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Character formulas in terms of R_β and R_m functions

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Abstract: The authors establish a set of fourteen character formulas in terms of R_β and R_m functions. Folsom [6] studied character formulas and Chaudhary [5] expressed those formulas in terms of continued fraction identities. Andrews *et al.* [2] introduced multivariate R-functions, which are further classified as R_α, R_β, and R_m (for $m = 1, 2, 3, \ldots$) functions by Srivastava *et al.* [10].
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1 Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. Let

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

For $a, q \in C$ with $|q| < 1$, let

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1}$$

and

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}, k \in N. \tag{2}$$

Ramanujan (see [8, 9]) defined the general theta function $\mathfrak{f}(a, b)$ as follows:

$$\mathfrak{f}(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty, |ab| < 1. \tag{3}$$

Ramanujan consider the following three special case and $f(a, b)$:

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left\{(-q;q^2)_\infty\right\}^2 (q^2;q^2)_\infty, \tag{4}$$

$$\psi(q) = \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5}$$

and

$$f(-q) = \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \tag{6}$$

He also defined

$$\chi(-q) = (q;q^2)_\infty. \tag{7}$$

The Rogers–Ramanujan identity are defined by

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q,-q^5)}, \tag{8}$$

and

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)}. \tag{9}$$

The well-known Rogers–Ramanujan continued fraction is defined by

$$R(q) = q^{\frac{1}{5}}\frac{H(q)}{G(q)} = \frac{q^{\frac{1}{5}}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots} \qquad , |q| < 1. \tag{10}$$

Theorem 1.1. *Suppose that* $|q| < 1$. *Then*

$$(q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-}\cdots, \tag{11}$$

and

$$C(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots \tag{12}$$

By introducing the general family $R(s,t,l,u,v,w)$, Andrews *et al.* [2] investigated a number of interesting double-summation hypergeometric q-series representations for several families of partitions and further explored the rôle of double series in combinatorial-partition identities:

$$R(s,t,l,u,v,w)=\sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn}\, r(l,u,v,w;n), \tag{13}$$

where

$$r(l,u,v,w:n)=\sum_{j=0}^{[\frac{n}{u}]}(-1)^j\ \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}\ (q^{uv};q^{uv})_j}. \tag{14}$$

We also recall the following interesting special cases of (13) (see, [2, p. 106, Theorem 3]):

$$R(2,1,1,1,2,2)=(-q;q^2)_\infty, \tag{15}$$

$$R(2,2,1,1,2,2)=(-q^2;q^2)_\infty \tag{16}$$

and

$$R(m,m,1,1,1,2)=\frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{17}$$

Recently, Srivastava *et al.* (see [10]) has introduced three notations:

$$R_\alpha=R(2,1,1,1,2,2);\, R_\beta=R(2,2,1,1,2,2);\, R_m=R(m,m,1,1,1,2);\, m\in\mathbb{N}. \tag{18}$$

for multivaraite R-functions, which we shall use for computation of our main results in Section 2.

Ever since the year 2015, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities.

Here, in this paper, our main objective is to establish a set of fourteen character formulas interms of R_β and R_m functions.

In order to established our main results, we used the identities studied by chaudhary (see, Theorem 3 from [5]).

$$\begin{aligned}f(-q)=&-4q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\big(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\,13)}+q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\,13)}\big)+4q\widehat{\beta}_{12,1}(\tau)\\&+\frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty\{(q^{16};q^{16})_\infty\}^2}\\&\cdot\left\{\frac{1}{1-}\,\frac{q^8}{1+}\,\frac{q^8(1-q^8)}{1-}\,\frac{q^{24}}{1+}\,\frac{q^{16}(1-q^{16})}{1-}\,\frac{q^{40}}{1+}\,\frac{q^{24}(1-q^{24})}{1-}\cdots\right\}^2\\&\cdot\left\{\frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-}\cdots\right\}^3\\&\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\};\end{aligned} \tag{19}$$

$$\phi(q) = -2q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\big(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\,13)}+q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\,13)}\big)+2q\widehat{\beta}_{12,1}(\tau)$$

$$+\frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty\{(q^{16};q^{16})_\infty\}^2}$$

$$\cdot\left\{\frac{1}{1-}\,\frac{q^8}{1+}\,\frac{q^8(1-q^8)}{1-}\,\frac{q^{24}}{1+}\,\frac{q^{16}(1-q^{16})}{1-}\,\frac{q^{40}}{1+}\,\frac{q^{24}(1-q^{24})}{1-}\cdots\right\}^2$$

$$\cdot\left\{\frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-}\cdots\right\}^3$$

$$\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}; \tag{20}$$

$$\chi(-q) = -q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\big(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\,13)}+q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\,13)}\big)$$

$$+q\widehat{\beta}_{12,1}(\tau)+\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{\{(q^{12};q^{12})_\infty\}^2\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}$$

$$\cdot\left\{\frac{1}{1-}\,\frac{q^3}{1+}\,\frac{q^3(1-q^3)}{1-}\,\frac{q^9}{1+}\,\frac{q^6(1-q^6)}{1-}\,\frac{q^{15}}{1+}\,\frac{q^9(1-q^9)}{1-}\cdots\right\}$$

$$\cdot\left\{\frac{1}{1-}\,\frac{q^{10}}{1+}\,\frac{q^{10}(1-q^{10})}{1-}\,\frac{q^{30}}{1+}\,\frac{q^{20}(1-q^{20})}{1-}\,\frac{q^{50}}{1+}\,\frac{q^{30}(1-q^{30})}{1-}\cdots\right\}^2$$

$$\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2\left\{1+\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2; \tag{21}$$

$$\upsilon(q) = -q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\big(\mathrm{tr}_{L(\bigwedge_{(-2)};\,13)}+\mathrm{tr}_{L(\bigwedge_{(2)};\,13)}\big)+q\widehat{\beta}_{12,-2}(\tau)$$

$$+\frac{(q^4;q^4)_\infty}{\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}$$

$$\cdot\left\{\frac{1}{1-}\,\frac{q^{10}}{1+}\,\frac{q^{10}(1-q^{10})}{1-}\,\frac{q^{30}}{1+}\,\frac{q^{20}(1-q^{20})}{1-}\,\frac{q^{50}}{1+}\,\frac{q^{30}(1-q^{30})}{1-}\cdots\right\}^2$$

$$\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2\cdot\left\{1+\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2; \tag{22}$$

$$\rho(q) = -\frac{1}{2}\cdot\widehat{\Theta}_{6}^{-1}\cdot q^{L_0}\big(\mathrm{tr}_{L(\bigwedge_{(-1)};\,7)}+\mathrm{tr}_{L(\bigwedge_{(1)};\,7)}\big)+\frac{1}{2}\widehat{\beta}_{6,-1}(\tau)+\frac{3}{2}\frac{1}{(q^2;q^2)_\infty}$$

$$\cdot\left\{\frac{1}{1-}\,\frac{q^3}{1+}\,\frac{q^3(1-q^3)}{1-}\,\frac{q^9}{1+}\,\frac{q^6(1-q^6)}{1-}\,\frac{q^{15}}{1+}\,\frac{q^9(1-q^9)}{1-}\cdots\right\}^2; \tag{23}$$

$$\sigma(-q) = q^2\cdot\widehat{\Theta}_{36}^{-1}\cdot q^{L_0}\big(q^{\frac{3}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(3)};\,37)}+q^{-\frac{3}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(15)};\,37)}\big)-q^2\widehat{\beta}_{36,3}(\tau)$$

$$+\frac{\{(q^2,q^{10};q^{12})_\infty\}^2(q^6;q^{12})_\infty}{(q;q^2)_\infty}$$

$$\cdot\left\{\frac{1}{1-}\,\frac{q^6}{1+}\,\frac{q^6(1-q^6)}{1-}\,\frac{q^{18}}{1+}\,\frac{q^{12}(1-q^{12})}{1-}\,\frac{q^{30}}{1+}\,\frac{q^{18}(1-q^{18})}{1-}\cdots\right\}; \tag{24}$$

$$A(q^2) = q\cdot\widehat{\Theta}_8^{-1}\mathrm{tr}_{L(\bigwedge_{(2)};\,9)}q^{L_0} - q\widehat{\eta}_{8,2}(\tau) - q(-q^2;q^2)_\infty(-q^4;q^4)_\infty$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q^4}{1+}\ \frac{q^4(1-q^4)}{1-}\ \frac{q^{12}}{1+}\ \frac{q^8(1-q^8)}{1-}\ \frac{q^{20}}{1+}\ \frac{q^{12}(1-q^{12})}{1-}\cdots\right\}; \tag{25}$$

$$\mu(q^4) = -\,2q\cdot\widehat{\Theta}_4^{-1}\mathrm{tr}_{L(\bigwedge_{(0)};\,5)}q^{L_0} + 2q\widehat{\eta}_{4,0}(\tau) + \frac{12\,(q^8;q^8)_\infty}{(q;q)_\infty(q^2;q^4)_\infty}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-}\cdots\right\}; \tag{26}$$

$$\phi(q^4) = -\,2q\cdot\widehat{\Theta}_{12}^{-1}\mathrm{tr}_{L(\bigwedge_{(4)};\,13)}q^{L_0} + 2q\widehat{\eta}_{12,4}(\tau) + \frac{(q^2,q^4,q^6;q^8)_\infty(q^{12};q^{24})_\infty\{(q^3;q^6)_\infty\}^2}{(q;q)_\infty}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-}\cdots\right\}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q^6}{1+}\ \frac{q^6(1-q^6)}{1-}\ \frac{q^{18}}{1+}\ \frac{q^{12}(1-q^{12})}{1-}\ \frac{q^{30}}{1+}\ \frac{q^{18}(1-q^{18})}{1-}\cdots\right\}; \tag{27}$$

$$\psi(q^4) = -\,q^3\cdot\widehat{\Theta}_{12}^{-1}\mathrm{tr}_{L(\bigwedge_{(0)};\,13)}q^{L_0} + q^3\widehat{\eta}_{12,0}(\tau) + \frac{(q^4,q^{12},q^{20},q^{24};q^{24})_\infty q^3}{(q^8,q^{16};q^{24})_\infty(q^3;q^3)_\infty}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-}\cdots\right\}; \tag{28}$$

$$\phi(q) = 2q\cdot\widehat{\Theta}_{10}^{-1}\mathrm{tr}_{L(\bigwedge_{(1)};\,11)}q^{L_0} - 2q\widehat{\eta}_{10,1}(\tau) + \frac{j(-q^2;q^5)}{j(q^2;q^{10})}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q^5}{1+}\ \frac{q^5(1-q^5)}{1-}\ \frac{q^{15}}{1+}\ \frac{q^{10}(1-q^{10})}{1-}\ \frac{q^{25}}{1+}\ \frac{q^{15}(1-q^{15})}{1-}\cdots\right\}; \tag{29}$$

$$\psi(q) = 2q\cdot\widehat{\Theta}_{10}^{-1}\mathrm{tr}_{L(\bigwedge_{(3)};\,11)}q^{L_0} - 2q\widehat{\eta}_{10,3}(\tau) - q\frac{j(-q;q^5)}{j(q^4;q^{10})}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q^5}{1+}\ \frac{q^5(1-q^5)}{1-}\ \frac{q^{15}}{1+}\ \frac{q^{10}(1-q^{10})}{1-}\ \frac{q^{25}}{1+}\ \frac{q^{15}(1-q^{15})}{1-}\cdots\right\}; \tag{30}$$

$$X(-q^2) = -2q\cdot\widehat{\Theta}_{40}^{-1}\cdot q^{L_0}\big(\mathrm{tr}_{L(\bigwedge_{(18)};\,41)} - \mathrm{tr}_{L(\bigwedge_{(2)};\,41)}\big) + 2q\widehat{\eta}_{40,18}(\tau)$$

$$-2q\widehat{\eta}_{40,2}(\tau) + \frac{(j(-q^2,q^{20})j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3)}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^8,q^{40})}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q^2}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^6}{1+}\ \frac{q^4(1-q^4)}{1-}\ \frac{q^{10}}{1+}\ \frac{q^6(1-q^6)}{1-}\cdots\right\}; \tag{31}$$

$$\chi(-q^2) = -2q^3\cdot\widehat{\Theta}_{40}^{-1}\cdot q^{L_0}\big(\mathrm{tr}_{L(\bigwedge_{(14)};\,41)} + q^2\cdot\mathrm{tr}_{L(\bigwedge_{(6)};\,41)}\big) + 2q^3\widehat{\eta}_{40,14}(\tau)$$

$$+2q^5\widehat{\eta}_{40,6}(\tau) + q^2\frac{(2q(q^{40};q^{40})_\infty^3 - j(-q^6,q^{20})^2 j(q^4,q^{40}))}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^{16},q^{40})}$$

$$\cdot\left\{\frac{1}{1-}\ \frac{q^2}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^6}{1+}\ \frac{q^4(1-q^4)}{1-}\ \frac{q^{10}}{1+}\ \frac{q^6(1-q^6)}{1-}\cdots\right\}. \tag{32}$$

2 Main results

In this section, we prove a set of fourteen identities which depict inter-relationships among q-product identities, continued fraction identities, character formulas, R_β, and R_m functions.

Theorem 3. *Each of the following relationships holds true:*

$$f(-q) = -4q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\left(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\,13)}+q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\,13)}\right)+4q\widehat{\beta}_{12,1}(\tau)$$
$$+\frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty\{(q^{16};q^{16})_\infty\}^2}$$
$$\cdot[R_8]^2\,[R_1]^3\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}; \tag{33}$$

$$\phi(q) = -2q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\left(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\,13)}+q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\,13)}\right)+2q\widehat{\beta}_{12,1}(\tau)$$
$$+\frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty\{(q^{16};q^{16})_\infty\}^2}$$
$$\cdot[R_8)]^2\,[R_1]^3\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}; \tag{34}$$

$$\chi(-q) = q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\left(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\,13)}+q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\,13)}\right)+q\widehat{\beta}_{12,1}(\tau)$$
$$+\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{\{(q^{12};q^{12})_\infty\}^2\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}\cdot[R_3]\,[R_{10}]^2$$
$$\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2\cdot\left\{1+\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2; \tag{35}$$

$$\upsilon(q) = -q\cdot\widehat{\Theta}_{12}^{-1}\cdot q^{L_0}\left(\mathrm{tr}_{L(\bigwedge_{(-2)};\,13)}+\mathrm{tr}_{L(\bigwedge_{(2)};\,13)}\right)+q\widehat{\beta}_{12,-2}(\tau)+\frac{(q^4;q^4)_\infty[R_{10}]^2}{\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}$$
$$\cdot\left\{\frac{1}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2\cdot\left\{1+\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\,\frac{q^{16}}{1+}\,\frac{q^{20}}{1+}\,\frac{q^{24}}{1+}\cdots\right\}^2; \tag{36}$$

$$\rho(q) = -\frac{1}{2}\cdot\widehat{\Theta}_{6}^{-1}\cdot q^{L_0}\left(\mathrm{tr}_{L(\bigwedge_{(-1)};\,7)}+\mathrm{tr}_{L(\bigwedge_{(1)};\,7)}\right)+\frac{1}{2}\widehat{\beta}_{6,-1}(\tau)+\frac{3}{2\,(q^2;q^2)_\infty}\left[R_3\right]^2; \tag{37}$$

$$\sigma(-q) = q^2\cdot\widehat{\Theta}_{36}^{-1}\cdot q^{L_0}\left(q^{\frac{3}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(3)};\,37)}+q^{-\frac{3}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(15)};\,37)}\right)$$
$$-q^2\widehat{\beta}_{36,3}(\tau)+\frac{(q^2,q^{10};q^{12})_\infty^2(q^6;q^{12})_\infty}{(q;q^2)_\infty}\,R_6; \tag{38}$$

$$A(q^2) = q\cdot\widehat{\Theta}_{8}^{-1}\cdot\mathrm{tr}_{L(\bigwedge_{(2)};9)}q^{L_0}-q\cdot\widehat{\eta}_{8,2}(\tau)-q(-q^4;q^4)_\infty\,R_4\,R_\beta; \tag{39}$$

$$\mu(q^4) = -2q\cdot\widehat{\Theta}_{4}^{-1}\cdot\mathrm{tr}_{L(\bigwedge_{(0)};\,5)}q^{L_0}+2q\,\widehat{\eta}_{4,0}(\tau)+\frac{12\,(q^8;q^8)_\infty R_1}{(q;q)_\infty(q^2;q^4)_\infty}; \tag{40}$$

$$\phi(q^4) = -2q\cdot\widehat{\Theta}_{12}^{-1}\cdot\mathrm{r}_{L(\bigwedge_{(4)};\,13)}q^{L_0}+2q\,\widehat{\eta}_{12,4}(\tau)$$
$$+\frac{(q^2,q^4,q^6;q^8)_\infty(q^{12};q^{24})_\infty(q^3;q^6)_\infty^2}{(q;q)_\infty}R_1\,R_6; \tag{41}$$

$$\psi(q^4) = -\, q^3 \cdot \widehat{\Theta}_{12}^{-1} \cdot \mathrm{tr}_{L(\bigwedge_{(0)};\,13)}\, q^{L_0} + q^3\, \widehat{\eta}_{12,0}(\tau) + \frac{(q^4, q^{12}, q^{20}, q^{24}; q^{24})_\infty}{(q^8, q^{16}; q^{24})_\infty} \frac{q^3}{(q^3; q^3)_\infty} R_1; \quad (42)$$

$$\phi(q) = 2q \cdot \widehat{\Theta}_{10}^{-1} \cdot \mathrm{tr}_{L(\bigwedge_{(1)};\,11)} q^{L_0} - 2q\, \widehat{\eta}_{10,1}(\tau) + \frac{j(-q^2; q^5)}{j(q^2; q^{10})} R_5; \quad (43)$$

$$\psi(q) = 2q \cdot \widehat{\Theta}_{10}^{-1} \cdot \mathrm{tr}_{L(\bigwedge_{(3)};\,11)} q^{L_0} - 2q\, \widehat{\eta}_{10,3}(\tau) - q\frac{j(-q; q^5)}{j(q^4; q^{10})} R_5; \quad (44)$$

$$X(-q^2) = -2q \cdot \widehat{\Theta}_{40}^{-1} \cdot q^{L_0} \big(\mathrm{tr}_{L(\bigwedge_{(18)};\,41)} - \mathrm{tr}_{L(\bigwedge_{(2)};\,41)}\big) + 2q\, \widehat{\eta}_{40,18}(\tau)$$
$$-2q\, \widehat{\eta}_{40,2}(\tau) + \frac{(j(-q^2, q^{20}) j(q^{12}, q^{40}) + 2q(q^{40}; q^{40})_\infty^3)}{(q^{20}; q^{20})_\infty (q^{40}; q^{40})_\infty j(q^8, q^{40})} R_2; \quad (45)$$

$$\chi(-q^2) = -2q^3 \cdot \widehat{\Theta}_{40}^{-1} \cdot q^{L_0} \big(\mathrm{tr}_{L(\bigwedge_{(14)};\,41)} + q^2 \cdot \mathrm{tr}_{L(\bigwedge_{(6)};\,41)}\big) + 2q^3 \widehat{\eta}_{40,14}(\tau)$$
$$+2q^5 \widehat{\eta}_{40,6}(\tau) + q^2 \frac{(2q(q^{40}; q^{40})_\infty^3 - j(-q^6, q^{20})^2 j(q^4, q^{40}))}{(q^{20}; q^{20})_\infty (q^{40}; q^{40})_\infty j(q^{16}, q^{40})} R_2. \quad (46)$$

Proof. We prove our identities with the help of elemintary and existing results. First of all we prove our first identity (33), using identities (11) and (18) (for $m = 1$) into (19), after little algebra, we obtain our identity (33). Secondly, we attempt to prove identity (34), applying (11) and (18) (for $m = 8$) into (20), after simplification, we obtain identity (34). Further, applying the identity (11) along with (18) (for $m = 3, 10$) into (21), we get the identity (35); and applying the identity (11) along with (18) (for $m = 10$) into (22) proves the identity (36). Again, using the identity (11) along with (18) (for $m = 3$) into the identity (24) yields the identity (37). Similarly, by using the identity (18) (for $m = 6, 4, 1$) along with the identity (11), in the three corresponding identities (24), (25) and (26), respectively, we obtain our desired assertions (38), (39) and (40).

We can prove identities (41) to (56) easily, by using similar argument as in proofs of the above identities (33) to (40). □

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