Factorials as repdigits in base $b$

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Abstract: Let $b \in \{2, 3, \ldots, 9\}$. In this paper, we show that the solutions of the equation $(x)_b = m!$ are $(11)_5 = 3!$, $(33)_7 = (44)_5 = 4!$, where $(x)_b$ has at least two digits.

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1 Introduction

Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence given by $F_n = F_{n-1} + F_{n-2}$, for $n \geq 0$, with $F_0 = 0$ and $F_1 = 1$. The first few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

The following Table 1 shows several generalizations of the Fibonacci sequence.

Finding special properties in these sequences is a very interesting problem. A number of mathematicians studied equations involving the above sequences, repdigits and factorials. For example,

- Marques and Lengyel [8] solved the equation $T_n = m!$.
- Irmak [3] found the solutions of the equation $w_n = m!$, where $w_n$ is the $n$-th term of Perrin or Padovan sequence.
### Sequence | Recurrence relation | Initial conditions
--- | --- | ---
Lucas | \( L_n = L_{n-1} + L_{n-2} \) | \( L_0 = 2 \) and \( L_1 = 1 \)
Pell | \( P_n = 2P_{n-1} + P_{n-2} \) | \( P_0 = 0 \) and \( P_1 = 1 \)
Pell–Lucas | \( Q_n = 2Q_{n-1} + Q_{n-2} \) | \( Q_0 = 2 \) and \( Q_1 = 2 \)
Balancing | \( B_n = 6B_{n-1} - B_{n-2} \) | \( B_0 = 0 \) and \( B_1 = 1 \)
Jacobsstal | \( J_n = J_{n-1} + 2J_{n-2} \) | \( J_0 = 0 \) and \( J_1 = 1 \)
Tribonacci | \( T_n = T_{n-1} + T_{n-2} + T_{n-3} \) | \( T_0 = 0 \), \( T_1 = 1 \), and \( T_2 = 1 \)
Perrin | \( R_n = R_{n-2} + R_{n-3} \) | \( R_0 = 3 \), \( R_1 = 0 \), and \( R_2 = 2 \)
Padovan | \( P_n = P_{n-2} + P_{n-3} \) | \( P_0 = 1 \), \( P_1 = 1 \), and \( P_2 = 1 \).

Table 1. Generalizations of the Fibonacci sequence

- Marques and Togbé [9] handled the equation
  \[ F_nF_{n+1} \ldots F_{n+k-1} = d \left( \frac{10^m - 1}{9} \right). \]
- The equation
  \[ L_nL_{n+1} \ldots L_{n+k-1} = d \left( \frac{10^m - 1}{9} \right) \]
  was solved by Irmak and Togbé [4].

There are also several results including sum of the members of linear recurrences (given in the table) which are repdigits (see the papers [1, 2, 10–14]).

Motivated by these papers, it is natural to ask the following question:

**What factorials are repdigits in base \( b \)?**

We answer this question by proving the following theorem.

**Theorem 1.1.** Let \( b \in \{2, 3, \ldots, 9\} \) and \( x, m \) be positive integers. The solutions of the equation

\[ (x)_b = m! \]

are given by \( (11)_5 = 3! \), \( (33)_5 = (44)_5 = 4! \).

To prove this theorem, we will characterize and use the 2-adic values \( \nu_2(b^n - 1) \). The \( p \)-adic order, \( \nu_p(r) \) of \( r \) is the exponent of the highest power of a prime \( p \) which divides \( r \).

### 2 Auxiliary results

Now, we will give the 2-adic order of the term \( b^n - 1 \), for \( b \in \{2, 3, \ldots, 9\} \) by proving the following theorem. It is obvious that \( \nu_2(b^k - 1) = 0 \) for even \( b \).
Theorem 2.1. For $k \geq 1$, we have

(i) 
\[ \nu_2 \left( 3^k - 1 \right) = \begin{cases} 
\nu_2 (k) + 2, & \text{if } k \text{ is even} \\
1, & \text{if } k \text{ is odd}
\end{cases} \]

(ii) 
\[ \nu_2 \left( 5^k - 1 \right) = \begin{cases} 
\nu_2 (k) + 2, & \text{if } k \text{ is even} \\
1, & \text{if } k \text{ is odd}
\end{cases} \]

(iii) 
\[ \nu_2 \left( 7^k - 1 \right) = \begin{cases} 
\nu_2 (k) + 1, & \text{if } k \text{ is even} \\
1, & \text{if } k \text{ is odd}
\end{cases} \]

(iv) 
\[ \nu_2 \left( 9^k - 1 \right) = \begin{cases} 
\nu_2 (k) + 3, & \text{if } k \text{ is even} \\
1, & \text{if } k \text{ is odd}
\end{cases} \]

Proof. Firstly, we will deal with the first 2-adic order. Assume that $k$ is even positive integer. To prove it, we need to show that

\[ 3^{2k_s} - 1 \equiv 2^{k+2} \quad (\text{mod } 2^{k+3}). \]

We will use the induction on $s$ to prove the congruence. Firstly, we will deal with the basic case $s = 1$. So, we want to prove that

\[ 3^{2k} - 1 \equiv 2^{k+2} \quad (\text{mod } 2^{k+3}). \]

Now, we will use the induction on $k$. Obviously, the congruence holds for $k = 1$. Then we suppose that the congruence

\[ 3^{2k} - 1 \equiv 2^{k+2} \quad (\text{mod } 2^{k+3}) \quad (1) \]

is true for $k$. Our aim is to show that $3^{2^{k+1}} - 1 \equiv 2^{k+3} \quad (\text{mod } 2^{k+4})$. The congruence (1) implies that

\[ 3^{2k} = 2^{k+2} + 1 + l_1 2^{k+3}, \quad (2) \]

for some $l_1$. So, we deduce that

\[ 3^{2^{k+1}} = \left( 3^{2k} \right)^2 = \left( 2^{k+2} + 1 + l_1 2^{k+3} \right)^2 \]

\[ = 2^{2(k+3)} l_1^2 + 2^{2(k+3)} l_1 + 2^{2(k+2)} + 2^{k+4} l_1 + 2^{k+3} + 1 \]

and this gives

\[ 3^{2^{k+1}} - 1 \equiv 2^{k+3} \quad (\text{mod } 2^{k+4}) \]

as desired. Here, we used the facts that $2k + 6 \geq k + 3$ and $2k + 4 \geq k + 3$, for $k \geq 1$.

By the induction hypothesis, $3^{2^{k} - 1} \equiv 2^{k+2} \quad (\text{mod } 2^{k+3})$ holds for $s$. It means that there exists the integer $l_2$ such that $3^{2^{k} s} = 2^{k+2} s + l_2 2^{k+2} + 1$. From this and (2), we deduce that

\[ 3^{2^{k}(s+1)} = 3^{2^{k} s} \cdot 3^{2^{k}} \]

\[ = \left( 2^{k+2} s + l_2 2^{k+2} + 1 \right) \left( 2^{k+2} + 1 + l_1 2^{k+3} \right) . \]
So,

\[ 3^{2^{k(s+1)}} - 1 \equiv 2^{k+2} (s + 1) \pmod{2^{k+3}} \]

follows.

From now on, suppose that \( k \) is odd. Our aim is to show that

\[ 3^{2^{w+1}} - 1 \equiv 2 \pmod{4} \]  \( (3) \)

We use the induction method again. It is easy to see that the congruence holds for \( w = 1 \). Then assume that it is true for \( w \). If we multiply the congruence (3) with 9, then the congruence

\[ 3^{2^{w+3}} - 1 \equiv 2 \pmod{4} \]

holds as claimed. This finished the proof. The remaining items can be similarly proven. Therefore, we leave the details to the reader.

The following lemma gives the upper and lower bounds for the term \( \nu_p(k!) \). To prove this, we refer to Lemma 2.2 in [7].

**Lemma 2.1.** For any integer \( k \geq 1 \) and \( p \) prime, we have

\[ \frac{k}{p-1} - \frac{\log k}{\log p} \leq \nu_p(k!) \leq \frac{k-1}{p-1}, \]

where \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \).

### 3 Proof of Theorem

Assume that \( (x)_b = d \left( \frac{b^k-1}{b-1} \right) \) for \( d \in \{1, 2, \ldots, 9\} \) and \( b \in \{2, \ldots, 9\} \). By using Lemma 2.1 with Theorem 2.1, we have

\[ m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor \leq \nu_2(m!) = \nu_2 \left( d \frac{b^k-1}{b-1} \right) \leq \nu_2(k) + 3 + \nu_2(d) \leq \nu_2(k) + 6. \]

It means that \( 2^{m-6-[\frac{\log m}{\log 2}]} \) divides \( k \). Then

\[ 2^{m-6-[\frac{\log m}{\log 2}]} \leq k \]  \( (4) \)

follows. It is known that \( m! < \left( \frac{m}{2} \right)^m \). This fact gives

\[ (k - 1) \log b < \log \left( d \frac{b^k-1}{b-1} \right) < m \left( \log \frac{m}{2} \right). \]  \( (5) \)

Combining (4) and (5), we arrive at

\[ 2^{m-6-[\frac{\log m}{\log 2}]} < \frac{m \log \left( \frac{m}{2} \right)}{\log b} + 1. \]

This inequality implies that \( m \leq 14 \) for \( b \in \{2, \ldots, 9\} \). We use a simple routine written in Mathematica which gives the solutions as listed in Theorem 1.1. The proof of our main result is complete. \( \square \)
References


