms, originally developed by Baker [1], later refined [8, 10, 13]. Luca [9] (and others) considers odd numbers x_j only. A cycle is represented by the structure $\langle \ell_1, \ldots, \ell_k \rangle$ where ℓ_j is the maximum power of 2 that divides $3x_j + 1$. In Luca's notation Steiner has proved that the only $\langle 1, \ldots, 1, \ell \rangle$ is $\langle 2 \rangle$. Luca proves a more general result i.e. the number of cycles of a particular structure type is finite.

Steiner's proof [15] of the (non-)existence of 1-cycles assumes a 1-cycle consisting of an increasing subsequence of k odd numbers (starting with $x_0 = a2^k - 1$), followed by a decreasing subsequence of ℓ even numbers (starting with $x_k = a3^k - 1$) down to $x_{k+\ell} = x_0$. From $x_{k+\ell} = \frac{x_k}{2^\ell}$ follows the equation $\frac{a3^k - 1}{2^\ell} = a2^k - 1$, and this leads to the kernel inequality

$$0 < 2^{k+\ell} - 3^k < 2^\ell, \tag{2}$$

with as only solution $k = \ell = 1$. Simons and de Weger [14] prove that for $m \leq 75$ no m-cycles exist and they present explicit bounds for the cycle length for $m > 75$. They prove (see also Brox [2]):

Theorem 1. *For each m the original Collatz sequence has a finite number of m-cycles, and for the cycle length an explicit m-dependent upper bound exists.*

As the example above shows, Steiner's assumption about the expression of the start numbers of the odd, and even subsequence $x_0 = a2^k - 1, x_k = a3^k - 1$ is no longer true, so his proof and the proof of Simons and de Weger (which are based on these expressions) cannot simply be generalized to higher-order Collatz sequences.

There is however computational evidence that Theorem 1 is true for higher-order Collatz sequences. For starting values $< 10^6$ the higher-order sequence of Equation (1) has 11 cycles, and similar computational evidence, i.e., some cycles with "small" numbers was found for $q > 1$. Every cycle for $q = 1$ corresponds to a cycle of q-folds if $q > 1$, however also cycles with numbers $\not\equiv 0 \pmod q$ can exist.

1.3 Main result

We generalize the approach of Simons and de Weger in a non-trivial way. A proof of Theorem 1 for 1-cycles of higher-order Collatz sequences and a list of existing of 1-cycles for small

q is presented in this paper. We discuss a possible generalization to m-cycles. We consider higher-order Collatz sequences in $\mathbb{Z}_+$ defined by:

$$x_{n+1} = \frac{x_n}{2} \text{ if } x_n \equiv 0 \pmod 2 \text{ and } x_{n+1} = \frac{x_n + 2x_{n-1} + q}{2} \text{ if } x_n \equiv 1 \pmod 2 \qquad (3)$$

with $q = 1$ or an odd prime. Our main result is:

Theorem 2 (**Main Theorem**). *Consider the higher-order Collatz sequence of Equation (3).*

1. *For each m there is a finite number of m-cycles.*
2. *The cycle length of an m-cycle is upper bounded by an explicit function of m.*
3. *For $q = 1$ there are no 1-cycles.*
4. *For $q = 3, 7, 11$ there are no other 1-cycles than listed in the table in section 4.*
5. *For $q = 5, 13, 17, 19$ there are no 1-cycles.*
6. *For $19 < q \leq 997$ 1-cycles are exceptional (numerical result, no theoretical proof).*

We start with an analysis for $q = 1$ and deal with $q > 1$ later.

2 Generalization of Steiner's proof for higher-order Collatz sequences

2.1 Rephrasing Steiner's proof for the original Collatz sequence

Steiner's proof can be rephrased without a priori using the expression $x_0 = a2^k - 1, x_k = a3^k - 1$. A cycle of k odd numbers, followed by ℓ even numbers starting with x_0 increases up to $x_k = \frac{3^k x_0 + 3^k - 2^k}{2^k} = 2^\ell x_0$. So we find

$$x_0 = \frac{3^k - 2^k}{2^{k+\ell} - 3^k} \qquad (4)$$

which can be rewritten as

$$2^k(2^\ell x_0 + 1) = 3^k(x_0 + 1). \qquad (5)$$

from which follows $x_0 = a2^k - 1$ and $2^\ell x_0 = a3^k - 1$. The expressions $x_0 = a2^k - 1, x_k = a3^k - 1$ are a result of the analysis. For higher-order Collatz sequences this line of analysis can be applied to find an appropriate expression for x_0.

2.2 An expression for x_0 in 1-cycles of the higher-order Collatz sequence with $q = 1$

Assume that for the sequence of Equation (1), there exists a 1-cycle, consisting of k odd numbers followed by ℓ even numbers. For ease of analysis we take x_1 odd, and $x_0 = 2x_1$. So $x_1, \ldots, x_k$ are odd numbers, $x_{k+1}, \ldots, x_{k+\ell}$ are even numbers, and $\frac{x_{k+1}}{2^{\ell-1}} = x_{k+\ell} = x_0$. From Equation (1) we find for $0 \leq n \leq k+1$

$$x_n = \frac{a_n x_0 + b_n}{2^n}, \tag{6}$$

where a_n, b_n are solutions of difference equations $a_{n+1} = a_n + 4a_{n-1},\ a_0 = a_1 = 1$, and $b_{n+1} = b_n + 4b_{n-1} + 2^n,\ b_0 = b_1 = 0$. For $n \geq 0$ we have

$$a_n = \frac{1}{\sqrt{17}}\left[\left(\frac{1+\sqrt{17}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{17}}{2}\right)^{n+1}\right], \tag{7}$$

$$b_n = \frac{\sqrt{17}+3}{2\sqrt{17}}\left(\frac{1+\sqrt{17}}{2}\right)^{n} + \frac{\sqrt{17}-3}{2\sqrt{17}}\left(\frac{1-\sqrt{17}}{2}\right)^{n} - 2^n. \tag{8}$$

Application of the cycle condition $x_{k+1} = \frac{a_{k+1}x_0 + b_{k+1}}{2^{k+1}} = 2^{\ell-1}x_0$ leads to a new expression (we define $\overline{b}_n = b_n + 2^n$ for comparison with Equation (4)) for x_0:

$$x_0 = \frac{\overline{b}_{k+1} - 2^{k+1}}{2^{k+\ell} - a_{k+1}} \tag{9}$$

This is a necessary condition for a 1-cycle and consequently this equation can result in a rational or negative x_0. E.g. $k = 3, \ell = 2$ leads to the sequence $(x_0 = \frac{22}{3}, \frac{11}{3}, \frac{29}{3}, \frac{27}{3}, \frac{44}{3}, \frac{22}{3})$, and $k = 2, \ell = 1$ leads to the sequence $(x_0 = -6,\ -3,\ -7,\ -6)$.

Equation (9) can be rewritten as

$$2^{k+1}(2^{\ell-1}x_0 + 1) = a_{k+1}x_0 + \overline{b}_{k+1}, \tag{10}$$

from which follows $a_{k+1}x_0 = c2^{k+1} - \overline{b}_{k+1}$.

We checked numerically that in general $(a_{k+1},\overline{b}_{k+1}) = 1$, so there is no simple expression $x_0 = w.2^k - d$ similar to $x_0 = a2^k - 1$ for the original Collatz sequence. Computational evidence suggests that for the higher-order Collatz sequence of Equation (1) the constant -1 in the expression $x_0 = a2^k - 1$ becomes a variable, depending on k.

2.3 The nonconstant term in the expression for x_1

Because of the choice $x_0 = 2x_1$, we analyze x_1 as the first odd number in a sequence. We are looking for an expression such that $x_1, \ldots, x_k$ are odd, and x_{k+1} is even. Indeed we have:

Lemma 3. *Consider the higher-order Collatz sequence of Equation (1). For $k \geq 1$ there exists a $d_{k+1} \equiv 1 \pmod 2$ with $1 \leq d_{k+1} \leq 2^{k+1} - 1$ such that for $x_1 = d_{k+1},\ x_0 = 2x_1$ the numbers $x_1 \ldots x_k$ are odd, and x_{k+1} is the first even number.*

Proof. From the sequence $(6,\ 3,\ 8)$ we easily find $d_2 = 3$.
We define $y_{j,k}$, $z_{j,k}$ through the initial conditions $y_{1,2} = 4,\ y_{2,2} = 10,\ z_{1,2} = 1,\ z_{2,2} = 3$, and $y_{j,k}$, $z_{j,k},\ d_{k+1}$ for $k \geq 2$ through the recurrence relations

$$y_{k+1,k} = \frac{y_{k,k} + 2 \cdot y_{k-1,k}}{2}, \tag{11}$$

$$z_{k+1,k} = \frac{z_{k,k} + 2 \cdot z_{k-1,k} + 1}{2}, \tag{12}$$

$$\text{if } z_{k+1,k} \equiv 0 \pmod 2, \text{ then } d_{k+1} = z_{1,k}, \tag{13}$$

$$\text{and for } 1 \leq j \leq k+1 \; z_{j,k+1} = y_{j,k} + z_{j,k}, \tag{14}$$

$$\text{if } z_{k+1,k} \equiv 1 \pmod 2, \text{ then } d_{k+1} = y_{1,k} + z_{1,k}, \tag{15}$$

$$\text{and for } 1 \leq j \leq k+1 \; z_{j,k+1} = z_{j,k}, \tag{16}$$

$$\text{for } 1 \leq j \leq k+1 \; y_{j,k+1} = 2 \cdot y_{j,k}. \tag{17}$$

We calculate d_3, d_4, d_5.

$$\begin{array}{rcll}
k=2 & \to & y_{3,2} = 9 & z_{3,2} = 3 \\
z_{3,2} \equiv 1 \pmod 2 & \to & d_3 = y_{1,2} + z_{1,2} = 5 & (= z_{1,2} + 2^2) \\
 & & z_{1,3} = z_{1,2} = 1 & y_{1,3} = 2y_{1,2} = 8 \\
 & & z_{2,3} = z_{2,2} = 3 & y_{2,3} = 2y_{2,2} = 20 \\
 & & z_{3,3} = z_{3,2} = 3 & y_{3,3} = 2y_{3,2} = 18
\end{array}$$

$$\begin{array}{rcll}
k=3 & \to & y_{4,3} = 29 & z_{4,3} = 5 \\
z_{4,3} \equiv 1 \pmod 2 & \to & d_4 = y_{1,3} + z_{1,3} = 9 & (= z_{1,3} + 2^3) \\
 & & z_{1,4} = z_{1,3} = 1 & y_{1,4} = 2y_{1,3} = 16 \\
 & & z_{2,4} = z_{2,3} = 3 & y_{2,4} = 2y_{2,3} = 40 \\
 & & z_{3,4} = z_{3,3} = 3 & y_{3,4} = 2y_{3,3} = 36 \\
 & & z_{4,4} = z_{4,3} = 5 & y_{4,4} = 2y_{4,3} = 58
\end{array}$$

$$\begin{array}{rcll}
k=4 & \to & y_{5,4} = 65 & z_{5,4} = 6 \\
z_{5,4} \equiv 0 \pmod 2 & \to & d_5 = z_{1,4} = 1 & (= z_{1,4}) \\
 & & z_{1,5} = y_{1,4} + z_{1,4} = 17 & y_{1,5} = 2y_{1,4} = 32 \\
 & & \ldots & \ldots
\end{array}$$

In general, there are two cases for $z_{k+1,k}$.

Case 1. $z_{k+1,k} \equiv 0 \pmod 2$. Set $x_j = z_{j,k}, j = 1, \ldots, k+1$, and $x_0 = 2x_1$. Now $x_1, \ldots, x_k$ are odd, x_{k+1} is even, and $(x_j, j = 1, \ldots, k+1)$ satisfy $x_{j+1} = \frac{x_j + 2x_{j-1} + 1}{2}$. We conclude that $d_{k+1} = x_1 = z_{1,k}$.

Case 2. $z_{k+1,k} \equiv 0 \pmod 1$. Set $x_j = z_{j,k}, j = 1, \ldots k+1$, and $x_0 = 2x_1$. Now $x_1, \ldots, x_{k+1}$ are odd. Set $y_1 = x_1 + 2^k$, and $y_0 = 2y_1$, and for $j = 1, \ldots, k+1$ define $y_{j+1} = \frac{y_j + 2y_{j-1} + 1}{2}$. For the maximal power of 2 that divides $y_j - x_j$ we have $2^{k+1-j} | y_j - x_j$. This implies that $y_{k+1} - x_{k+1}$ has maximal factor 2^0, i.e., y_{k+1} is even, while $y_1, \ldots y_k$ are odd. We conclude that $d_{k+1} = y_1 = x_1 + 2^k = z_{1,k} + y_{1,k}$.

The adjustment $d_{k+1} = y_{1,k} + z_{1,k}$ takes place at most k times, so $d_{k+1} \leq 2^{k+1} - 1$ which proves this lemma. $\square$

We computed $d_2 = 3, d_3 = 5, d_4 = 9, d_5 = 1, d_6 = 49, d_7 = 81, d_8 = 17, d_9 = 145, d_{10} = 913, \ldots$. For $k = 4$ we have $d_5 = 1$, and the sequence $(x_0 = 2, 1, 3, 3, 5, 6)$, etc.

2.4 The general expression for x_1

Note that $x_1 = d_{k+1} + 2^{k+1}$ also leads to k odd numbers, and x_{k+1} as the first even number. Hence the general expression for x_1 as the beginning of a sequence of k odd numbers, followed by an even number is $x_1 = w.2^{k+1} + d_{k+1}$ with constant w and d_{k+1} defined in Lemma 3.

3 1-cycles for the higher-order Collatz sequence with $q = 1$

3.1 The kernel inequality for 1-cycles

Once x_1 (and $x_0 = 2x_1$) are known, we have an expression for x_{k+1} by substitution of the expression for x_0 into Equation (6)

$$x_{k+1} = \frac{a_{k+1}(2^{k+2}w + 2d_{k+1}) + b_{k+1}}{2^{k+1}}. \tag{18}$$

From the cycle condition $\frac{x_{k+1}}{x_1} = 2^\ell$ we find

$$\frac{a_{k+1}(2^{k+2}w + 2d_{k+1}) + b_{k+1}}{2^{k+1}w + d_{k+1}} = 2^{k+\ell+1}, \tag{19}$$

which can be rewritten as

$$(2^{k+2}w + 2d_{k+1})(2^{k+\ell} - a_{k+1}) = b_{k+1}. \tag{20}$$

This leads to the higher-order Collatz kernel inequality (compare Equation (2))

$$0 < 2^{k+\ell} - a_{k+1} < \frac{b_{k+1}}{2d_{k+1}}. \tag{21}$$

To find a theoretical upper bound for k for 1-cycles of the higher-order Collatz sequence, we need for $2^{k+\ell} - a_{k+1}$ (as a function of k) an upper bound from Equation (21), and a lower bound from transcendental number theory.

3.2 An upper bound for $2^{k+\ell} - a_{k+1}$

For an effective theoretical upper bound for $2^{k+\ell} - a_{k+1}$, a lower bound for d_{k+1} that is exponential in k is sufficient. Computational evidence suggests that d_{k+1} grows exponentially with increasing k. From Lemma 3 we find that $z_{1,k}$ is a non-decreasing function of k, and $d_{k+1} \geq z_{1,k}$. The next Lemmas 4, 5, 6, and 7 supply an exponential lower bound for d_{k+1}.

Lemma 4. *Consider a recurring sequence $\{Z_j\}$ for $j \geq 1$ defined by*

$$Z_{j+1} = \frac{Z_j}{2} + Z_{j-1} \tag{22}$$

with initial conditions $Z_0, Z_1 \in \mathbb{Z}_+$. Suppose that for $j = 0, \dots, k$ $Z_k \in \mathbb{Z}_+$.
Then $4^{\lfloor \frac{k}{2} \rfloor} \leq 10(\max(Z_0, Z_1))^2$.

Proof. Let $\alpha = \frac{1+\sqrt{17}}{4}$, $\beta = \frac{1-\sqrt{17}}{4}$ be the roots of the characteristic equation $x^2 - \frac{x}{2} - 1 = 0$. Then the solution of Equation (22) is

$$Z_j = \frac{2}{\sqrt{17}}((Z_1 - \beta Z_0)\alpha^j + (\alpha Z_0 - Z_1)\beta^j). \tag{23}$$

Since $Z_k \in \mathbb{Z}$, $\sqrt{17}Z_k \in \mathbb{Z}(\frac{1+\sqrt{17}}{2})$, we find

$$2^k\sqrt{17}Z_k = 2\left((Z_1 - \beta Z_0)\left(\frac{1+\sqrt{17}}{2}\right)^k + (\alpha Z_0 - Z_1)\left(1 - \frac{\sqrt{17}}{2}\right)^k\right). \tag{24}$$

Both sides of this equation represent a quadratic integer in $\mathbb{Z}(\frac{1+\sqrt{17}}{2})$, so we have

$$2^k \mid 2((Z_1 - \beta Z_0)\left(\frac{1+\sqrt{17}}{2}\right)^k + (\alpha Z_0 - Z_1)\left(1 - \frac{\sqrt{17}}{2})^k\right). \tag{25}$$

Further we have $(\frac{1+\sqrt{17}}{2}\frac{1-\sqrt{17}}{2})^{\lfloor \frac{k}{2} \rfloor} = (-4)^{\lfloor \frac{k}{2} \rfloor} \mid 2^k$. Note that $\mathbb{Z}(\frac{1+\sqrt{17}}{2})$ is a unique (Euclidean) factorization domain [6], and that $(\frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}) = 1$. As a consequence we have

$$\left(\frac{1+\sqrt{17}}{2}\right)^{\lfloor \frac{k}{2} \rfloor} \mid 2^k \mid 2(\alpha Z_0 - Z_1). \tag{26}$$

Computing norms in $\mathbb{Z}(\frac{1+\sqrt{17}}{2})$, we find ($\|.\|$ denotes the absolute value)

$$\begin{aligned} 4^{\lfloor \frac{k}{2} \rfloor} &= \left\| N(\frac{1+\sqrt{17}}{2}) \right\|^{\lfloor \frac{k}{2} \rfloor} \\ &\leq \|2N(\alpha Z_0 - Z_1)\| \\ &= \| -4Z_0^2 - 2Z_0Z_1 + 4Z_1^2\| \\ &\leq 10(\max(Z_0, Z_1))^2. \end{aligned} \tag{27}$$

This completes the proof. □

Lemma 5. *Let $y_{j,k}, z_{j,k}, d_k$ be defined as in Equations (11) and further. Then $y_{j,j} < 2^{2j}, z_{j,j} < 2^{2j}$ for $2 \leq j \leq k+1$.*

Proof. The proof is by induction. First part is for $y_{j,j}$. For $j = 2$ $y_{j-1,j} = y_{1,2} = 4 < 2^3$, and $y_{j,j} = y_{2,2} = 10 < 2^4$. Assume that $y_{j-1,j} < 2^{2j-1}, y_{j,j} < 2^{2j}$ for some $j \geq 2$.

Using the appropriate recurrence relation from Equations (11) and further, we find

$$\begin{aligned} y_{j+1,j} &= \frac{y_{j,j} + 2y_{j-1,j}}{2} < \frac{2^{2j} + 2.2^{2j-1}}{2} = 2^{2j}, \\ y_{j,j+1} &= 2y_{j,j} < 2^{2j+1}, \\ y_{j+1,j+1} &= 2y_{j+1,j} < 2^{2j+1} < 2^{2j+2}. \end{aligned}$$

The second part is for $z_{j,j}$, and uses the result for $y_{i,j}$. For $j = 2$, $z_{j-1,j} = z_{1,2} = 1 < 2^3$, and $z_{j,j} = z_{2,2} = 3 < 2^4$. Assume that $z_{j-1,j} < 2^{2j-1}, z_{j,j} < 2^{2j}$ for some $j \geq 2$. Using the appropriate recurrence relation from Equations (11) and further, we find

$$z_{j+1,j} = \frac{z_{j,j} + 2z_{j-1,j} + 1}{2} < \frac{2^{2j} + 2^{2j} + 1}{2} < 2^{2j+1}.$$

The worst case for $z_{j,j+1}, z_{j+1,j+1}$ is $z_{j+1,j} \equiv 0 \pmod 2$. Consequently

$$z_{j,j+1} \leq y_{j,j} + z_{j,j} < 2^{2j+1},$$

and

$$z_{j+1,j+1} \leq y_{j+1,j} + z_{j+1,j} < 2^{2j} + 2^{2j+1} < 2^{2j+2}.$$

This completes the proof. □

We now assume that indices $1 < i < k$ exist such that $z_{k+1,k}$ is even, $z_{k,k-1} \ldots z_{i+2,i+1}$ are odd, and $z_{i+1,1}$ is even. Then (using the appropriate recurrence relation) we have

$$\begin{array}{lllll} z_{k+1,k} \equiv 0 & d_{k+1} = z_{1,k} & z_{1,k+1} = y_{1,k} + z_{1,k} & \ldots & z_{k+1,k+1} = y_{k+1,k} + z_{k+1,k} \\ z_{k,k-1} \equiv 1 & d_k = y_{1,k-1} + z_{1,k-1} & z_{1,k} = z_{1,k-1} & z_{k-1,k} = z_{k-1,k-1} & z_{k,k} = z_{k,k-1} \\ z_{k-1,k-2} \equiv 1 & d_{k-1} = y_{1,k-2} + z_{1,k-2} & z_{1,k-1} = z_{1,k-2} & z_{k-2,k-1} = z_{k-2,k-2} & z_{k-1,k-1} = z_{k-1,k-2} \\ z_{k-2,k-3} \equiv 1 & d_{k-2} = y_{1,k-3} + z_{1,k-3} & z_{1,k-2} = z_{1,k-3} & z_{k-3,k-2} = z_{k-3,k-3} & z_{k-2,k-2} = z_{k-2,k-3} \\ \ldots & & & & \\ z_{i+2,i+1} \equiv 1 & d_{i+2} = y_{1,i+1} + z_{1,i+1} & z_{1,i+2} = z_{1,i+1} & \ldots & z_{i+2,i+2} = z_{i+2,i+1} \\ z_{i+1,i} \equiv 0 & d_{i+1} = z_{1,i} & z_{1,i+1} = y_{1,i} + z_{1,i} & \ldots & z_{i+1,i+1} = y_{i+1,i} + z_{i+1,i} \end{array}$$

From the last column we find that $z_{j+1,j+1} = z_{j+1,j}$ for $i+1 \leq j \leq k-1$, and the combination of the last two columns shows $z_{j+1,j} = z_{j+1,j+2}$ for $i+1 \leq j \leq k-2$. Putting $Z_j = z_{j,j} + 1$ we find for $j = i+3 \ldots k-1$ the recurrence relation

$$Z_{j+1} = \frac{Z_j}{2} + Z_{j-1} \tag{28}$$

with initial conditions $Z_{i+1} = z_{i+1,i+1} + 1,\ Z_{i+2} = z_{i+2,i+2} + 1$.

From the second and the third column we find

$$d_{k+1} = z_{1,k} = z_{1,k-1} = \ldots = z_{1,i+1} = y_{1,i} + z_{1,i} > y_{1,i} = 2^i \tag{29}$$

Lemma 6. $d_{k+1} > 2^{0.25k-3.75}$.

Proof. We apply Lemmas 4 and 5 to find an inequality relation between k and i

$$2^{k-1} \leq 4^{\lfloor \frac{k}{2} \rfloor} \leq 10(\max(Z_{i+1}, Z_{i+2}))^2 \leq 10.2^{10}.2^{4i} < 2^{4i+14}. \tag{30}$$

Hence $i(k) > \frac{k-15}{4}$. Substituting this lower bound in Equation (29) supplies the required exponential lower bound for d_{k+1} as a function of $k \geq 1$ □

Note that this lower bound is valid under the assumption that indices $1 < i < k$ exist such that $z_{k+1,k}$ is even, $z_{k,k-1} \ldots z_{i+2,i+1}$ are odd, and $z_{i+1,1}$ is even. Lemma 4 shows that for every two positive integers Z_0, Z_1 the sequence of integer terms $\{Z_j\}$ is finite. The next lemma shows that the last integer term must be odd.

Lemma 7. *Consider a recurring sequence $\{Z_j\}$ for $j \geq 1$ defined by*

$$Z_{j+1} = \frac{Z_j}{2} + Z_{j-1} \tag{31}$$

with initial conditions Z_0, and $Z_1 \equiv 0 \pmod 2 \in \mathbb{Z}_+$. Then there exists an index $k \geq 2$ with $Z_k \equiv 1 \pmod 2$.

Proof. A consequence of Lemma 4 is the existence of a maximal k with $Z_j \in \mathbb{Z}_+$ for $0 \leq j \leq k$. Hence $Z_{k+1} > 0 \notin \mathbb{Z}_+$, and this requires that $Z_k \equiv 1 \pmod 2$. □

We now consider the sequence $(z_{3,2},\ z_{4,3}, \ldots)$. $z_{3,2} = 3$, and $z_{4,3} = 5$. By definition this sequence consists of subsequences of odd, and even $z_{j+1,j}$. Lemma 7 proves the existence of a (smallest) k with $z_{k,k-1}$ is odd, and $z_{k+1,k}$ is even. We now distinguish two cases:

Case 1. All subsequences of even $z_{j+1,j}$ are finite. Then the assumption for the proof of the lower bound for d_{k+1} is satisfied and Lemma 6 is true for all j.

Case 2. There exists a infinite subsequence of even $z_{j+1,j}$. Then there is a maximal k such that $z_{k,k-1}$ is odd, and $z_{k+1,k}$ is even. For the next number we find

$$d_{k+2} = z_{1,k+1} = y_{1,k} + z_{1,k} = y_{1,k} + d_{k+1} > 2^k + 2^{0.25k-3.75} > 2^{0.25(k+1)-3.75} \tag{32}$$

By induction Lemma 6 is true for $j \geq k+1$, and consequently true for all j.

We conclude that Lemma 6 is true for every sequence $(z_{3,2},\ z_{4,3}, \ldots)$. From Lemma 6 and Equations (21), (8) we find the upper bound

$$0 < 2^{k+\ell} - a_{k+1} < 17.232 \cdot \left(\frac{1+\sqrt{17}}{2}\right)^{0.816 \cdot k}. \tag{33}$$

and boundaries for $\ell(k)$.

Corollary 8. *If a k, ℓ 1-cycle exists, then there exists a k-dependent minimal, and maximal value for ℓ.*

Lemma 33 supplies for $|(k+\ell)\log 2 + \log\sqrt{17} - (k+2)\log\frac{1+\sqrt{17}}{2}|$ a negative exponential upper bound in k. Then Corollary 8 supplies a negative exponential upper bound in the cycle length $k+\ell$. To find an upper bound for the cycle length we need an appropriate lower bound from transcendental number theory.

3.3 An upper bound for $|(k+\ell)\log 2 + \log\sqrt{17} - (k+2)\log\frac{1+\sqrt{17}}{2}|$

Inserting Equation (7) in Equation (33) results in a lower bound for negative values, and an upper bound for positive values

$$-\frac{1}{\sqrt{17}}\left(\frac{1-\sqrt{17}}{2}\right)^{k+2} < 2^{k+\ell} - \frac{1}{\sqrt{17}}\left(\frac{1+\sqrt{17}}{2}\right)^{k+2}, \tag{34}$$

and

$$2^{k+\ell} - \frac{1}{\sqrt{17}}\left(\frac{1+\sqrt{17}}{2}\right)^{k+2} < 17.233\cdot\left(\frac{1+\sqrt{17}}{2}\right)^{0.816k} - \frac{1}{\sqrt{17}}\left(\frac{1-\sqrt{17}}{2}\right)^{k+2}. \tag{35}$$

For odd, and even k $-\frac{1}{\sqrt{17}}(\frac{1-\sqrt{17}}{2})^{k+2} \leq \frac{1}{\sqrt{17}}(\frac{\sqrt{17}-1}{2})^{k+2}$. Inserting this in Equations (34) and (35), we have after multiplication with $\sqrt{17}(\frac{2}{1+\sqrt{17}})^{k+2}$

$$-\left(\frac{1-\sqrt{17}}{1+\sqrt{17}}\right)^{k+2} < 2^{k+\ell}\sqrt{17}\left(\frac{2}{1+\sqrt{17}}\right)^{k+2} - 1 < 17.233\cdot\sqrt{17}\frac{\left(\frac{2}{1+\sqrt{17}}\right)^{2}}{\left(\frac{1+\sqrt{17}}{2}\right)^{0.184k}} + \left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)^{k+2} \tag{36}$$

Using $(\frac{1+\sqrt{17}}{2})^{0.184} > 1.18$, and $\frac{\sqrt{17}+1}{\sqrt{17}-1} > 1.18$ we find

$$1 - \frac{1}{1.18^{k+2}} < 2^{k+\ell}\sqrt{17}\left(\frac{2}{1+\sqrt{17}}\right)^{k+2} < 1 + \frac{12}{1.18^{k+2}}. \tag{37}$$

For $k \geq 1$ $\log(1 - \frac{1}{1.18^{k+2}}) > -\frac{3}{1.18^{k+2}}$. Taking logs leads to

$$-\frac{12}{1.18^{k}} < -\frac{3}{1.18^{k+2}} < (k+\ell)\log 2 + \log\sqrt{17} - (k+2)\log\frac{1+\sqrt{17}}{2} < \frac{12}{1.18^{k}}. \tag{38}$$

If $k \geq 16$ then $\frac{12}{1.18^{k}} < 1$. From these bounds we find bounds for ℓ as a function of $k \geq 16$

$$0.357k + 0.310 \ < \ \ell \ < \ 0.357k + 2.113. \tag{39}$$

3.4 A lower bound for $|(k+\ell)\log 2 + \log\sqrt{17} - (k+2)\log\frac{1+\sqrt{17}}{2}|$

For a lower bound the theorem of Rhin's [13] cannot be used since it applies to a linear form in two logarithms. Matveev [10] has developed a lower bound for a linear form in three logarithms. Mignotte [12] (Proposition 5.2) has improved Matveev's lower bound. Evertse [5] has proved from Matveev's approach:

Lemma 9. *Let $\gamma_1 \dots \gamma_n$ be algebraic numbers from a field $\mathbb{K}$ of degree D, distinct from 0 and 1, with height $h(\gamma_1)\dots h(\gamma_n)$. Take $\log\gamma_1 \dots \log\gamma_n$ to be any determination of their logarithms. Let $b_1 \dots b_n$ be non-zero integers such that $\Lambda = |b_1\log\gamma_1 + \dots + b_n\log\gamma_n|$. Let $A_1 \dots A_n$ be real numbers > 1 with $\log A_i \geq \max(Dh(\gamma_i), |\log\gamma_i|, 0.16)$. Set $B = \max(|b_i|)$. Then $\log|\Lambda| > -2\cdot 30^{n+4}\cdot(n+1)^6\cdot D^2\cdot\log(eD)\cdot\log A_1\cdot\dots\cdot\log A_n\cdot\log(eB)$.*

We use Evertse's lemma to derive a lower bound for $|(k+\ell)\log 2+\log\sqrt{17}-(k+2)\log\frac{1+\sqrt{17}}{2}|$. Note that $n=3,\ D=2,\ \log A_1 = 4,\ \log A_2 = 34,\ \log A_3 = 8,\ B = k+\ell$. This results in

$$|(k+\ell)\log 2 + \log\sqrt{17} - (k+2)\log\frac{1+\sqrt{17}}{2}| > (e(k+\ell))^{-1.32\cdot 10^{18}} \tag{40}$$

3.5 An upper bound for k

Confronting the bounds from Equations (38) with the bounds from Equation (40) results in the upper bound $k < k_{\max} = 3.89 \cdot 10^{20}$. We can now apply a reduction technique based on a generalized lemma of Baker and Davenport [4]

Lemma 10. *Let $A > 0, B > 1, \kappa > 0, \mu > 0 \in \mathbb{R}$. Suppose $M \in \mathbb{N}$. Let $\frac{p}{q}$ be a convergent of the continued fraction expansion of κ such that $q > 6M$, and let $\epsilon = \|\mu q\| - M\|\kappa q\|$, where $\|.\|$ denotes the distance to the nearest integer.*

1. *If $\epsilon > 0$ then there is no solution in integers m, n of the inequality*

$$0 < m\kappa - n + \mu < A \cdot B^{-m} \tag{41}$$

   with $\dfrac{\log(\frac{Aq}{\epsilon})}{\log B} \leq m \leq M$.

2. *If $\epsilon < 0$, let $r = \lfloor \mu q + \frac{1}{2} \rfloor$. If $p - q + r = 0$, then there is no solution in integers m, n of the Inequality (41) with $\max\left(\dfrac{\log(3Aq)}{\log B}, 1\right) \leq m \leq M$.*

From Equation (39) we have $k > \dfrac{k+\ell-2.113}{1.357}$, which implies

$$1.18^{\frac{2.113}{1.357}} \cdot (1.118^{\frac{1}{1.357}})^{-(k+\ell)} > 1.18^{-k}.$$

Using this, and dividing by $\log \frac{1+\sqrt{17}}{2}$, we can rewrite Equation (38) in the format of Equation (41)

$$\begin{aligned} -\frac{3}{\log \frac{1+\sqrt{17}}{2}} \cdot 1.18^{-k} &< (k+\ell)\frac{\log 2}{\log \frac{1+\sqrt{17}}{2}} + \frac{\log \sqrt{17}}{\log \frac{1+\sqrt{17}}{2}} - (k+2) \\ &< \frac{1}{\log \frac{1+\sqrt{17}}{2}} \cdot 12 \cdot 1.18^{\frac{2.113}{1.357}} \cdot (1.118^{\frac{1}{1.357}})^{-(k+\ell)} \end{aligned}$$

i.e.,

$$-4.441 \cdot (1.18)^{-k} < (k+\ell) \cdot 0.7369 + 1.506 - (k+2) < 16.5081 \cdot 1.129726^{-(k+\ell)}. \tag{42}$$

We distinguish two cases:

Case 1. $0 < (k+\ell) \cdot 0.7369 + 1.506 - (k+2) < 16.5081 \cdot 1.129726^{-(k+\ell)}$. We now can apply Lemma 10 with $\kappa = 0.7369,\ \mu = 1.506,\ A = 16.5081,\ B = 1.12972$. From $k_{\max}$, and Equation (39) we find $k + \ell < 6.224 \cdot 10^{20} = M$. The continued fraction of κ is $(0, 1, 2, 1, 4, 40, 1, 6, 18, 2, 4, 3, 1, 1, 2, 8, 2, 1, 1, 1, 4, 1, 4, 1, 2, 1, 1, 7, 4, 3, 1, 1, 2, 6, 1, 8, 1, 1, 4, 12, 1, 1, 1, 1, 3, 1, 1, 1, 3, 19, 3, 1, 81, 1, 24, 1, 2, 4, 2083, 4, \ldots)$. The first convergent $\frac{p_n}{q_n}$ with $q_n > 6 \cdot M$ is $q_{49} = 5.797 \cdot 10^{21}$. For $q = q_{49}, \epsilon = 0.034 > 0$, and $\dfrac{\log(\frac{Aq}{\epsilon})}{\log B} = 461.5$. Then Lemma 10 states that $k + \ell \leq 461$. Subsequently we applied Lemma 10 with $M = 461$ to find $k + \ell \leq 128$.

Case 2. $-4.441 \cdot (1.18)^{-k} < (k+\ell) \cdot 0.7369 + 1.506 - (k+2) < 0$. After division by κ, and redefining μ we find in this case $0 < 1.357 \cdot k + 0.6703 - (k+\ell) < 6.027 \cdot (1.18)^{-k}$, and now again Lemma 10 is applicable. Doing a similar calculation we found (initially with $M = k_{\max}$) that $k \leq 331$, and through repetition that $k \leq 86$.

Using Equation (39), we find from combining these cases the following corollary.

Corollary 11. *If the higher-order Collatz sequence of Equation (1) has a* 1*-cycle, then* $k \leq 94$*.*

3.6 Non-existence of 1-cycles

Theorem 12. *The higher-order Collatz sequence of Equation (1) has no* 1*-cycles.*

Proof. Corollary 11 requires that $k \leq 94$. We checked numerically that if $2 \leq k \leq 100$, for all ℓ values that satisfy Equation (39), then Equation (9) has no solution $x_0 \in \mathbb{Z}_+$. □

This proves Theorem 2 (2).

3.7 Existence of m-cycles

We found the following $m > 1$-cycles for start values $x_1 < 10^6,\ x_0 = 2x_1$ (Table 1).

# cycles	# odd elem.	# even elem.	m	$x_{\min}$	$x_{\max}$
11	5	4	2	1	8
	6	5	4	77	273
	8	5	3	157	1 004
	10	5	2	3 185	50 960
	10	5	2	4 017	32 136
	18	14	10	11 037	142 868
	11	7	4	11 687	166 213
	11	7	4	12 711	144 620
	11	7	4	12 817	116 660
	11	7	4	13 847	177 240
	11	7	4	15 377	139 960

Table 1. $m > 1$-cycles for start values $x_1 < 10^6,\ x_0 = 2x_1$

Note that these solutions (k, ℓ) do not refer to convergents to $\log_2 3$.

4 1-cycles for higher-order Collatz sequence with $q > 1$

For Equation (3) the same difference equation, and initial conditions for a_n apply. For b_n we now have $b_{n+1} = b_n + 4b_{n-1} + q \cdot 2^n,\ b_0 = b_1 = 0$, with the solution

$$b_n = q\Big[\frac{\sqrt{17}+3}{2\sqrt{17}}\Big(\frac{1+\sqrt{17}}{2}\Big)^n + \frac{\sqrt{17}-3}{2\sqrt{17}}\Big(\frac{1-\sqrt{17}}{2}\Big)^n - 2^n\Big]. \tag{43}$$

With this new expression for b_n the expressions for x_n (Equation (6)), and for x_0 (Equation (9)), and the kernel equation (21) are valid. Lemma 3 is valid for $q = 1$. For $q > 1$ the initial conditions for $z_{1,2}$ and $z_{2,2}$ change, i.e., if $q \equiv 1 \pmod 4$ then $z_{1,2} = 1,\ z_{2,2} = \frac{q+5}{2}$, and if $q \equiv 3 \pmod 4$ then $z_{1,2} = 1,\ z_{2,2} = \frac{q+15}{2}$. So d_{k+1} is a function of q as shown in the Table 2 below. We take $q \in \{3, 5, 7, 11, 13, 17, 19\}$.

q	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
3	1	7	11	3	19	115	51	435	691
5	3	1	13	5	53	21	85	213	469
7	1	3	15	7	23	55	119	503	247
11	1	7	3	11	27	123	187	59	827
13	3	1	5	13	61	29	221	349	605
17	3	5	9	17	1	97	33	417	161
19	1	7	11	19	35	3	67	195	963

Table 2. Nonconstant term $d_{k+1}(q)$ for different q values

Note that $d_5(q) = q$ refers to the sequence of q-multiples of $(1, 3, 3, 5, 6)$. Lemma 5 uses the definition $Z_j = z_{jj} + 1$, which now becomes $Z_j = z_{jj} + q$. The upper bound $z_{jj} < 2^{2j}$ then requires a $j_{\min}(q) \geq 3$.

Lemma 6 remains valid because the lower bound in Equation (29) is independent of $z_{1,i}$.

The overall effect of $q > 1$ is an extra factor q in Equation (6), and in the bounds of Equation (38) that determines $k_{\max}$. We calculated for the worst case $q = 19$ that if $k \geq 33$ then $\frac{12 \cdot q}{1.18^k} < 1$, and Equation (39) remains valid. The effect of "small" q on $k_{\max}$ is negligible, and this implies that the reduced upper bound for k of Corollary 11 applies for $q \in \{3, 5, 7, 11, 13, 17, 19\}$. For larger values of q Equation (38) remains applicable. With Equation (39) an upper bound for the cycle length as a function of m is given.

This proves Theorem 2 (1). (See Section 6, Remark 2).

Theorem 13. *The higher-order Collatz sequence of Equation (3) has for* $q \in \{3, 5, 7, 11, 13, 17, 19\}$ *the following* 1*-cycles: for* $q = 3\ (x_1 = 1, 11),\ q = 7\ (x_1 = 3),\ q = 11\ (x_1 = 1)$.

Proof. Corollary 11 requires that $k \leq 94$. We checked numerically that if $2 \leq k \leq 100$ for all ℓ values that satisfy Equation (39), Equation (3) has no other solutions $x_0 \in \mathbb{Z}_+$ than those mentioned in the theorem. □

This proves Theorem 2 $(3, 4)$. Apart from trivial m-cycles (q-multiples) we found the following m-cycles for start values $x_1 < 10^6,\ x_0 = 2x_1$ (Table 3).

q	# $\not\equiv q$ cycles	# odd elem.	# even elem.	m	$x_{\min}$	$x_{\max}$
3	3	1	2	1	1	4
		3	2	1	11	44
		3	4	3	6	244
5	0					
7	2	2	2	1	3	12
		2	3	2	9	36
11	6	1	3	1	1	8
		11	11	6	3	103
		8	7	5	2 557	13 076
		25	18	12	2 107	24 184
		8	7	5	2 499	11 103
		8	7	5	2 307	13 103
13	7	2	4	2	3	24
		5	5	3	19	152
		5	5	3	21	124
		8	7	5	2 323	16 509
		8	7	5	2 523	17 924
		8	7	5	2 603	20 824
		8	7	5	2 921	18 684
17	2	15	12	7	29	496
		18	16	11	13 383	377 818
19	58	10	8	3	3	127
		...	...	...	...	...
		11	7	4	250 167	4 002 672

Table 3. m-cycles for start values $x_1 < 10^6,\ x_0 = 2x_1$
for different q values

This proves Theorem 2 (5). For $19 < q \leq 997$, 1-cycles are exceptional. We found numerically (without a theoretical upper bound $k_{\max}$) a 1-cycle for $q = 23, 59, 71, 191, 227, 251, 331, 503, 883$. This proves Theorem 2 (6).

5 On proving the existence of m-cycles

The approach for finding 1-cycles can be generalized to m-cycles. A necessary and sufficient condition for the existence of an m-cycle is the existence of a solution $x_i \in \mathbb{Z}_+$ of the system (for $q = 1$):

$$\begin{pmatrix} -a_{k_1+1} & 2^{k_1+\ell_1} & & \\ & -a_{k_2+1} & 2^{k_2+\ell_2} & \\ & & \ddots & \\ 2^{k_m+\ell_m} & & & -a_{k_m+1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_{k_1+\ell_1} \\ \vdots \\ x_{k_m+\ell_m} \end{pmatrix} = \begin{pmatrix} b_{k_1+1} \\ b_{k_2+1} \\ \vdots \\ b_{k_m+1} \end{pmatrix}, \qquad (44)$$

where $k_i, \ell_i,\ (i = 1, \ldots, m)$ is the length of the i-th pair of subsequences, and a_i, b_i are defined by Equations (7), (8). For example, for $m = 2,\ k_1 = 4,\ \ell_1 = 1,\ k_2 = 1,\ \ell_2 = 3$ we have the solution $x_0 = 2,\ x_5 = 6$ for the 2-cycle $((x_0 = 2),\ 1,\ 3,\ 3,\ 5,\ 6,\ 3,\ 8,\ 4,\ 2)$. Let $K = \sum_{i=1}^{m} k_i$, Ł $= \sum_{i=1}^{m} \ell_i$. Then a lower and an upper bound must be found for $2^{K+L} - \prod_{i=1}^{m} a_{k_i+1}$. In principle, for each d_{k_i+1} a lower bound can be found, since the beginning of the next odd subsequence follows from the foregoing pair of subsequences. We leave this for further research.

6 Remarks

Remark 1. For the original Collatz sequence, odd numbers form an increasing subsequence, and even numbers form a decreasing subsequence. An m-cycle is defined as a cycle with m (even) local maxima and m (odd) local minima. For higher-order Collatz sequences there can exist odd maxima that can "overrule" even maxima. As an example, for $q = 11$ there exists the cycle $(6, 3, 13, 15, 26, 13, 38, 19, 53, 51, 84, 42, 21, 58, 29, 78, 103, 96, 48, 24, 12)$ with 5 even local maxima and 2 odd local maxima. Following our definition $m = 6$, so the definition of m-cycles for the original Collatz sequence must be amended.

Remark 2. The proof of Theorem 2 (1) differs from the proof for the original Collatz sequence in [14]. The expression for the starting odd numbers requires a nonconstant term d_{k+1}, leading to a different kernel equation (21) with a non-trivial upper bound analysis. For the resulting linear log form, the simple upper bound reduction based on convergents is not applicable.

Remark 3. A natural generalization is to apply $x_{n+1} = \dfrac{ax_n + (3-a)x_{n-1} + q}{2}$ if x_n is odd, with a an odd number. Then an analysis for a new d_{k+1} is required. Also the number 17, with the property that $\mathbb{Z}(\frac{1+\sqrt{17}}{2})$ is a unique factorization domain, no longer holds. So an amended Lemma 4 is required. Computational evidence suggests that in such cases the number of m-cycles is finite. We leave this for further research.

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