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Incomplete generalized Vieta–Pell and Vieta–Pell–Lucas polynomials

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Abstract: In this paper, we introduce the incomplete Vieta–Pell and Vieta–Pell–Lucas polynomials. We give some properties, the recurrence relations and the generating function of these polynomials with suggestions for further research.
Keywords: Binet's formula, Generating function, Incomplete generalized Vieta–Pell polynomials, Incomplete generalized Vieta–Pell–Lucas polynomials.
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1 Introduction

Fibonacci sequences are one of the most well-known sequences of numbers. These number sequences and their generalizations have found application in many fields (see, e.g., [2, 5, 6, 7, 8, 14]). Many researchers have presented many features by defining the polynomials of these generalizations [4, 9, 10, 18]. There is still scope for generalizations of the dimensions of the associated continued fraction algorithms [17].

One of these expansions was introduced by Filipponi [3]. Ramírez gave the incomplete k-Fibonacci and k-Lucas numbers and their polynomials [12, 13]. Powers of such numbers also offer scope for extensions and development of related identities [16]. Catarino et al. gave the incomplete k-Pell, k-Pell–Lucas and Modified k-Pell numbers [1]. Uygun et al. defined generalized Vieta–Pell and Vieta–Pell–Lucas polynomial sequences [20].

Definition 1.1. *The generalized Vieta–Pell polynomials $P_{k,n}(x)$ are defined by*

$$P_{k,n}(x) = 2^k x P_{k,n-1}(x) - P_{k,n-2}(x)$$

with $P_{k,0}(x) = 0$ and $P_{k,1}(x) = 1$ [20].

Definition 1.2. *The generalized Vieta–Pell–Lucas polynomials $Q_{k,n}(x)$ are given by*

$$Q_{k,n}(x) = 2^k x Q_{k,n-1}(x) - Q_{k,n-2}(x)$$

with $Q_{k,0}(x) = 2$ and $Q_{k,1}(x) = 2^k x$ [20].

The characteristic equation of these polynomial sequences is

$$r^2 - 2^k x r + 1 = 0.$$

The roots of this equation are

$$\alpha(x) = \frac{2^k x + \sqrt{2^{2k}x^2 - 4}}{2}, \beta(x) = \frac{2^k x - \sqrt{2^{2k}x^2 - 4}}{2}.$$

The Binet formulas for these sequences are

$$P_{k,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}$$

and

$$Q_{k,n}(x) = \alpha^n(x) + \beta^n(x). \text{ [10]}$$

These polynomials are given in an explicit closed form by using the generating function respectively [20]:

$$P_{k,n}(x) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-j}{j} (-1)^j (2^k x)^{n-2j-1}$$
$$Q_{k,n}(x) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-j} \binom{n-1-j}{j} (-1)^j (2^k x)^{n-2j} \tag{1.1}$$

In this work, we describe the incomplete generalized Vieta–Pell and incomplete generalized Vieta–Pell–Lucas polynomials and give new relations, identities and the generating function of these polynomials.

Lemma 1.1 (see [11], p. 592). *For $n > 1$, let $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ be a complex sequence with*

$$s_n = a s_{n-1} + b s_{n-2} + r_n$$

where a, b are complex numbers. Then the generating function $U(t)$ for $\{s_n\}$ is

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2},$$

where $G(t)$ is the generating function for $\{r_n\}$.

2 The incomplete generalized Vieta–Pell polynomials

Definition 2.1. *The incomplete generalized Vieta–Pell polynomials are described by*

$$P_{k,n}^{l}(x) = \sum_{j=0}^{l} \binom{n-1-j}{j} (-1)^j (2^k x)^{n-2j-1}, 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor \tag{2.1}$$

In Table 1, we give sample values for the incomplete generalized Vieta–Pell polynomials.

n/l	0	1	2	3
1	1			
2	$2^k x$			
3	$2^{2k}x^2$	$2^{2k}x^2 - 1$		
4	$2^{3k}x^3$	$2^{3k}x^3 - 2(2^k)x$		
5	$2^{4k}x^4$	$2^{4k}x^4 - 3(2^{2k})x^2$	$2^{4k}x^4 - 3(2^{2k})x^2 + 1$	
6	$2^{5k}x^5$	$2^{5k}x^5 - 4(2^{3k})x^3$	$2^{5k}x^5 - 4(2^{3k})x^3 + 3(2^k)x$	
7	$2^{6k}x^6$	$2^{6k}x^6 - 5(2^{4k})x^4$	$2^{6k}x^6 - 5(2^{4k})x^4 + 6(2^{2k})x^2$	$2^{6k}x^6 - 5(2^{4k})x^4 + 6(2^{2k})x^2 - 1$

Table 1. Sample values for $P_{k,n}^{l}(x)$

When $k = 1$, $P_{1,n}^{\left\lfloor \frac{n-1}{2} \right\rfloor}(x) = P_n$, we get the Vieta–Pell polynomials [19].
From (2.1), we can write the following results:

$$P_{k,n}^{0}(x) = (2^k x)^{n-1}, n \ge 1$$

$$P_{k,n}^{1}(x) = (2^k x)^{n-1} - (n-2)(2^k x)^{n-3}, n \ge 3$$

$$P_{k,n}^{2}(x) = (2^k x)^{n-1} - (n-2)(2^k x)^{n-3} + \frac{(n-4)(n-3)}{2}(2^k x)^{n-5}, n \ge 5$$

$$P_{k,n}^{\left\lfloor \frac{n-1}{2} \right\rfloor}(x) = P_{k,n}(x), n \ge 1$$

$$P_{k,n}^{\left\lfloor \frac{n-3}{2} \right\rfloor}(x) = \begin{cases} P_{k,n}(x) + \frac{n2^k x}{2}, & \text{if } n \ge 3 \text{ and } n \text{ is even} \\ P_{k,n}(x) + 1, & \text{if } n \ge 3 \text{ and } n \text{ is odd} \end{cases}$$

Theorem 2.1. *We have the recurrence relation for* $P_{k,n}^{l}(x)$

$$P_{k,n+2}^{l+1}(x) = (2^k x) P_{k,n+1}^{l+1}(x) - P_{k,n}^{l}(x), 0 \le l \le \left\lfloor \frac{n-2}{2} \right\rfloor \tag{2.2}$$

From (2.1), we can obtain the relation

$$P_{k,n+2}^{l}(x) = (2^k x) P_{k,n+1}^{l}(x) - P_{k,n}^{l}(x) - \binom{n-1-l}{l}(2^k x)^{n-1-2l} \tag{2.3}$$

Proof. From Definition 2.1, we have

$$(2^k x)P_{k,n+1}^{l+1}(x) - P_{k,n}^{l}(x)$$

$$= (2^k x)\sum_{j=0}^{l+1}\binom{n-j}{j}(-1)^j(2^k x)^{n-2j} - \sum_{j=0}^{l}\binom{n-1-j}{j}(-1)^j(2^k x)^{n-2j-1}$$

$$= \sum_{j=0}^{l+1}\binom{n-j}{j}(-1)^j(2^k x)^{n-2j+1} - \sum_{j=1}^{l+1}\binom{n-j}{j-1}(-1)^{j-1}(2^k x)^{n-2j+1}$$

$$= \sum_{j=0}^{l+1}\binom{n-j}{j}(-1)^j(2^k x)^{n-2j+1} + \sum_{j=1}^{l+1}\binom{n-j}{j-1}(-1)^{j}(2^k x)^{n-2j+1}$$

$$= (-1)^j(2^k x)^{n-2j+1}\left(\sum_{j=0}^{l+1}\binom{n-j}{j} + \binom{n-j}{j-1}\right) - \binom{n}{-1}(2^k x)^{n+1}$$

$$= \sum_{j=0}^{l+1}\binom{n-j+1}{j}(-1)^j(2^k x)^{n-2j+1} - 0 = P_{k,n+2}^{l+1}(x)$$

This completes the proof. □

Theorem 2.2. *We have the following equality*

$$\sum_{j=0}^{s}\binom{s}{j}P_{k,n+j}^{l+j}(x)\,(2^k x)^j(-1)^{j+1} = P_{k,n+2s}^{l+s}(x)(-1)^{s+1}, \qquad 0 \le l \le \frac{n-s-1}{2} \tag{2.4}$$

Proof. We proved by induction on s. For $s = 0$ and $s = 1$, the result clearly holds. Now assume that the result is true for $i \le s$. We prove it for $s + 1$:

$$\sum_{j=0}^{s+1}\binom{s+1}{j}P_{k,n+j}^{l+j}(x)(2^k x)^j(-1)^{j+1}$$

$$= \sum_{j=0}^{s+1}\left[\binom{s}{j} + \binom{s}{j-1}\right]P_{k,n+j}^{l+j}(x)(2^k x)^j(-1)^{j+1}$$

$$= \sum_{j=0}^{s+1}\binom{s}{j}P_{k,n+j}^{l+j}(x))(2^k x)^j(-1)^{j+1} + \sum_{j=0}^{s+1}\binom{s}{j-1}P_{k,n+j}^{l+j}(x)(2^k x)^j(-1)^{j+1}$$

$$= (-1)^{s+1}P_{k,n+2s}^{l+s}(x) + \binom{s}{s+1}P_{k,n+s+1}^{l+s+1}(x)(2^k x)^{s+1}(-1)^{s+2}$$

$$+ \sum_{j=-1}^{s}\binom{s}{j}P_{k,n+j+1}^{l+j+1}(x)(2^k x)^{j+1}(-1)^{j+2}$$

$$= (-1)^{s+1}P_{k,n+2s}^{l+s}(x) + 0 + \sum_{j=0}^{s}\binom{s}{j}P_{k,n+j+1}^{l+j+1}(x)(2^k x)^{j+1}(-1)^{j+2} + \binom{s}{-1}P_{k,n}^{l}(x)(2^k x)^0(-1)^1$$

$$= (-1)^{s+1}P_{k,n+2s}^{l+s}(x) - (2^k x)\sum_{j=0}^{s}\binom{s}{j}P_{k,n+j+1}^{l+j+1}(x)(2^k x)^j(-1)^{j+1} + 0$$

$$= (-1)^{s+1}P_{k,n+2s}^{l+s}(x) - (2^k x)P_{k,n+2s+1}^{l+s+1}(x)(-1)^{s+1}$$

$$= (-1)^{s+2}\left((2^k x)P_{k,n+2s+1}^{l+s+1}(x) - P_{k,n+2s}^{l+s}(x)\right) = (-1)^{s+2}P_{k,n+2s+2}^{l+s+1}(x).$$ □

Theorem 2.3. *For* $n \geq 2l+2$, *we have*

$$\sum_{j=0}^{s-1} P_{k,n+j}^{l}(x)(2^k x)^{s-1-j} = -P_{k,n+s+1}^{l+1}(x) + (2^k x)^s P_{k,n+1}^{l+1}(x).$$

Proof. We prove the assertion by induction on s. For $s=1$, it is easily seen that the result is true. Assume that the assertion is true for $i<s$. We prove it for s:

$$\begin{aligned}\sum_{j=0}^{s} P_{k,n+j}^{l}(x)(2^k x)^{s-j} &= (2^k x)\sum_{j=0}^{s-1} P_{k,n+j}^{l}(x)(2^k x)^{s-j-1} + P_{k,n+s}^{l}(x)\\ &= (2^k x)\left(-P_{k,n+s+1}^{l+1}(x) + (2^k x)^s P_{k,n+1}^{l+1}(x)\right) + P_{k,n+s}^{l}(x)\\ &= -(2^k x)P_{k,n+s+1}^{l+1}(x) + P_{k,n+s}^{l}(x) + (2^k x)^{s+1} P_{k,n+1}^{l+1}(x)\\ &= -P_{n+s+2}^{l+1}(x) + (2^k x)^{s+1} P_{k,n+1}^{l+1}(x).\end{aligned}$$

□

Theorem 2.4. *We have the following equality*

$$P'_{k,n}(x) = (2^k)\left(\frac{nQ_{k,n}(x) - (2^k x)P_{k,n}(x)}{(2^k x)^2 - 4}\right).$$

Proof. From Binet's formula, we obtain

$$P'_{k,n}(x) = \frac{\left(n\alpha^{n-1}(x)\alpha'(x) - n\beta^{n-1}(x)\beta'(x)\right)(\alpha(x)-\beta(x)) - (\alpha'(x)-\beta'(x))(\alpha^n(x)-\beta^n(x))}{(\alpha(x)-\beta(x))^2}$$

where $\alpha(x) = \frac{2^k x + \sqrt{2^{2k}x^2 - 4}}{2}$ and $\beta(x) = \frac{2^k x - \sqrt{2^{2k}x^2 - 4}}{2}$.

Taking the derivatives of $\alpha(x)$ and $\beta(x)$, we have

$$\alpha'(x) = \frac{2^k \alpha(x)}{\alpha(x)-\beta(x)} \text{ and } \beta'(x) = \frac{-2^k \beta(x)}{\alpha(x)-\beta(x)}.$$

If we put away $\alpha'(x)$ and $\beta'(x)$ in the equation above:

$$P'_{k,n}(x) = \frac{\left(n\alpha^{n-1}(x)\frac{2^k\alpha(x)}{\alpha(x)-\beta(x)} + n\beta^{n-1}(x)\frac{2^k\beta(x)}{\alpha(x)-\beta(x)}\right)(\alpha(x)-\beta(x))}{(\alpha(x)-\beta(x))^2} - \frac{\left(\frac{2^k\alpha(x)}{\alpha(x)-\beta(x)} + \frac{2^k\beta(x)}{\alpha(x)-\beta(x)}\right)(\alpha^n(x)-\beta^n(x))}{(\alpha(x)-\beta(x))^2}$$

$$\begin{aligned}P'_{k,n}(x) &= \frac{2^k n(\alpha^n(x)+\beta^n(x))}{(\alpha(x)-\beta(x))^2} - \frac{2^k(\alpha(x)+\beta(x))(\alpha^n(x)-\beta^n(x))}{(\alpha(x)-\beta(x))^3}\\ &= \frac{2^k n Q_{k,n}(x)}{(2^k x)^2 - 4} - \frac{2^k 2^k x P_{k,n}(x)}{(2^k x)^2 - 4}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.5. *We have the relation*

$$\sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} j\binom{n-1-j}{j}(2^k x)^{n-1-2j}(-1)^j = \frac{\left(\left(\left(2^k x\right)^2 - 4\right)n + 4\right)P_{k,n}(x) - n(2^k x)Q_{k,n}(x)}{2\left((2^k x)^2 - 4\right)}. \quad (2.5)$$

Proof. From equation (1.1), we get

$$(2^k x)P_{k,n}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} (2^k x)^{n-2j}(-1)^j.$$

If the derivative of the above equation is taken, we obtain

$$2^k P_{k,n}(x) + 2^k x P'_{k,n}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2j) \binom{n-1-j}{j} (-1)^j (2^k x)^{n-2j-1} 2^k$$

$$= nP_{k,n}(x)2^k - 2 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} j \binom{n-1-j}{j} (-1)^j (2^k x)^{n-2j-1} 2^k$$

From Theorem 2.4, we get

$$2^k P_{k,n}(x) + 2^k x \left(\frac{2^k n Q_{k,n}(x) - 2^{2k} x P_{k,n}(x)}{2^{2k} x^2 - 4} \right)$$

$$= nP_{k,n}(x)2^k - 2 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} j \binom{n-1-j}{j} (-1)^j (2^k x)^{n-2j-1} 2^k$$

$$\frac{P_{k,n}(x) + x \left(\frac{2^k n Q_{k,n}(x) - 2^{2k} x P_{k,n}(x)}{2^{2k} x^2 - 4} \right) - nP_{k,n}(x)}{-2}$$

$$= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} j \binom{n-1-j}{j} (2^k x)^{n-1-2j} (-1)^j$$

$$= \frac{(2^{2k} x^2 - 4)P_{k,n}(x) + x 2^k n Q_{k,n}(x) - 2^{2k} x^2 P_{k,n}(x) - n(2^{2k} x^2 - 4)P_{k,n}(x)}{-2(2^{2k} x^2 - 4)}$$

$$= \frac{P_{k,n}(x)(2^{2k} x^2 - 4 - 2^{2k} x^2 - n 2^{2k} x^2 + 4n) + Q_{k,n}(x)(x 2^k n)}{-2(2^{2k} x^2 - 4)}.$$

Thus, if some algebraic operations are done, the equation (2.5) is obtained. □

Theorem 2.6. *We get the following equality*

$$\sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} P_{k,n}^l(x) = \begin{cases} \dfrac{-4P_{k,n}(x) + n(2^k x)Q_{k,n}(x)}{2(2^{2k} x^2 - 4)}, & \text{if } n \text{ is even;} \\ \dfrac{(2^{2k} x^2 - 8)P_{k,n}(x) + n 2^k x Q_{k,n}(x)}{2(2^{2k} x^2 - 4)}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof.

$$\sum_{l=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} P_{k,n}^{l}(x) = \binom{n-1-0}{0}(-1)^{0}(2^{k}x)^{n-0-1}$$
$$+\left[\binom{n-1-0}{0}(-1)^{0}(2^{k}x)^{n-0-1}+\binom{n-1-1}{1}(-1)^{1}(2^{k}x)^{n-2-1}\right]+\cdots$$
$$+\left[\binom{n-1-0}{0}(-1)^{0}(2^{k}x)^{n-0-1}+\binom{n-1-1}{1}(-1)^{1}(2^{k}x)^{n-2-1}+\cdots\right.$$
$$\left.+\binom{n-1-\left\lfloor\frac{n-1}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2^{k}x)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1}\right]$$
$$=\left(\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)\binom{n-1-0}{0}(-1)^{0}(2^{k}x)^{n-1}+\left\lfloor\frac{n-1}{2}\right\rfloor\binom{n-1-1}{1}(-1)^{1}(2^{k}x)^{n-3}+\cdots$$
$$+\binom{n-1-\left\lfloor\frac{n-1}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2^{k}x)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1}$$
$$=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(\left\lfloor\frac{n-1}{2}\right\rfloor+1-i\right)\binom{n-1-i}{i}(2^{k}x)^{n-1-2i}(-1)^{i}$$
$$=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)\binom{n-1-i}{i}(2^{k}x)^{n-1-2i}(-1)^{i}-\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} i\binom{n-1-i}{i}(2^{k}x)^{n-1-2i}(-1)^{i}$$
$$=\left(\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)P_{k,n}(x)-\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} i\binom{n-1-i}{i}(2^{k}x)^{n-1-2i}(-1)^{i}.$$

From Theorem 2.6, we obtain

$$=\left(\left\lfloor\frac{n-1}{2}\right\rfloor+1\right)P_{k,n}(x)-\left(\frac{\left(((2^{k}x)^{2}-4)n+4\right)P_{k,n}(x)-n2^{k}xQ_{k,n}(x)}{2((2^{k}x)^{2}-4)}\right).$$

.

If we use Theorem 2.5, the proof is completed. □

Theorem 2.7. *The generating function for* $P_{k,n}^{l}(x)$ *is given by*

$$B_{k,l}(x)=\sum_{i=0}^{\infty}P_{k.i}^{l}(x)t^{i}$$
$$=t^{2l+1}\left[P_{k,2l+1}(x)+\left(P_{k,2l+2}(x)-(2^{k}x)P_{k,2l+1}(x)\right)t-\frac{t^{2}}{\left(1-(2^{k}x)t\right)^{l+1}}\right]\left(1-(2^{k}x)t+t^{2}\right)^{-1}.$$

Proof. Suppose that l is a fixed positive integer. From (2.1) and (2.3), for $0 \le n < 2l+1$, we get

$$P_{k,n}^{l}(x) = 0$$

$$P_{k,2l+1}^{l}(x) = P_{k,2l+1}(x)$$

$$P_{k,2l+2}^{l}(x) = P_{k,2l+2}(x)$$

and

$$P_{k,n}^{l} = (2^k x)P_{k,n-1}^{l}(x) - P_{k,n-2}^{l}(x) - \binom{n-3-l}{l}(2^k x)^{n-3-2l}.$$

If we show $s_o = P_{k,2l+1}^{l}(x)$ and $s_1 = P_{k,2l+2}^{l}(x)$ then we have $s_n = P_{k,n+2l+1}^{l}(x)$.
Similarly, if $r_0 = r_1 = 0$ then we have $r_n = \binom{n+l-1}{n-2}(2^k x)^{n-2}$.
The generating function for $\{r_n\}$ is

$$G(t) = \frac{t^2}{(1-(2^k x)t)^{l+1}}$$

(see [15, p. 355]).
Thus, from Lemma 1.3, we obtain the generating function $B_{k,l}(x)$ for $\{s_n\}$. □

3 The incomplete generalized Vieta–Pell–Lucas Polynomials

Definition 3.1. *The incomplete generalized Vieta–Pell Lucas polynomials are given by*

$$Q_{k,n}^{l}(x) = \sum_{j=0}^{l} \frac{n}{n-j}\binom{n-j}{j}(-1)^j (2^k x)^{n-2j},\ 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor. \quad (3.1)$$

In Table 2, we give sample values for the incomplete generalized Vieta–Pell–Lucas polynomials. For $x = 1$, $Q_{1,n}^{\lfloor \frac{n}{2} \rfloor}(x) = Q_n$. ($Q_n$: n. Vieta–Pell–Lucas number), we get an incomplete Vieta–Pell Lucas number.

n/l	0	1	2	3
1	$2^k x$			
2	$2^{2k}x^2$	$2^{2k}x^2 - 2$		
3	$2^{3k}x^3$	$2^{3k}x^3 - 3(2^k)x$		
4	$2^{4k}x^4$	$2^{4k}x^4 - 4(2^{2k})x^2$	$2^{4k}x^4 - 4(2^{2k})x^2 + 2$	
5	$2^{5k}x^5$	$2^{5k}x^5 - 5(2^{3k})x^3$	$2^{5k}x^5 - 5(2^{3k})x^3 + 5(2^k)x$	
6	$2^{6k}x^6$	$2^{6k}x^6 - 6(2^{4k})x^4$	$2^{6k}x^6 - 6(2^{4k})x^4 + 9(2^{2k})x^2$	$2^{6k}x^6 - 6(2^{4k})x^4 + 9(2^{2k})x^2 - 2$
7	$2^{7k}x^7$	$2^{7k}x^7 - 7(2^{5k})x^5$	$2^{7k}x^7 - 7(2^{5k})x^5 + 14(2^{3k})x^3$	$2^{7k}x^7 - 7(2^{5k})x^5 + 14(2^{3k})x^3 - 7(2^k)x$

Table 2. Sample values for $Q_{k,n}^{l}(x)$

From (3.1), we can write the following results:

$$Q_{k,n}^{0}(x) = (2^k x)^n \,, (n \geq 1)$$

$$Q_{k,n}^{1}(x) = (2^k x)^n - n(2^k x)^{n-2} \,, (n \geq 2)$$

$$Q_{k,n}^{2}(x) = (2^k x)^n - n(2^k x)^{n-2} + \tfrac{n(n-3)}{2}(2^k x)^{n-4} \,, (n \geq 4)$$

$$Q_{k,n}^{\left\lfloor\frac{n}{2}\right\rfloor}(x) = Q_{k,n}(x) \,, (n \geq 1)$$

$$Q_{k,n}^{\left\lfloor\frac{n-2}{2}\right\rfloor}(x) = \begin{cases} Q_{k,n}(x) - 2, & \text{if n} \geq 2 \text{ and even;} \\ Q_{k,n}(x) - n(2^k x), & \text{if n} \geq 2 \text{ and odd.} \end{cases}$$

Theorem 3.1. *We have the recurrence relation*

$$Q_{k,n}^{l}(x) = -P_{k,n-1}^{l-1}(x) + P_{k,n+1}^{l}(x); \;\; 0 \leq l \leq \left\lfloor\frac{n}{2}\right\rfloor \tag{3.2}$$

Proof. Applying Definition 2.1 to the right-hand side (RHS) of Eq. (3.2), we obtain

$$\begin{aligned}
\text{(RHS)} &= -\sum_{j=0}^{l-1}\binom{n-2-j}{j}(2^k x)^{n-2-2j}(-1)^j + \sum_{j=0}^{l-1}\binom{n-j}{j}(-1)^j(2^k x)^{n-2j} \\
&= -\sum_{j=1}^{l-1}\binom{n-1-j}{j-1}(2^k x)^{n-2j}(-1)^{j-1} + \sum_{j=0}^{l}\binom{n-j}{j}(-1)^j(2^k x)^{n-2j} \\
&= \sum_{j=1}^{l}\binom{n-1-j}{j-1}(2^k x)^{n-2j}(-1)^j + \sum_{j=0}^{l}\binom{n-j}{j}(-1)^j(2^k x)^{n-2j} \\
&= \sum_{j=0}^{l}\left(\binom{n-1-j}{j-1} + \binom{n-j}{j}\right)(2^k x)^{n-2j}(-1)^j - \binom{n-1}{-1} \\
&= \sum_{j=0}^{j}\frac{n}{n-j}\binom{n-j}{j}(2^k x)^{n-2j}(-1)^j = Q_{k,n}^{l}(x).
\end{aligned}$$

□

Theorem 3.2. *We have the recurrence relation for* $Q_{k,n}^{l}(x)$

$$Q_{k,n+2}^{l+1}(x) = (2^k x)Q_{k,n+1}^{l+1}(x) - Q_{k,n}^{l}(x), \;\; 0 \leq l \leq \left\lfloor\frac{n}{2}\right\rfloor. \tag{3.3}$$

Proof. From (3.3), we can obtain the recurrence relation

$$Q_{k,n+2}^{l+1}(x) = (2^k x)Q_{k,n+1}^{l+1}(x) - Q_{k,n}^{l}(x) + \tfrac{n}{n-l}\tbinom{n-l}{l}(2^k x)^{n-2l}. \tag{3.4}$$

The proof is clear from (3.2) and (2.2). □

Theorem 3.3. *We have the following relation*

$$-(2^k x)Q_{k,n}^{l}(x) = P_{k,n+2}^{l}(x) + P_{k,n-2}^{l-2}(x), \; 0 \leq l \leq \left\lfloor\frac{n}{2}\right\rfloor.$$

Proof. By (3.2), we obtain

$$P_{k,n+2}^{l}(x) = -Q_{k,n+1}^{l}(x) - P_{k,n}^{l-1}(x)$$
$$P_{k,n-2}^{l-2}(x) = P_{k,n}^{l-1}(x) - Q_{k,n-1}^{l-1}(x).$$

If we add the equations, we have

$$P_{k,n+2}^{l}(x) + P_{k,n-2}^{l-2}(x) = -Q_{k,n+1}^{l}(x) - Q_{k,n-1}^{l-1}(x).$$

From (3.3), we get

$$P_{k,n+2}^{l}(x) + P_{k,n-2}^{l-2}(x) = -(2^k x) Q_{k,n}^{l}(x). \qquad \square$$

Theorem 3.4. *We have the following equality*

$$\sum_{i=0}^{s} \binom{s}{i} (2^k x)^i Q_{k,n+i}^{l+i}(x)\,(-1)^{i+1} = Q_{k,n+2s}^{l+s}(x), 0 \le l \le \frac{n-s}{2}.$$

Proof. By using (3.2) and (2.4), we get

$$\sum_{i=0}^{s} \binom{s}{i} (2^k x)^i Q_{k,n+i}^{l+i}(x)\,(-1)^{i+1} = \sum_{i=0}^{s} \binom{s}{i} \left[P_{k,n+1+i}^{l+i}(x) - P_{k,n-1+i}^{l-1+i}(x)\right](2^k x)^i\,(-1)^{i+1}$$

$$= \sum_{i=0}^{s} \binom{s}{i} \left[P_{k,n+1+i}^{l+i}(x)\right](2^k x)^i\,(-1)^{i+1} - \sum_{i=0}^{s} \binom{s}{i} \left[P_{k,n-1+i}^{l-1+i}(x)\right](2^k x)^i\,(-1)^{i+1}$$

$$P_{k,n+2s+1}^{l+s}(x) - P_{k,n+2s-1}^{l+s-1}(x) = Q_{k,n+2s}^{l+s}(x). \qquad \square$$

Theorem 3.5. *For* $n \ge 2l+1$, *we have*

$$\sum_{i=0}^{s-1} Q_{k,n+i}^{l}(x)\,(2^k x)^{s-1-i}(-1)^{i+1} = Q_{k,n+s+1}^{l+1}(x) - (2^k x)^s Q_{k,n+1}^{l+1}(x).$$

Proof. By using equation (3.3) and induction on s, we get proof of the theorem. $\square$

Theorem 3.6. *The generating function for* $Q_{k,n}^{l}(x)$ *is given by*

$$A_{k,l}(x) = \sum_{i=0}^{\infty} Q_{k.i}^{l}(x) t^i$$
$$= t^{2l}\left[Q_{k,2l}(x) + \left(Q_{k,2l+1}(x) - (2^k x) Q_{k,2l}(x)\right) t - \frac{t^2(2-t)}{(1-(2^k x)t)^{l+1}}\right]\left(1 - (2^k x)t + t^2\right)^{-1}.$$

Proof. Let l be a fixed positive integer. From (3.1) and (3.4), for $0 \le n < 2l$, we get

$$Q_{k,n}^{l}(x) = 0,$$
$$Q_{k,2l}^{l}(x) = Q_{k,2l}(x),$$
$$Q_{k,2l+1}^{l}(x) = Q_{k,2l+1}(x),$$

and

$$Q_{k,n}^{l}(x) = (2^k x) Q_{k,n-1}^{l}(x) - Q_{k,n-2}^{l}(x) + \frac{n-2}{n-2-l}\binom{n-2-l}{n-2-2l}(2^k x)^{n-2-2l}.$$

If we show $s_o = Q_{k,2l}^{l}(x), s_1 = Q_{k,2l+1}^{l}(x)$, then we have $s_n = Q_{k,n+2l}^{l}(x)$.

Similarly, if $r_0 = r_1 = 0$, then we get

$$r_n = \binom{n+2l-2}{n+l-2} (2^k x)^{n+2l-2}.$$

The generating function for $\{r_n\}$ is

$$G(t) = \frac{t^2(2-t)}{(1-(2^k x)t)^{l+1}}.$$

(see [10, p. 355]).

Thus, from Lemma 1.1, we obtain the generating function $A_{k,l}(x)$ for $\{s_n\}$. □

Finally, there are still further bridges to be built among the properties of these inter-related sequences of numbers and their associated polynomials [4, 18].

4 Conclusion

In this work, we gave definitions of the incomplete Vieta–Pell and Vieta–Pell-Lucas polynomials. Then we introduced some properties, the recurrence relations and the generating function of these polynomials with suggestions for further research.

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