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Multicomponent hybrid numbers: On algebraic properties and matrix representations of hybrid-hyperbolic numbers

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Abstract: In this study, the hybrid-hyperbolic numbers are introduced. This number system is a more general form of the hybrid number system, which is an interesting number system, as well as a number system that includes multicomponent number systems (i.e., complex-hyperbolic, dual-hyperbolic and bihyperbolic numbers). In this paper, we give algebraic properties of hybrid-hyperbolic numbers. In addition, 2×2 and 4×4 hyperbolic matrix representations of hybrid-hyperbolic numbers are given and some properties of them are examined.

Keywords: Hybrid numbers, Complex-hyperbolic numbers, Dual-hyperbolic numbers, Bihyperbolic numbers, Hybrid-hyperbolic numbers.
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1 Introduction

In mathematics, complex, dual and hyperbolic numbers are two dimensional number systems. In literature, there are many algebraic, geometric and even physical investigations for these number systems. A hyperbolic number is written in the form $w = x + jy$, where x and y are real numbers and hyperbolic unit j satisfies $j^2 = 1$. The set of hyperbolic numbers $\mathbb{H}$ is defined as

$$\mathbb{H} = \{w = x + jy : x, y \in \mathbb{R}, j^2 = 1\}.$$

The hyperbolic modulus is

$$\|w\| = \sqrt{|w\overline{w}|} = \sqrt{x^2 - y^2}$$

(for details, see [1, 2]). Also, number systems such as generalized complex numbers [3], hypercomplex numbers [4], etc., that contain these number systems together have been introduced. The hybrid number system, on the other hand, is a special and useful number system that combines complex, hyperbolic and dual numbers introduced by Özdemir in [5]. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K} = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}.$$

The multiplication table of hybrid numbers base elements is given below.

.	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	$h+1$	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

Table 1. Multiplication table of hybrid numbers

Also in this study, algebraic and geometric properties, classifications and matrix representations of hybrid numbers are mentioned. For details about hybrid numbers, see [5].

In [6] multicomponent number systems are given and four-component number systems and eight-component number system are examined. Four-component number systems such as dual-hyperbolic, dual-complex, complex-hyperbolic defined in four-dimensional spaces are defined. A wide literature has been formed by examining dual-complex, dual-hyperbolic, complex-hyperbolic, bicomplex and bihyperbolic numbers by researchers [6–12].

A dual-hyperbolic number is defined as below, [7]

$$\mathbb{D}_{\mathbb{H}} = \{w = w_1 + jw_2 | w_1, w_2 \in \mathbb{D}, j^2 = 1, j \neq \pm 1\}.$$

The set of complex-hyperbolic numbers can be obtained as [8]

$$\mathbb{C}_{\mathbb{H}} = \{w = z_1 + jz_2 | z_1, z_2 \in \mathbb{C}, j^2 = 1, j \neq \pm 1\}.$$

The set of bihyperbolic numbers can be obtained as [9]

$$\mathbb{H}_2 = \{w = h_1 + jh_2 | h_1, h_2 \in \mathbb{H}, j^2 = 1, j \neq \pm 1\}.$$

In this study, a new eight-component number system is introduced, inspired by hybrid numbers and multicomponent number systems. This number system is called *hybrid-hyperbolic numbers* because it is a number system that includes the complex-hyperbolic, dual-hyperbolic and bihyperbolic numbers described above. Consequently, this number system is the general form of many number systems and will give effective results in both theory and practice. Also, algebraic properties of hybrid hyperbolic numbers are examined and some characterizations are given. Finally, hyperbolic matrix representations of hybrid hyperbolic numbers are given and the properties of these matrices are examined.

2 Hybrid-hyperbolic numbers

In this section, we define hybrid-hyberbolic numbers and investigate their algebraic properties. Also, the terms *conjugate*, *inner product*, *norm* and *inverse* for hybrid-hyperbolic numbers are defined.

Let us deal with a statement of the form $H_0 + H_1 i + H_2 \varepsilon + H_3 h$ which is a linear combination of the hybrid units $\{1, i, \varepsilon, h\}$, where the H_0, H_1, H_2 and H_3 are hyperbolic numbers. Since $\{1, i, \varepsilon, h\}$ are hybrid units, they provide the following multiplication rules

$$i^2 = -1, \quad \varepsilon^2 = 0, \quad h^2 = 1, \quad ih = -hi = \varepsilon + i \tag{1}$$

Consequently, we can give the following definition.

Definition 2.1. *The set of hybrid-hyperbolic numbers, denoted by* $\mathbb{KH}$*, is represented as*

$$\mathbb{KH} = \{W = H_0 + H_1 i + H_2 \varepsilon + H_3 h : H_0, H_1, H_2, H_3 \in \mathbb{H}\}.$$

Let us examine the unit j for hyperbolic numbers $H_k = a_k + jb_k \in \mathbb{H}$, where $0 \leq k \leq 3$. Since the theory is to construct a number system that includes multicomponent number systems, j and h have been considered as different ($j \neq h$) hyperbolic units. Also, since multicomponent units for multicomponent numbers are commutative and given by complex-hyperbolic numbers (with $ij = ji$), dual-hyperbolic numbers (with $j\varepsilon = \varepsilon j$), bihyperbolic numbers (with $jh = hj$). Hence, $\{1, ji, j\varepsilon, jh\}$ containing complex-hyperbolic, dual-hyperbolic and bihyperbolic units are called *hybrid-hyperbolic units* and the following product rules are valid:

$$\begin{array}{lll} (ji)^2 = -1, & (ji)(j\varepsilon) = (1-h), & (ji)(jh) = \varepsilon + i \\ (j\varepsilon)^2 = 0, & (j\varepsilon)(ji) = h + 1, & (j\varepsilon)(jh) = -\varepsilon \\ (jh)^2 = 1, & (jh)(ji) = -\varepsilon - i, & (jh)(j\varepsilon) = \varepsilon \end{array}$$

On the other hand, W can be given in the following forms.

1. $\{W = w_1 + jw_2, j^2 = 1\}$, where w_1 and w_2 are hybrid numbers.
   Let $H_k = a_k + jb_k \in \mathbb{H}$, where $0 \leq k \leq 3$. Then, we have

   $$W = (a_0 + jb_0) + (a_1 + jb_1)i + (a_2 + jb_2)\varepsilon + (a_3 + jb_3)h.$$

   By editing this equation, we get

   $$W = (a_0 + a_1 i + a_2 \varepsilon + a_3 h) + j(b_0 + b_1 i + b_2 \varepsilon + b_3 h).$$

   Finally, it can be written in the form of $W = w_1 + jw_2$, where $j^2 = 1$ for hybrid numbers $w_1 = a_0 + a_1 i + a_2 \varepsilon + a_3 h$ and $w_2 = b_0 + b_1 i + b_2 \varepsilon + b_3 h$.

2. $W = a_0+a_1i+a_2\varepsilon+a_3h+b_0j+b_1ij+b_2\varepsilon j+b_3hj$, where $a_k, b_k \in \mathbb{R}, k = 0,1,2,3$. Hence, a four-component number system is obtained, and the multiplication table of these units can be constructed using the multiplication rules of hybrid units and hybrid-hyperbolic units.

W can be written total of a scalar part $S(W) = H_0$ and a vector part $V(W) = H_1i + H_2\varepsilon + H_3h$. Therefore, we have

$$W = S(W) + V(W).$$

We define addition, scalar multiplication and multiplication on hybrid-hyperbolic numbers as follows:

$$W_1 + W_2 = (H_0 + H_0^*) + (H_1 + H_1^*)i + (H_2 + H_2^*)\varepsilon + (H_3 + H_3^*)h$$

$$\lambda W = (\lambda H_0) + (\lambda H_1)i + (\lambda H_2)\varepsilon + (\lambda H_3)h$$

$$\begin{aligned} W_1W_2 \;=\; & [H_0H_0^* - H_1H_1^* + H_1H_2^* + H_2H_1^* + H_3H_3^*] \\ & + [H_0H_1^* + H_1H_0^* + H_1H_3^* - H_3H_1^*]i \\ & + [H_0H_2^* + H_1H_3^* + H_2H_0^* - H_2H_3^* - H_3H_1^* + H_3H_2^*]\varepsilon \\ & + [H_0H_3^* - H_1H_2^* + H_2H_1^* - H_3H_0^*]h \end{aligned} \tag{2}$$

where $W_1 = H_0+H_1i+H_2\varepsilon+H_3h$ and $W_2 = H_0^*+H_1^*i+H_2^*\varepsilon+H_3^*h$ are any hybrid-hyperbolic numbers and λ is any hyperbolic number. Using the product rules of hybrid units in Table 1, the product of hybrid-hyperbolic numbers is calculated.

Theorem 2.1. *The set of hybrid-hyperbolic numbers a module over the ring of hyperbolic numbers with the addition and multiplication operations which are defined above.*

Remark 2.1. *According to Equalities 2, the hybrid-hyperbolic numbers are not commutative.*

Corollary 2.1.1. *Hybrid-hyperbolic numbers are generalized versions of some special numbers. These are*

1. *If hyperbolic numbers $H_1 = H_3 = 0$, W is a dual-hyperbolic or hyperbolic-dual number [7].*

2. *If hyperbolic numbers $H_2 = H_3 = 0$, W is a complex-hyperbolic or hyperbolic-complex number [8].*

3. *If hyperbolic numbers $H_1 = H_2 = 0$, W is a bihyperbolic number [9].*

Definition 2.2. *A hybrid-hyperbolic number $W = H_0 + H_1i + H_2\varepsilon + H_3h$ has three conjugates such that*

1. $W^{\dagger 1} = H_0 - H_1i - H_2\varepsilon - H_3h = \overline{w_1} + j\overline{w_2}$, *hybrid conjugation,*

2. $W^{\dagger 2} = \overline{H_0} + \overline{H_1}i + \overline{H_2}\varepsilon + \overline{H_3}h = w_1 - jw_2$, *hyperbolic conjugation,*

3. $W^{\dagger 3} = \overline{H_0} - \overline{H_1}i - \overline{H_2}\varepsilon - \overline{H_3}h = \overline{w_1} - j\overline{w_2}$, *coupled conjugation.*

Theorem 2.2. *Let $W = H_0 + H_1 i + H_2 \varepsilon + H_3 h$ be a hybrid-hyperbolic number. Then, we have*

1. *W is a hyperbolic number $\Leftrightarrow W^{\dagger 1} = W$,*
2. *W is a hybrid number $\Leftrightarrow W^{\dagger 2} = W$,*
3. *W is a pure hybrid-hyperbolic number $\Leftrightarrow W^{\dagger 3} = -W^{\dagger 2}$.*

Proof. 1. Suppose that $W^{\dagger 1} = W$. Therefore, we have

$$\begin{aligned}
&H_0 + H_1 i + H_2 \varepsilon + H_3 h = H_0 - H_1 i - H_2 \varepsilon - H_3 h, \\
&\Rightarrow -H_1 = H_1, -H_2 = H_2, -H_3 = H_3, \\
&\Rightarrow H_1 = 0, H_2 = 0, H_3 = 0, \\
&\Rightarrow W = H_0.
\end{aligned}$$

Consequently, W is a hyperbolic number.

2. Suppose that $W^{\dagger 2} = W$. Let $H_0 = a_0 + jb_0$, $H_1 = a_1 + jb_1$, $H_2 = a_2 + jb_2$ and $H_3 = a_3 + jb_3 \in \mathbb{H}$. Therefore, we get

$$\begin{aligned}
&H_0 + H_1 i + H_2 \varepsilon + H_3 h = \overline{H_0} + \overline{H_1} i + \overline{H_2} \varepsilon + \overline{H_3}, \\
&\Rightarrow \overline{H_0} = H_0, \overline{H_1} = H_1, \overline{H_2} = H_2, \overline{H_3} = H_3, \\
&\Rightarrow a_0 + jb_0 = a_0 - jb_0, a_1 + jb_1 = a_1 - jb_1, \\
&a_2 + jb_2 = a_2 - jb_2, a_3 + jb_3 = a_3 - jb_3, \\
&\Rightarrow b_0 = -b_0, b_1 = -b_1, b_2 = -b_2, b_3 = -b_3, \\
&\Rightarrow b_0 = b_1 = b_2 = b_3 = 0, \\
&\Rightarrow W = a_0 + a_1 i + a_2 \varepsilon + a_3 h.
\end{aligned}$$

Consequently, W is a hybrid number.

3. Suppose that $W^{\dagger 3} = -W^{\dagger 2}$. Therefore, we get

$$\begin{aligned}
&\overline{H_0} - \overline{H_1} i - \overline{H_2} \varepsilon - \overline{H_3} = -\overline{H_0} + \overline{H_1} i + \overline{H_2} \varepsilon + \overline{H_3}, \\
&\Rightarrow \overline{H_0} = -\overline{H_0}, \\
&\Rightarrow \overline{H_0} = 0, \\
&\Rightarrow H_0 = 0.
\end{aligned}$$

Consequently, W is a pure hybrid-hyperbolic number. □

2.1 Inner product, norm of hybrid-hyperbolic numbers

In this section, inner product and norm definitions of hybrid hyperbolic numbers are given and these concepts are examined.

Definition 2.3. *The inner product in the hybrid-hyperbolic numbers is defined as follows:*

$$g(W_1, W_2) = \frac{1}{2}(W_1^{\dagger 1} W_2 + W_2^{\dagger 1} W_1) \tag{3}$$

According to Equation 3, we get

$$g(W_1, W_2) = H_0 H_0^* + H_1 H_1^* - H_1 H_2^* - H_2 H_1^* - H_3 H_3^*, \tag{4}$$

where $W_1 = H_0 + H_1 i + H_2 \varepsilon + H_3 h$ *and* $W_2 = H_0^* + H_1^* i + H_2^* \varepsilon + H_3^* h$.

It should be noted that the result of Equation 4 is a hyperbolic number. The inner product satisfies the following properties:

i) $g(W_1, W_2) = g(W_2, W_1), \quad W_1, W_2 \in \mathbb{KH}$

ii) $g(W_1, W_2 + W_3) = g(W_1, W_2) + g(W_1, W_3), \quad W_3 \in \mathbb{KH}$

iii) $\lambda g(W_1, W_2) = g(\lambda W_1, W_2) + g(W_1, \lambda W_2), \quad \lambda \in \mathbb{H}$

iv) $g(W_1, W_1) = W_1^{\dagger 1} W_1 \in \mathbb{H}$

According to i), ii), and iii), the inner product is a symmetric bilinear form. But iv) shows us that is not positive definite. The inner product of the hybrid-hyperbolic numbers is a generalized inner product for the complex-hyperbolic, dual-hyperbolic and bihyperbolic number systems. If this situation is further generalized, we have the following Table 2.

$g(W_1, W_2)$	W_2 complex-hyperbolic	W_2 dual-hyperbolic	W_2 bi-hyperbolic
W_1 complex-hyperbolic	$H_0 H_0^* + H_1^* H_1^*$	$H_0 H_0^* - H_1 H_2^*$	$H_0 H_0^*$
W_1 dual-hyperbolic	$H_0 H_0^* - H_2 H_1^*$	$H_0 H_0^*$	$H_0 H_0^*$
W_1 bi-hyperbolic	$H_0 H_0^* - H_2 H_1^*$	$H_0 H_0^* - H_1^* H_1^*$	$H_0 H_0^* - H_3 H_3^*$

Table 2. Special cases of inner products of hybrid-hyperbolic numbers

We can write the inner product of two hybrid-hyperbolic numbers as follows:

$$g(W_1, W_2) = g(w_1, w_1^*) + g(w_2, w_2^*) + j(g(w_1, w_2^*) + g(w_1^*, w_2)), \tag{5}$$

where $W_1 = w_1 + j w_2$ and $W_2 = w_1^* + j w_2^*$. In Equation 5, if W_1 is taken instead of W_2, we get

$$g(W_1, W_1) = g(w_1, w_1) + g(w_2, w_2) + 2j g(w_1, w_2).$$

Definition 2.4. *Let* $W = H_0 + H_1 i + H_2 \varepsilon + H_3 h$ *be a hybrid-hyperbolic number. The norm is denoted by* N_W. *Therefore, we can write*

$$N_W = \sqrt{|g(W, W)|} = \sqrt{|H_0^2 + (H_1 - H_2)^2 - H_2^2 - H_3^2|}.$$

The special versions of this norm may correspond to one of the modules of dual-hyperbolic, complex-hyperbolic and bihyperbolic numbers.

Definition 2.5. *The inverse of the hybrid-hyperbolic number* $W = H_0 + H_1 i + H_2\varepsilon + H_3 h$, $N(W) \neq 0$ *is defined as*

$$W^{-1} = \frac{W^{\dagger 1}}{N(W)^2}.$$

Lemma 2.3. *Let* $W = w_1 + jw_2$ *be a hybrid-hyperbolic number, where* w_1 *and* w_2 *are hybrid numbers. Then, the norm of* W *be given by*

$$N_W = \sqrt[4]{|(g(w_1, w_1) + g(w_2, w_2))^2 - 4g(w_1, w_2)^2|}.$$

Proof. According to the definition of norm, we have

$$N_W = \sqrt{|g(W, W)|} = \sqrt{|g(w_1 + jw_2, w_1 + jw_2)|}.$$

If the statement is regulated

$$\begin{aligned} N_W &= \sqrt{|g(w_1, w_1) + g(w_2, w_2) + j(g(w_1, w_2) + g(w_2, w_1)|}, \\ &= \sqrt{|g(w_1, w_1) + g(w_2, w_2) + 2jg(w_1, w_2)|}, \\ &= \sqrt[4]{|(g(w_1, w_1) + g(w_2, w_2))^2 - 4g(w_1, w_2)^2|}. \end{aligned}$$

□

Remark 2.2. *Let* $W = w_1 + jw_2$ *be a hybrid-hyperbolic number, where* w_1 *and* w_2 *are hybrid numbers. Then* W *is a unit hybrid-hyperbolic number*

$$g(w_1, w_2)^2 = \frac{(g(w_1, w_1) + g(w_2, w_2))^2 \mp 1}{4}.$$

Proof. According to Lemma 2.3, we have

$$\begin{aligned} N_W = 1 &\Leftrightarrow |(g(w_1, w_1) + g(w_2, w_2))^2 - 4g(w_1, w_2)^2| = 1 \\ &\Leftrightarrow (g(w_1, w_1) + g(w_2, w_2))^2 - 4g(w_1, w_2)^2 = \mp 1 \\ &\Leftrightarrow g(w_1, w_2)^2 = \frac{(g(w_1, w_1) + g(w_2, w_2))^2 \mp 1}{4}. \end{aligned}$$

□

3 Hyperbolic matrix representations of hybrid-hyperbolic numbers

3.1 2 × 2 Hyperbolic matrix representations of hybrid-hyperbolic numbers

In this section, we get 2×2 hyperbolic matrix representations of hybrid-hyperbolic numbers and some properties of them are examined.

Theorem 3.1. *Let* $W = H_0 + H_1 i + H_2\varepsilon + H_3 h \in \mathbb{KH}$ *be an arbitrary hybrid-hyperbolic number, where* $H_0, H_1, H_2, H_3, \in \mathbb{H}$. *Then the hyperbolic matrix*

$$\sigma(W) = \begin{bmatrix} H_0 + jH_2 & (H_1 - H_2) + jH_3 \\ (H_2 - H_1) + jH_3 & H_0 - jH_2 \end{bmatrix},$$

which corresponds to W, *is called* 2×2 *hyperbolic matrix representation of* W.

Proof. Assume the hyperbolic matrix set which can be represented as

$$M_2(\mathbb{H}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{H} \right\}$$

or

$$M_2(\mathbb{H}) = \left\{ \begin{bmatrix} z_1 + jz_1^* & z_2 + jz_2^* \\ z_3 + jz_3^* & z_4 + jz_4^* \end{bmatrix} : z_i, z_i^* \in \mathbb{R}, \quad j^2 = 1, \quad i = 1, 2, 3, 4 \right\}$$

Then, we can write

$$\begin{aligned} a_0 &= \frac{z_1 + z_4}{2}, \quad b_0 = \frac{z_1^* + z_4^*}{2} \\ a_1 &= \frac{z_2 - z_3}{2} + \frac{z_1 - z_4}{2}, \quad b_1 = \frac{z_1^* - z_4^*}{2} + \frac{z_2^* - z_3^*}{2} \\ a_2 &= \frac{z_1 - z_4}{2}, \quad b_2 = \frac{z_1^* - z_4^*}{2} \\ a_3 &= \frac{z_2 + z_3}{2}, \quad b_3 = \frac{z_2 * + z_3 *}{2}. \end{aligned}$$

Therefore, we have

$$A = \begin{bmatrix} (a_0 + a_2) + j(b_0 + b_2) & (a_1 - a_2 + a_3) + j(b_1 - b_2 + b_3) \\ (a_2 - a_1 + a_3) + j(b_2 - b_1 + b_3) & (a_0 - a_2) + j(b_0 - b_2) \end{bmatrix} \tag{6}$$

According to Equation 6, we have

$$\sigma(W) = \begin{bmatrix} H_0 + jH_2 & (H_1 - H_2) + jH_3 \\ (H_2 - H_1) + jH_3 & H_0 - jH_2 \end{bmatrix}$$

where $H_i = a_i + jb_i \in \mathbb{H}, \ \ i = 0, 1, 2, 3.$ □

Remark 3.1. *Let us take into account the function*

$$\sigma : \mathbb{KH} \to M_2(\mathbb{H})$$

$$W = H_0 + H_1 i + H_2 \varepsilon + H_3 h \to \sigma = \begin{bmatrix} H_0 + jH_2 & (H_1 - H_2) + jH_3 \\ (H_2 - H_1) + jH_3 & H_0 - jH_2 \end{bmatrix}.$$

The function σ satisfies the properties

$$\sigma(W_1 + W_2) = \sigma(W_1) + \sigma(W_2), \quad \sigma(W_1 W_2) = \sigma(W_1)\sigma(W_2),$$

where W_1 and W_2 are any hybrid-hyperbolic numbers. Also, it can be easily seen that σ is a bijection. Consequently, σ is a linear isomorphism.

Theorem 3.2. *Let $W = H_0 + H_1 i + H_2 \varepsilon + H_3 h \in \mathbb{KH}$ be a hybrid-hyperbolic number. Then the following equalities hold:*

1. $|\det(\sigma(W))| = (N_W)^2 = |H_0^2 + (H_1 - H_2)^2 - H_2^2 - H_3^2|$
2. *W is invertible if and only if $\sigma(W)$ is invertible, then $\sigma\left(W^{-1}\right) = \left(\sigma(W)\right)^{-1}$.*

Proof. 1. Let

$$\sigma = \begin{bmatrix} H_0 + jH_2 & (H_1 - H_2) + jH_3 \\ (H_2 - H_1) + jH_3 & H_0 - jH_2 \end{bmatrix}$$

be a hyperbolic matrix representation of W. Then, we get

$$\begin{aligned} |\det(\sigma(W))| &= \begin{vmatrix} H_0 + jH_2 & (H_1 - H_2) + jH_3 \\ (H_2 - H_1) + jH_3 & H_0 - jH_2 \end{vmatrix} \\ &= |H_0^2 + (H_1 - H_2)^2 - H_2^2 - H_3^2| \\ &= (N_W)^2 \end{aligned}$$

2. According to Definition 2.5, W is invertible if and only if $N_W \neq 0$. Then, we have W is invertible $\Leftrightarrow N_W \neq 0 \Leftrightarrow \det(\sigma(W)) \neq 0 \Leftrightarrow \sigma(W)$ is invertible.
Assume that W and $\sigma(W)$ are invertible. Therefore, we have

$$WW^{-1} = W^{-1}W = 1$$

Since σ is a linear isomorphism, we can write

$$\sigma(W)\sigma(W^{-1}) = \sigma(WW^{-1}) = \sigma(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$\sigma(W^{-1})\sigma(W) = \sigma(W^{-1}W) = \sigma(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Consequently, we have $\sigma(W^{-1}) = (\sigma(W))^{-1}$. □

Theorem 3.3. *Let $W = H_0 + H_1 i + H_2 \varepsilon + H_3 h \in \mathbb{KH}$ be a hybrid-hyperbolic number. Then, the following equations are provided:*

1. $\sigma(W^{\dagger 1}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\sigma(W))^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,
2. $\sigma(W^{\dagger 2}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sigma(W)^{\triangle} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,
3. $\sigma(W^{\dagger 3}) = (\sigma(W))^{\triangledown}$,

where $W^{\dagger 1}$ is hybrid-conjugate , $W^{\dagger 2}$ is hyperbolic-conjugate, $\sigma(W)^{\triangle}$ is hyperbolic-conjugate of matrix $\sigma(W)$, $W^{\dagger 3}$ is coupled-conjugate and $(\sigma(W))^{\triangledown} = (\sigma(W)^{\triangle})^T$.

Proof. 1. Let $W = H_0 + H_1 i + H_2 \varepsilon + H_3 h$ be a hybrid-hyperbolic number. The hybrid conjugation $W^{\dagger 1} = H_0 - H_1 i - H_2 \varepsilon - H_3 h$ of hybrid-hyperbolic number. The hyperbolic matrix representation of this is:

$$\sigma(W^{\dagger 1}) = \begin{bmatrix} H_0 - jH_2 & (-H_1 + H_2) - jH_3 \\ (-H_2 + H_1) - jH_3 & H_0 + jH_2 \end{bmatrix}.$$

Also, we have

$$(\sigma(W))^T = \begin{bmatrix} H_0 + jH_2 & (H_2 - H_1) + jH_3 \\ (H_1 - H_2) + jH_3 & H_0 - jH_2 \end{bmatrix}.$$

As a result of basic operations results, we obtain

$$\sigma(W^{\dagger 1}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\sigma(W))^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2. Let $W = H_0 + H_1 i + H_2 \varepsilon + H_3 h$ be a hybrid-hyperbolic number. The hyperbolic conjugation $W^{\dagger 3} = \overline{H_0} - \overline{H_1} i - \overline{H_2} \varepsilon - \overline{H_3} h$ of hybrid-hyperbolic number. The hyperbolic matrix representation of this is:

$$\sigma(W^{\dagger 2}) = \begin{bmatrix} \overline{H_0} + j\overline{H_2} & (\overline{H_1} - \overline{H_2}) + j\overline{H_3} \\ (\overline{H_2} - \overline{H_1}) + j\overline{H_3} & \overline{H_0} - j\overline{H_2} \end{bmatrix}.$$

On the other hand, we get

$$\sigma(W)^{\triangle} = \begin{bmatrix} (H_0 + jH_2)^{\dagger 2} & ((H_1 - H_2) + jH_3)^{\dagger 2} \\ ((H_2 - H_1) + jH_3)^{\dagger 2} & (H_0 - jH_2)^{\dagger 2} \end{bmatrix}$$

$$= \begin{bmatrix} H_0^{\dagger 2} + (jH_2)^{\dagger 2} & (H_1 - H_2)^{\dagger 2} + (jH_3)^{\dagger 2} \\ (H_2 - H_1)^{\dagger 2} + (jH_3)^{\dagger 2} & H_0^{\dagger 2} + (-jH_2)^{\dagger 2} \end{bmatrix}.$$

After this step, general conjugate will be used instead of $\dagger 2$ since the conjugates of the hyperbolic numbers are taken.

$$\sigma(W)^{\triangle} = \begin{bmatrix} (\overline{H_0} - j\overline{H_2}) & (\overline{H_1} - \overline{H_2}) - j\overline{H_3}) \\ (\overline{H_2} - \overline{H_1}) - j\overline{H_3} & \overline{H_0} + j\overline{H_2} \end{bmatrix}.$$

Consequently, we have $\sigma(W^{\dagger 2}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sigma(W)^{\triangle} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$

3. Let $W = H_0 + H_1 i + H_2 \varepsilon + H_3 h$ be a hybrid-hyperbolic number. The coupled conjugation $W^{\dagger 3} = \overline{H_0} - \overline{H_1} i - \overline{H_2} \varepsilon - \overline{H_3} h$ of hybrid-hyperbolic number. The hyperbolic matrix representation of this is:

$$\sigma(W^{\dagger 3}) = \begin{bmatrix} \overline{H_0} - j\overline{H_2} & (-\overline{H_1} + \overline{H_2}) - j\overline{H_3} \\ (-\overline{H_2} + \overline{H_1}) - j\overline{H_3} & \overline{H_0} + j\overline{H_2} \end{bmatrix}$$

Moreover, we can write

$$\sigma(W)^{\triangledown} = \begin{bmatrix} (\overline{H_0} - j\overline{H_2}) & (\overline{H_1} - \overline{H_2}) - j\overline{H_3}) \\ (\overline{H_2} - \overline{H_1}) - j\overline{H_3} & \overline{H_0} + j\overline{H_2} \end{bmatrix}^T$$

$$= \begin{bmatrix} (\overline{H_0} - j\overline{H_2}) & (\overline{H_2} - \overline{H_1}) - j\overline{H_3}) \\ (\overline{H_1} - \overline{H_2}) - j\overline{H_3}) & \overline{H_0} + j\overline{H_2} \end{bmatrix}.$$

Hence equality is shown. □

3.2 4×4 Hyperbolic matrix representations of hybrid-hyperbolic numbers

Let $W = H_0 + H_1 i + H_2\varepsilon + H_3 h$ be a hybrid-hyperbolic number. We introduce the linear maps

$$\begin{aligned} \zeta^{+}: \ & \mathbb{KH} && \to \mathbb{KH} \\ & Q && \to \zeta^{+}(Q) = WQ \end{aligned} \tag{7}$$

and

$$\begin{aligned} \zeta^{-}: \ & \mathbb{KH} && \to \mathbb{KH} \\ & Q && \to \zeta^{-}(Q) = QW \end{aligned} \tag{8}$$

Then, we have

$$\begin{aligned} \zeta^{+}(1) &= W1 = H_0 + H_1 i + H_2\varepsilon + H_3 h \\ \zeta^{+}(i) &= Wi = (H_2 - H_1) + (H_0 - H_3)i - H_3\varepsilon + H_2 h \\ \zeta^{+}(\varepsilon) &= W\varepsilon = H_1 + (H_0 + H_3)\varepsilon - H_1 h \\ \zeta^{+}(h) &= Wh = H_3 + H_1 i + (H_1 - H_2)\varepsilon + H_0 h \end{aligned} \tag{9}$$

and

$$\begin{aligned} \zeta^{-}(1) &= 1W = H_0 + H_1 i + H_2\varepsilon + H_3 h \\ \zeta^{-}(i) &= iW = (H_2 - H_1) + (H_0 + H_3)i + H_3\varepsilon - H_2 h \\ \zeta^{-}(\varepsilon) &= \varepsilon W = H_1 + (H_0 - H_3)\varepsilon + H_1 h \\ \zeta^{-}(h) &= hW = H_3 - H_1 i + (H_2 - H_1)\varepsilon + H_0 h \end{aligned} \tag{10}$$

Therefore, we get the following hyperbolic matrix representations

$$Z^{+}(W) = \begin{bmatrix} H_0 & H_2 - H_1 & H_1 & H_3 \\ H_1 & H_0 - H_3 & 0 & H_1 \\ H_2 & -H_3 & H_0 + H_3 & H_1 - H_2 \\ H_3 & H_2 & -H_1 & H_0 \end{bmatrix}$$

and

$$Z^{-}(W) = \begin{bmatrix} H_0 & H_2 - H_1 & H_1 & H_3 \\ H_1 & H_0 + H_3 & 0 & -H_1 \\ H_2 & H_3 & H_0 - H_3 & H_2 - H_1 \\ H_3 & -H_2 & H_1 & H_0 \end{bmatrix},$$

where ζ^{+}, ζ^{-} are linear maps and $H_i = a_i + jb_i \in \mathbb{H},\ \ i = 0, 1, 2, 3.$

Thus, the product of two hybrid hyperbolic numbers with the help of these matrices can be given by the matrix vector product as:

$$WQ = Z^{+}(W)Q = \begin{bmatrix} H_0 & H_2 - H_1 & H_1 & H_3 \\ H_1 & H_0 - H_3 & 0 & H_1 \\ H_2 & -H_3 & H_0 + H_3 & H_1 - H_2 \\ H_3 & H_2 & -H_1 & H_0 \end{bmatrix} \begin{bmatrix} H_0^* \\ H_1^* \\ H_2^* \\ H_3^* \end{bmatrix}$$

and

$$QW = Z^{-}(W)Q = \begin{bmatrix} H_0 & H_2 - H_1 & H_1 & H_3 \\ H_1 & H_0 + H_3 & 0 & -H_1 \\ H_2 & H_3 & H_0 - H_3 & H_2 - H_1 \\ H_3 & -H_2 & H_1 & H_0 \end{bmatrix} \begin{bmatrix} H_0^* \\ H_1^* \\ H_2^* \\ H_3^* \end{bmatrix}$$

where $W = H_0 + H_1 i + H_2\varepsilon + H_3 h$ and $Q = H_0^* + H_1^* i + H_2^*\varepsilon + H_3^* h$ hybrid-hyperbolic numbers.

Theorem 3.4. *Let $W = H_0+H_1i+H_2\varepsilon+H_3h$ and $Q = H_0^*+H_1^*i+H_2^*\varepsilon+H_3^*h$ hybrid-hyperbolic numbers in $\mathbb{KH}$ and λ be a hyperbolic number.*

1. $W = Q \Leftrightarrow Z^+(W) = Z^+(Q) \Leftrightarrow Z^-(W) = Z^-(W)$
2. $Z^+(W + R) = Z^+(W) + Z^+(Q), \quad Z^-(W + R) = Z^-(W) + Z^-(Q)$
3. $Z^+(\lambda W) = \lambda Z^+(W), \quad Z^-(\lambda W) = \lambda Z^-(W)$
4. $Z^+(WQ) = Z^+(W)Z^+(Q), \quad Z^-(WQ) = Z^-(Q)Z^-(W)$
5. $Z^+(W^{-1}) = [Z^+(W)]^{-1}, \quad Z^-(W^{-1}) = [Z^-(W)]^{-1}$
6. $\det Z^+(W) = (N_W)^4, \quad \det Z^-(W) = (N_W)^4, \quad trace[Z^+(W)] = trace[Z^-(W)] = 4H_0.$

Proof. All properties except for the fourth and the fifth oneare easily shown. Now let us consider the 4 and 5 properties:

4. Let $W = H_0 + H_1i + H_2\varepsilon + H_3h$ and $Q = H_0^* + H_1^*i + H_2^*\varepsilon + H_3^*h$ hybrid-hyperbolic numbers in $\mathbb{KH}$. Then, we can write

$$WQ = A + Bi + C\varepsilon + Dh,$$

where $A = H_0H_0^* - H_1H_1^* + H_1H_2^* + H_2H_1^* + H_3H_3^*$, $B = H_0H_1^* + H_1H_0^* + H_1H_3^* - H_3H_1^*$, $C = H_0H_2^* + H_1H_3^* + H_2H_0^* - H_2H_3^* - H_3H_1^* + H_3H_2^*$ and $D = H_0H_3^* - H_1H_2^* + H_2H_1^* - H_3H_0^*$. Therefore, we get

$$Z^+(WQ) = \begin{bmatrix} A & C-B & B & D \\ B & A-C & 0 & B \\ C & -D & A+C & B-C \\ D & C & -B & A \end{bmatrix}$$

$$= \begin{bmatrix} H_0 & H_2-H_1 & H_1 & H_3 \\ H_1 & H_0-H_3 & 0 & H_1 \\ H_2 & -H_3 & H_0+H_3 & H_1-H_2 \\ H_3 & H_2 & -H_1 & H_0 \end{bmatrix} \begin{bmatrix} H_0^* & H_2*-H_1* & H_1* & H_3* \\ H_1* & H_0*-H_3* & 0 & H_1* \\ H_2* & -H_3* & H_0*+H_3* & H_1*-H_2* \\ H_3* & H_2* & -H_1* & H_0* \end{bmatrix}$$

$= Z^+(W)Z^+(Q)$

and

$$Z^-(WQ) = \begin{bmatrix} A & C-B & B & D \\ B & A+C & 0 & -B \\ C & D & A-C & C-B \\ D & -C & B & A \end{bmatrix}$$

$$= \begin{bmatrix} H_0^* & H_2*-H_1* & H_1* & H_3* \\ H_1* & H_0*+H_3* & 0 & -H_1* \\ H_2* & H_3* & H_0*-H_3* & H_2*-H_1* \\ H_3* & -H_2* & H_1* & H_0* \end{bmatrix} \begin{bmatrix} H_0 & H_2-H_1 & H_1 & H_3 \\ H_1 & H_0+H_3 & 0 & -H_1 \\ H_2 & H_3 & H_0-H_3 & H_2-H_1 \\ H_3 & -H_2 & H_1 & H_0 \end{bmatrix}$$

$= Z^-(Q)Z^-(R).$

5. According to definition inverse of hybrid-hyperbolic numbers, we have

$$WW^{-1} = W^{-1}W = 1$$

Then, we can write

$$Z^{+}(W)Z^{+}(W^{-1}) = Z^{+}(WW^{-1}) = Z^{+}(1) = I_4$$

and

$$Z^{+}(W^{-1})Z^{+}(W) = Z^{+}(W^{-1}W) = Z^{+}(1) = I_4.$$

Consequently, we have $Z^{+}(W^{-1}) = [Z^{+}(W)]^{-1}$. Similarly, it can be seen that $Z^{-}(W^{-1}) = [Z^{-}(W)]^{-1}$. □

Besides the hyperbolic matrix representations of hybrid hyperbolic numbers, the 4×4 real representation can be given as follows.

It is known that each hybrid number $q = a+bi+c\varepsilon+dh$ and hyperbolic number $h = h_1+jh_2$ can be represented by the two 2×2 real matrices

$$\begin{bmatrix} a+c & (b-c)+d \\ (c-b)+d & a-c \end{bmatrix}$$

and

$$\begin{bmatrix} h_1 & h_2 \\ h_2 & h_1 \end{bmatrix}.$$

Therefore, every hybrid-hyperbolic number

$$W = (a_0 + jb_0) + (a_1 + jb_1)i + (a_2 + jb_2)\varepsilon + (a_3 + jb_3)h$$

can be represented by a 4×4 real matrix.

$$W \leftrightarrow X_W = \begin{bmatrix} a_0+a_2 & b_0+b_2 & (a_1-a_2)+a_3 & (b_1-b_2)+b_3 \\ b_0+b_2 & a_0+a_2 & (b_1-b_2)+b_3 & (a_1-a_2)+a_3 \\ (a_2-a_1)+a_3 & (b_2-b_1)+b_3 & a_0-a_2 & b_0-b_2 \\ (b_2-b_1)+b_3 & (a_2-a_1)+a_3 & b_0-b_2 & a_0-a_2 \end{bmatrix}.$$

Example 3.1. *Let $W = (2+3j)+(1-j)i+(3-4j)\varepsilon+(2+j)h$ be a hybrid-hyperbolic number. The 2×2 hyperbolic matrix representation is*

$$\sigma(W) = \begin{bmatrix} -2+6j & -1+5j \\ 3-j & 6+0j \end{bmatrix}.$$

The 4×4 hyperbolic matrix representation is

$$Z^{+}(W) = \begin{bmatrix} 2+3j & 2-3j & 1-j & 2+j \\ 1-j & 2j & 0 & 1-j \\ 3-4j & -2-j & 4+4j & -2+3j \\ 2+j & 3-4j & -1+j & 2+3j \end{bmatrix}$$

and

$$Z^{-}(W)=\begin{bmatrix} 2+3j & 2-3j & 1-j & 2+j \\ 1-j & 4+4j & 0 & -1+j \\ 3-4j & 2+j & 2j & 2-3j \\ 2+j & -3+4j & 1-j & 2+3j \end{bmatrix}.$$

The 4×4 *real matrix representation is*

$$X_W=\begin{bmatrix} 5 & -1 & 0 & 4 \\ -1 & 5 & 4 & 0 \\ 4 & -2 & -1 & 7 \\ -2 & 4 & 7 & -1 \end{bmatrix}.$$

$$\begin{aligned} N_W &= \sqrt{|(2+3j)^2+(1-j-3+4j)^2-(3-4j)^2-(2+j)^2|} \\ &= \sqrt{|-4+20j|} \\ &= \sqrt[4]{|16-400|} \\ &= \sqrt[4]{384} \\ &= \sqrt[4]{|\det X_W|}, \sqrt[2]{|\det \sigma_W|}. \end{aligned} \tag{11}$$

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