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On the Vieta–Jacobsthal-like polynomials

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Abstract: In this paper, we first introduce the generalization of the Vieta–Jacobsthal polynomial, which is called the Vieta–Jacobsthal-like polynomial. After that, we give the generating function, the Binet formula, and some well-known identities for this polynomial. Finally, we also present the relation between this polynomial and the previously famous Vieta-polynomials.

Keywords: Vieta–Jacobsthal polynomial, Vieta–Jacobsthal–Lucas polynomial, Generalized Vieta–Jacobsthal polynomial.
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1 Introduction

The theory of the Vieta polynomials was first introduced in 1991 by Robbins [6]. The recursive sequence of the Vieta-Fibonacci polynomial $V_n(x)$ and Vieta-Lucas polynomials $v_n(x)$ were introduced by Horadam [2]. These polynomials are defined by $V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$, and $v_n(x) = xv_{n-1}(x) - v_{n-2}(x)$ for $n \geq 2$, with the initial conditions $V_0(x) = 0, V_1(x) = 1$, and

$v_0(x) = 2, v_1(x) = x$, respectively. These polynomials are closely related to the well-known Chebyshev polynomials of the first and second kinds that are denoted by $T_n(x)$ and $U_n(x)$, respectively. The related features of Vieta and Chebyshev polynomials are given as

$$V_n(x) = U_n\left(\frac{1}{2}x\right) \quad \text{see [2]},$$

$$v_n(x) = 2T_n\left(\frac{1}{2}x\right) \quad \text{see [3, 6]}$$

For more application of the Chebyshev polynomials of the first and second kinds, see [4, 5], and the references therein. In recent years, the Vieta polynomials have been much attention to many authors. In 2013, Tasci and Yalcin [7] introduced the recurrence relation of Vieta–Pell polynomials $t_n(x)$ and Vieta–Pell–Lucas polynomials $s_n(x)$ as $t_0(x) = 0$, $t_1(x) = 1$, $t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x)$, for $n \geq 2$, and $s_0(x) = 2$, $s_1(x) = 2x$, $s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x)$, for $n \geq 2$, provide that $|x| > 1$. Tasci and Yalcin [7] obtained the Binet form and generating functions of Vieta–Pell and Vieta–Pell–Lucas polynomials. Also, they received some differentiation rules and the finite summation formulas. Moreover, the following relations are obtained.

$$s_n(x) = 2T_n(x), \quad \text{and} \quad t_{n+1}(x) = U_n(x).$$

Recently, Yalcin et al. [1] introduced and studied the Vieta–Jacobsthal polynomials $G_n(x)$ and Vieta–Jacobsthal–Lucas polynomials $g_n(x)$ which defined respectively by

$$\begin{aligned} G_n(x) &= G_{n-1}(x) - 2xG_{n-2}(x), \quad \text{for } n \geq 2, \\ g_n(x) &= g_{n-1}(x) - 2xg_{n-2}(x), \quad \text{for } n \geq 2, \end{aligned}$$

where $G_0(x) = 0$, $G_1(x) = 1$ and $g_0(x) = 2$, $g_1(x) = 1$. The first few terms of $\{G_n(x)\}_{n=0}^{\infty}$ are $0,\ 1,\ 1,\ -2x + 1,\ -4x + 1,\ 4x^2 - 6x + 1,\ 12x^2 - 8x + 1, \ldots$ and the first few terms of $\{g_n(x)\}_{n=0}^{\infty}$ are $2,\ 1,\ -4x + 1,\ -6x + 1,\ 8x^2 - 8x + 1,\ 20x^2 - 10x + 1$. The n-th terms of these polynomials sequences are called the Vieta-Jcobsthal and Vieta–Jacobsthal–Lucas polynomials, respectively. The Binet formulas for Vieta–Jacobsthal polynomials $G_n(x)$ and Vieta–Jacobsthal-Lucas polynomials $g_n(x)$ are given by

$$G_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha_3(x) - \beta(x)},$$

$$g_n(x) = \alpha^n(x) + \beta^n(x),$$

for all $n \geq 0$, where $\alpha(x) = \frac{1 + \sqrt{1 - 8x}}{2}$ and $\beta(x) = \frac{1 - \sqrt{1 - 8x}}{2}$ are the roots of the characteristic equation $r^2 - r + 2x = 0$. Moreover, they also introduced the generalization of the Vieta–Jacobsthal and Vieta–Jacobsthal polynomials, and many identities for these polynomials are derived.

In this paper, we investigate the generalization of the Vieta–Jacobsthal polynomial, which is called Vieta-Jacobthal-like polynomial. We give the generating function, the Binet formula, and some well-known identities for this polynomial. Moreover the relation between this polynomial and the Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials are also presented.

2 Vieta–Jacobsthal-like polynomials and some identities

In this section, we investigate some new generalization of the Vieta–Jacobsthal polynomials sequence that has the same recurrence relation as the Vieta–Jacobsthal polynomials sequence and the initial conditions are the combination of the Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials as follows:

Definition 2.1. *For any natural number n the Vieta–Jacobsthal-like polynomials sequence $\{W_n(x)\}_{n=0}^{\infty}$ is defined by*

$$W_n(x) = W_{n-1}(x) - 2xW_{n-2}(x), \quad \textit{for } n \geq 2, \tag{1}$$

with the initial conditions $W_0(x) = 2$ and $W_1(x) = 2$.

From Difinition 2.1, it easy to verify that $W_n(x) = G_n(x) + g_n(x)$, for all n. The first few terms of $\{W_n(x)\}_{n=0}^{\infty}$ are as follows:

$$\begin{aligned}
W_0(x) &= 2,\\
W_1(x) &= 2,\\
W_2(x) &= -4x + 2,\\
W_3(x) &= -8x + 2,\\
W_4(x) &= 8x^2 - 12x + 2,\\
W_5(x) &= 24x^2 - 16x + 2,\\
W_6(x) &= -16x^3 + 48x^2 - 20x + 2,\\
W_7(x) &= -64x^3 + 80x^2 - 24x + 2.\\
&\vdots
\end{aligned}$$

The characteristic equation of (1) is defined by

$$r^2 - r + 2x = 0, \tag{2}$$

and the roots of equation (2) are $\alpha(x) = \frac{1+\sqrt{1-8x}}{2}$ and $\beta(x) = \frac{1-\sqrt{1-8x}}{2}$. We note that $\alpha(x) + \beta(x) = 1$, $\alpha(x) - \beta(x) = \sqrt{1-8x}$, and $\alpha(x)\beta(x) = 2x$.

We first give the generating function for this polynomials sequence as the following Theorem.

Theorem 2.1 (The generating function). *Let $g(x,t) = \sum_{n=0}^{\infty} W_n(x)t^n$ be the generating function of Vieta–Jacobsthal-like polynomials sequence. Then*

$$g(x,t) = \frac{2}{1 - t + 2xt^2}.$$

Proof. Consider,

$$\begin{aligned}
g_W(x,t) &= \sum_{n=0}^{\infty} W_n(x)t^n\\
&= W_0(x) + W_1(x)t + W_2(x)t^2 + \cdots + W_n(x)t^n + \dots .
\end{aligned}$$

Then, we get

$$\begin{aligned}
-tg_W(x,t) &= -W_0(x)t - W_1(x)t^2 - W_2(x)t^3 - ... - W_{n-1}(x)t^n - ...\\
2xt^2 g_W(x,t) &= 2xW_0(x)t^2 + 2xW_1(x)t^3 + 2xW_4(x)t^2 + ... + 2xW_{n-2}(x)t^n + ...
\end{aligned}$$

Thus,

$$\begin{aligned}
g(x,t)(1-t+2xt^2) &= W_0(x) + (W_1(x) - W_0(x))\, t\\
&\quad + \sum_{n=2}^{\infty} (W_n(x) - W_{n-1} + 2xW_{n-2}(x))\, t^n\\
&= W_0(x) + (W_1(x) - W_0(x))\, t\\
&= 2 + (2-2)t\\
&= 2\\
g(x,t) &= \frac{2}{(1-t+2xt^2)}.
\end{aligned}$$

This completes the proof. □

In the following theorem, we give the Binet form for the Vieta–Jacobsthal-like polynomials.

Theorem 2.2 (Binet's formula)**.** *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials, then*

$$W_n(x) = A\alpha^n(x) + B\beta^n(x), \tag{3}$$

where $A = \frac{2(1-\beta(x))}{\alpha(x)-\beta(x)}$, $B = \frac{2(\alpha(x)-1)}{\alpha(x)-\beta(x)}$ and $\alpha(x)$, $\beta(x)$ are the roots of the characteristic equation (2)*.*

Proof. Since the roots of the characteristic equation (2) are distinct, we get that

$$W_n(x) = d_1\alpha^n(x) + d_2\beta^n(x), \text{ for all } n \geq 0,$$

for some real numbers d_1 and d_2. Take $n = 0, n = 1$ and then by solving the system of linear equations, we obtain

$$W_n(x) = \frac{2(1-\beta(x))}{\alpha(x)-\beta(x)}\alpha^n(x) + \frac{2(\alpha(x)-1)}{\alpha(x)-\beta(x)}\beta^n(x).$$

Setting $A = \dfrac{2(1-\beta(x))}{\alpha(x)-\beta(x)}$ and $B = \dfrac{2(\alpha(x)-1)}{\alpha(x)-\beta(x)}$, we get

$$W_n(x) = A\alpha^n(x) + B_3\beta^n(x).$$

This completes the proof. □

We note that $A + B = 2$, $AB = \dfrac{8x}{(\alpha(x)-\beta(x))^2}$, and $A - 1 = \dfrac{1}{\alpha(x)-\beta(x)} = 1 - B$. The other explicit forms of this polynomial are given in Theorem 2.3 and Theorem 2.4

Theorem 2.3 (Explicit closed form)**.** *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$W_n(x) = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}(-2x)^i, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 2.1, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n(x)t^n &= \frac{2}{1-(t-2xt^2)} \\
&= 2\sum_{n=0}^{\infty}(t-2xt^2)^n \\
&= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}t^{n-i}(-2xt^2)^i \\
&= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-2x)^i t^{n+i} \\
&= \sum_{n=0}^{\infty}\left[2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-i}{i}(-2x)^i\right]t^n.
\end{aligned}$$

From the equality of both sides, the desired result is obtained. This complete the proof. □

Theorem 2.4 (Explicit closed form)**.** *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$W_n(x) = 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2i+1}(1-8x)^i, \quad \textit{for } n \geq 1.$$

Proof. Consider,

$$\begin{aligned}
\alpha^{n+1}(x) - \beta^{n+1}(x) &= 2^{-(n+1)}[(1+\sqrt{1-8x})^{n+1} - (1-\sqrt{1-8x})^{n+1}] \\
&= 2^{-(n+1)}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}(\sqrt{1-8x})^i - \sum_{i=0}^{n+1}\binom{n+1}{i}(-\sqrt{1-8x})^i\right] \\
&= 2^{-n}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n+1}{2i+1}(\sqrt{1-8x})^{2i+1}
\end{aligned}$$

Thus,

$$\begin{aligned}
W_n(x) &= A\alpha^n(x) + B\beta^n(x) \\
&= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)} \\
&= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\sqrt{1-8x}} \\
&= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n+1}{2i+1}(1-8x)^i.
\end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$\sum_{k=0}^{n-1} W_k(x) = \frac{2 - W_{n+1}(x)}{2x}.$$

Proof. By using Binet formula (3), we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} W_k(x) &= \sum_{k=0}^{n-1} \left(A\alpha^k(x) + B\beta^k(x)\right) \\
&= A\frac{1-\alpha^n(x)}{1-\alpha(x)} + B\frac{1-\beta^n(x)}{1-\beta(x)} \\
&= \frac{(A\alpha(x) + B\beta(x)) - (A\alpha^{n+1}(x) + B\beta^{n+1}(x))}{\alpha(x)\beta(x)} \\
&= \frac{W_1(x) - W_{n+1}(x)}{2x} \\
&= \frac{2 - W_{n+1}(x)}{2x}.
\end{aligned}$$

Thus

$$\sum_{k=0}^{n-1} W_k(x) = \frac{2 - W_{n+1}(x)}{2x}.$$

This completes the proof. □

Since the derivative of the polynomials are alway exists, we can give the following formula.

Theorem 2.6 (Differentiation formula)**.** *The derivative of $W_n(x)$ is obtained as the following.*

$$\frac{d}{dx}W_n(x) = \frac{-4(n+1)g_n(x) + 8G_{n+1}(x)}{1-8x}, \tag{4}$$

where $G_n(x)$ and $g_n(x)$ are the n-th Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials, respectively.

Proof. The result is obtained by using Binet formula (3). □

Again, by using Binet formula (3), we obtain some well-known identities as follows.

Theorem 2.7 (Catalan's identity)**.** *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta-Jacobsthak-like polynomials. Then*

$$W_n^2(x) - W_{n+r}(x)W_{n-r}(x) = (2x)^{n-r+1}W_{r-1}^2(x), \quad \textit{for } n \geq r \geq 1. \tag{5}$$

Proof. Consider,

$$\begin{aligned}
&W_n^2(x) - W_{n+r}(x)W_{n-r}(x) \\
&\quad = (A\alpha^n(x) + B\beta^n(x))^2 - \left(A\alpha^{n+r}(x) + B\beta^{n+r}(x)\right)\left(A\alpha^{n-r}(x) + B\beta^{n-r}(x)\right) \\
&\quad = -AB\left(\alpha(x)\beta(x)\right)^{n-r}\left(\alpha^r(x) - \beta^r(x)\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{8x}{(\alpha(x)-\beta(x))^2}(2x)^{n-r}\left(\alpha^r(x)-\beta^r(x)\right)^2 \\
&= (2x)^{n-r+1}\left(2\frac{\alpha^r(x)-\beta^r(x)}{\alpha(x)-\beta(x)}\right)^2 \\
&= (2x)^{n-r+1}\left(A\alpha^{r-1}(x)+B\beta^{r-1}(x)\right)^2 \\
&= (2x)^{n-r+1}W_{r-1}^2(x).
\end{aligned}
$$

This completes the proof. □

Take $r=1$ in Catalan's identity (5), we obtain Cassini's identity as the following corollary.

Corollary 2.1 (Cassini's identity). *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$
W_n^2(x) - W_{n+1}(x)W_{n-1}(x) = 4(2x)^n, \quad \textit{for } n \geq 1.
$$

Proof. Take $r=1$ in Catalan's identity (5), we obtian

$$
W_n^2(x) - W_{n+1}(x)W_{n-1}(x) = (2x)^n W_0^2(x) = (2x)^n 2^2 = 4(2x)^n.
$$

Thus

$$
W_n^2(x) - W_{n+1}(x)W_{n-1}(x) = 4(2x)^n.
$$

This completes the proof. □

Theorem 2.8 (d'Ocagne's identity). *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials. Then*

$$
W_m(x)W_{n+1}(x) - W_{m+1}(x)W_n(x) = 2(2x)^{n+1}W_{m-n-1}(x), \quad \textit{for } m > n \geq 1. \tag{6}
$$

Proof. We will prove d'Ocagne's identity (6) by using Binet formula (3).
Consider,

$$
\begin{aligned}
W_m(x)&W_{n+1}(x) - W_{m+1}(x)W_n(x) \\
&= \left(A\alpha^m(x)+B\beta^m(x)\right)\left(A\alpha^{n+1}(x)+B\beta^{n+1}(x)\right) \\
&\quad - \left(A\alpha^{m+1}(x)+B\beta^{m+1}(x)\right)\left(A\alpha^n(x)+B\beta^n(x)\right) \\
&= -AB\left(\alpha(x)\beta(x)\right)^n\left(\alpha(x)-\beta(x)\right)\left(\alpha^{m-n}(x)-\beta^{m-n}(x)\right) \\
&= \frac{4}{(\alpha(x)-\beta(x))^2}(2x)^{n+1}\left(\alpha(x)-\beta(x)\right)\left(\alpha^{m-n}(x)-\beta^{m-n}(x)\right) \\
&= 2(2x)^{n+1}\left(2\frac{\alpha^{m-n}(x)-\beta^{m-n}(x)}{\alpha(x)-\beta(x)}\right) \\
&= 2(2x)^{n+1}\left(A\alpha^{m-n-1}(x)+B\beta^{m-n-1}(x)\right) \\
&= 2(2x)^{n+1}W_{m-n-1}(x).
\end{aligned}
$$

This completes the proof. □

Next, by using Binet's formula, we derive the relation between the Vieta–Jacobsthal-like polynomials, Vieta–Jacobsthal polynomials, and Vieta–Jacobsthal–Lucas polynomials.

Theorem 2.9. *Let $\{W_n(x)\}_{n=0}^{\infty}$, $\{G_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Jacobsthal-like, Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials respectively. Then*

(1) $W_n(x) = 2G_{n+1}(x) = G_n(x) + g_n(x), \quad \text{for } n \geq 0,$

(2) $W_{n+1}(x) - 2xW_{n-1}(x) = 2g_{n+1}(x), \quad \text{for } n \geq 1,$

(3) $W_n(x) + 2g_{n+1}(x) = 4G_{n+2}(x), \quad \text{for } n \geq 0,$

(4) $(1-8x)W_n(x) + 2g_{n+1}(x) = 4g_{n+2}(x), \quad \text{for } n \geq 0,$

(5) $g_{n+1}(x) - 2xg_{n-1}(x) = \dfrac{1}{2}(1-8x)W_{n-1}(x), \quad \text{for } n \geq 1,$

(6) $W_n(x)g_n(x) - W_{2n}(x) = 2(2x)^n, \quad \text{for } n \geq 0,$

(7) $W_m(x)g_n(x) - (2x)^n W_{m-n}(x) = 2G_{(m+n)+1}(x), \quad \text{for } m \geq n \geq 0,$

(8) $4g_n^2(x) - (1-8x)W_{n-1}^2(x) = 16(2x)^n, \quad \text{for } n \geq 1,$

(9) $W_{n-1}(x)g_n(x) = 2G_{2n}(x), \quad \text{for } n \geq 1,$

(10) $G_m(x)g_n(x) + G_n(x)g_m(x) = W_{(m+n)-1}(x), \quad \text{for } m, n \geq 1,$

(11) $4g_m(x)g_n(x) + (1-8x)W_{m-1}(x)W_{n-1}(x) = 8g_{m+n}(x), \quad \text{for } m, n \geq 1,$

(12) $W_n(x)g_n(x) - g_n^2(x) = G_{2n}(x), \quad \text{for } n \geq 0,$

(13) $W_{n-1}(x)g_n(x) + W_n(x)g_{n-1}(x) = 4G_{2n}(x) + 2(2x)^{n-1}, \quad \text{for } n \geq 1.$

Proof. The results (1)–(13) are obtained by using Binet's formula (3). □

3 Matrix form of Vieta–Jacobsthal-like polynomials

In this section, we establish some identities of Vieta–Jacobsthal-like and Vieta–Jacobsthal polynomials by using elementary matrix methods. Let Q_w be 2×2 matrix defined by

$$Q_w = \begin{bmatrix} -4x+2 & -4x \\ 2 & -4x \end{bmatrix}. \tag{7}$$

Then by using this matrix, we can deduce some identities of Vieta–Jacobsthal-like and Vieta–Jacobsthal polynomials.

Theorem 3.1. *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials and let Q_w be the 2×2 matrix defined by (7). Then*

$$Q_w^n = 2^{n-1} \begin{bmatrix} W_{2n}(x) & -2xW_{2n-1}(x) \\ W_{2n-1}(x) & -2xW_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Proof. For the explanation, the Mathematical induction method is used. It easy to see that the statement is true for $n = 1$. Suppose that the result is true for any positive integer k, then we also have the result is true for $k + 1$. Because

$$\begin{aligned} Q_w^{k+1} &= Q_w^k \cdot Q_w \\ &= 2^{k-1} \begin{bmatrix} W_{2k}(x) & -2xW_{2k-1}(x) \\ W_{2k-1}(x) & -2xW_{2k-2}(x) \end{bmatrix} \cdot \begin{bmatrix} -4x+2 & -4x \\ 2 & -4x \end{bmatrix} \\ &= 2^{(k+1)-1} \begin{bmatrix} W_{2k+2}(x) & -2xW_{2k+1}(x) \\ W_{2k+1}(x) & -2xW_{2k}(x) \end{bmatrix}. \end{aligned}$$

By Mathematical induction, we have that the result is true for each $n \in \mathbb{N}$, that is

$$Q_w^n = 2^{n-1} \begin{bmatrix} W_{2n}(x) & -2xW_{2n-1}(x) \\ W_{2n-1}(x) & -2xW_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$ □

Theorem 3.2. *Let $\{W_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal-like polynomials and let Q_w be the 2×2 matrix defined by* (7)*. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $2W_{2(m+n)}(x) = W_{2m}(x)W_{2n}(x) - 2xW_{2m-1}(x)W_{2n-1}(x)$,

(2) $2W_{2(m+n)-1}(x) = W_{2m}(x)W_{2n-1}(x) - 2xW_{2m-1}(x)W_{2n-2}(x)$,

(3) $2W_{2(m+n)-1}(x) = W_{2m-1}(x)W_{2n}(x) - 2xW_{2m-2}(x)W_{2n-1}(x)$,

(4) $2W_{2(m+n)-2}(x) = W_{2m-1}(x)W_{2n-1}(x) - 2xW_{2m-2}(x)W_{2n-2}(x)$.

Proof. By Theorem 3.1 and the property of power matrix $Q_w^{m+n} = Q_w^m Q_w^n$, we obtain the result. □

By Theorem 3.1 and $W_n(x) = 2G_{n+1}(x)$, we get the following corollary.

Corollary 3.1. *Let $\{G_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Jacobsthal polynomials and let Q_w be the 2×2 matrix defined by* (7)*. Then*

$$Q_w^n = 2^n \begin{bmatrix} G_{2n+1}(x) & -2xG_{2n}(x) \\ G_{2n}(x) & -2xG_{2n-1}(x) \end{bmatrix}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 3.1, we get

$$Q_w^n = 2^{n-1} \begin{bmatrix} W_{2n}(x) & -2xW_{2n-1}(x) \\ W_{2n-1}(x) & -2xW_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Since $W_n(x) = 2G_{n+1}(x)$, we get that

$$\begin{aligned} Q_w^n &= 2^{n-1} \begin{bmatrix} 2G_{2n+1}(x) & -4xG_{2n}(x) \\ 2G_{2n}(x) & -4xG_{2n-1}(x) \end{bmatrix} \\ &= 2^n \begin{bmatrix} G_{2n+1}(x) & -2xG_{2n}(x) \\ G_{2n}(x) & -2xG_{2n-1}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \end{aligned}$$

This completes the proof. □

By Theorem 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get the following corollary.

Corollary 3.2. *Let* $\{G_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta–Jacobsthal polynomials. Then for all integers* $m \geq 1$, $n \geq 1$, *the following statements hold.*

(1) $G_{2(m+n)+1}(x) = G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x)$,

(2) $G_{2(m+n)}(x) = G_{2m+1}(x)G_{2n}(x) - 2xG_{2m}(x)G_{2n-1}(x)$,

(3) $G_{2(m+n)}(x) = G_{2m}(x)G_{2n+1}(x) - 2xG_{2m-1}(x)G_{2n}(x)$,

(4) $G_{2(m+n)-1}(x) = G_{2m}(x)G_{2n}(x) - 2xG_{2m-1}(x)G_{2n-1}(x)$.

Proof. From Theorem 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get that

$$\begin{aligned}
G_{2(m+n)+1}(x) &= \frac{1}{2}W_{2(m+n)}(x) \\
&= \frac{1}{4}\left(W_{2m}(x)W_{2n}(x) - 2xW_{2m-1}(x)W_{2n-1}(x)\right) \\
&= \frac{1}{4}\left(2G_{2m+1}(x)2G_{2n+1}(x) - 4xG_{2m}(x)2G_{2n}(x)\right) \\
&= G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x).
\end{aligned}$$

Thus, we get that (1) holds.
By the same argument as above, we get that (2), (3) and (4) hold. This completes the proof. □

By Corollary 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get the following corollary.

Corollary 3.3. *Let* $\{W_n(x)\}_{n=0}^{\infty}$ *and* $\{G_n(x)\}_{n=0}^{\infty}$ *be the sequences of Vieta–Jacobsthal-like and Vieta–Jacobsthal polynomials respectively. Then for all integers* $m \geq 1$, $n \geq 1$, *the following statements hold.*

(1) $W_{2(m+n)}(x) = 2\left(G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x)\right)$,

(2) $W_{2(m+n)-1}(x) = 2\left(G_{2m+1}(x)G_{2n}(x) - 2xG_{2m}(x)G_{2n-1}(x)\right)$,

(3) $W_{2(m+n)-1}(x) = 2\left(G_{2m}(x)G_{2n+1}(x) - 2xG_{2m-1}(x)G_{2n}(x)\right)$,

(4) $W_{2(m+n)-2}(x) = 2\left(G_{2m}(x)G_{2n}(x) - 2xG_{2m-1}(x)G_{2n-1}(x)\right)$.

Proof. From Corollary 3.2 and $W_n(x) = 2G_{n+1}(x)$, we get that

$$\begin{aligned}
W_{2(m+n)}(x) &= 2G_{2(m+n)+1}(x) \\
&= 2\left(G_{2m+1}(x)G_{2n+1}(x) - 2xG_{2m}(x)G_{2n}(x)\right).
\end{aligned}$$

Thus, we get that (1) holds.
By the same argument as above, we get that (2), (3) and (4) hold. This completes the proof. □

4 Conclusion

In this work, we defined the Vieta–Jacobsthal-like polynomial. Then we gave the generating function, the Binet formula, and well-known identities for this polynomial. Moreover, we also presented the relation between this polynomial and the previously famous Vieta polynomials.

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