Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 4, 236–244
DOI: 10.7546/nntdm.2021.27.4.236-244

On Fibonacci quaternion matrix

Serpil Halici[1] and Ömür Deveci[2]

[1] Department of Mathematics, Faculty of Science and Arts
Pamukkale University, Denizli, Turkey
e-mail: shalici@pau.edu.tr

[2] Department of Mathematics, Faculty of Science and Letters
Kafkas University, Kars, Turkey
e-mail: odeveci36@hotmail.com

Received: 15 May 2021 **Revised:** 13 November 2021 **Accepted:** 18 November 2021

Abstract: In this study, we have defined Fibonacci quaternion matrix and investigated its powers. We have also derived some important and useful identities such as Cassini's identity using this new matrix.
Keywords: Quaternions, Recurrence relations, Matrices.
2020 Mathematics Subject Classification: 11B37, 11B39, 11R52.

1 Introduction

Horadam first described complex quaternions in 1963 [7]. Halici defined and studied Fibonacci quaternions in [6]. The author gave some identities including the Fibonacci quaternions. In [1], the authors derived some equalities involving terms of the Horadam sequence using a special matrix. In [2], the author studied Binet-like formulas for the generalized Fibonacci and Lucas numbers by using some matrix methods. In [3], the author investigated quaternion group algebra and representations of the quaternions. Shannon and Deveci investigated some variations on Fibonacci matrix graphs and quaternions sequences [4, 11]. In 2004, Mc Laughlin derived a new formula for the powers of any 2×2 matrix A. Here, the author studied the relationship of $y_n = Ty_{n-1} - Dy_{n-2}$ linear recurrence relation involving matrices that the values D and T are the determinant and trace of matrix A, respectively [9]. For $n \geq 0$, the author derived the n-th power of matrix A as

$$A^n = y_n A_0 + y_{n-1} A_1, \;\; A_0 = I_2, \;\; A_1 = A - TI_2, \tag{1}$$

$$A^n = \begin{pmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_{n-1} & y_n - ay_{n-1} \end{pmatrix}, \tag{2}$$

where $y_{-1} = 0, \;\; y_0 = 1, \;\; y_1 = T$. The author used this formula together with an existing formula for the n-th power of any 2-dimensional matrix, various matrix identities, formulae for the powers of special matrices, etc, to derive various combinatorial identities. In [12], Williams obtained the powers of the matrix A as

$$A^n = \alpha^n \frac{(A - \beta I)}{\alpha - \beta} - \beta^n \frac{(A - \alpha I)}{\alpha - \beta}, \;\; \alpha \neq \beta \tag{3}$$

and

$$A^n = \alpha^{n-1} \left(nA - (n-1)\alpha I \right), \;\; \alpha = \beta \tag{4}$$

where α and β are the eigenvalues of the matrix A. Melham and Shannon gave some summation identities using generalized matrices [10].

In this study, we both gave the powers of the Fibonacci quaternion matrix and examined some of its properties. We also obtained some fundamental identities involving Fibonacci quaternions using this new matrix.

Fibonacci numbers are the sequence of numbers $\{F_n\}_{\{n \geq 0\}}$ defined by the linear recurrence relation

$$F_0 = 0, \;\; F_1 = 1, \;\; F_{n+2} = F_{n+1} + F_n. \tag{5}$$

The Binet formula for the F_n is given by $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation [8]. It should be noted that other than the use of the Binet formula and generating function to produce Fibonacci numbers, matrices are also widely used. The n-th Fibonacci quaternion Q_n [6] is

$$Q_n = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k. \tag{6}$$

Using this type quaternions we define the following matrix that has not been studied before.

$$\mathbf{Q} = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}. \tag{7}$$

The remainder of the paper is organized as follows. In Section 2, we begin by the powers of the Fibonacci quaternion matrix **Q**. We examined and compared some matrix methods to calculate the positive powers of this matrix and then, we gave some fundamental properties of the matrix **Q**. In Section 3, we briefly explained what has been done.

2 The Fibonacci matrix

We first calculated the powers of the Fibonacci quaternion matrix in the following theorem.

Theorem 2.1. *For $n \geq 1$, we have*

$$\mathbf{Q}^n = \begin{pmatrix} y_n - y_{n-1}Q_0 & y_{n-1}Q_1 \\ y_{n-1}Q_1 & y_n - y_{n-1}Q_2 \end{pmatrix} \tag{8}$$

where

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i. \tag{9}$$

Proof. From induction method, for $n = 1$

$$\mathbf{Q}^1 = \begin{pmatrix} y_1 - y_0Q_0 & y_0Q_1 \\ y_0Q_1 & y_1 - y_0Q_2 \end{pmatrix} = \begin{pmatrix} T - Q_0 & Q_1 \\ Q_1 & T - Q_2 \end{pmatrix} = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix} \tag{10}$$

is obtained. Also, for $n = 2$,

$$\mathbf{Q}^2 = \begin{pmatrix} y_2 - y_1Q_0 & y_1Q_1 \\ y_1Q_1 & y_2 - y_1Q_2 \end{pmatrix}. \tag{11}$$

On the other hand, let us calculate the square of the matrix $\mathbf{Q}$. Here when paying attention to matrix multiplication, we considered both the quaternion product rules and the order of the product. Also, we made the indices of the products from small to large. Then,

$$\mathbf{Q}^2 = \begin{pmatrix} Q_1^2 + Q_2^2 & Q_1Q_2 + Q_0Q_1 \\ Q_1Q_2 + Q_0Q_1 & Q_0^2 + Q_1^2 \end{pmatrix}. \tag{12}$$

For the elements $A(2,2)$ of the matrices (11) and (12) by using the recurrence relation $y_n = Ty_{n-1} - Dy_{n-2}$ we get

$$y_2 - y_1Q_2 = (T^2 - D) - TQ_2 = -19 + 2i + 4j + 6k \tag{13}$$

and

$$Q_0^2 + Q_1^2 = (i + j + 2k)^2 + (1 + i + 2j + 3k)^2 = -19 + 2i + 4j + 6k, \tag{14}$$

respectively. Where

$$T = Q_0 + Q_2 = 1 + 3i + 4j + 7k, \quad D = Q_0Q_2 - Q_1^2 = -2 - 2i - 4j - 3k.$$

Now, Assume that for $n = k$, the claim is true. Then

$$\mathbf{Q}^k = \begin{pmatrix} y_k - y_{k-1}Q_0 & y_{k-1}Q_1 \\ y_{k-1}Q_1 & y_k - y_{k-1}Q_2 \end{pmatrix} \tag{15}$$

and so for $n = k + 1$

$$\mathbf{Q}^{k+1} = \begin{pmatrix} y_k - y_{k-1}Q_0 & y_{k-1}Q_1 \\ y_{k-1}Q_1 & y_k - y_{k-1}Q_2 \end{pmatrix} \begin{pmatrix} y_1 - y_0Q_0 & y_0Q_1 \\ y_0Q_1 & y_1 - y_0Q_2 \end{pmatrix} \tag{16}$$

is written. Thus, we have

$$\mathbf{Q}^{k+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} A &= (y_k - y_{k-1}Q_0)(y_1 - Q_0) + y_{k-1}Q_1^2, \\ B &= (y_k - y_{k-1}Q_0)Q_1 + y_{k-1}Q_1(y_1 - Q_2), \\ C &= y_{k-1}Q_1(y_1 - Q_0) + Q_1(y_k - y_{k-1}Q_2), \\ D &= Q_1y_{k-1}Q_1 + (y_k - y_{k-1}Q_2)(y_1 - y_0Q_2). \end{aligned}$$

If we pay attention to element A of the matrix (17), then we have

$$A = y_kQ_2 - y_{k-1}(Q_0Q_2 - Q_1^2) = y_kQ_2 - y_{k-1}D. \tag{18}$$

That is, the $a_{11} = a$ element of the matrix

$$\mathbf{Q}^k\mathbf{Q} = \mathbf{Q}^{k+1} = \begin{pmatrix} y_k - y_{k-1}Q_0 & y_{k-1}Q_1 \\ y_{k-1}Q_1 & y_k - y_{k-1}Q_2 \end{pmatrix} \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}$$

is

$$a = (y_k - y_{k-1}Q_0)Q_2 + y_{k-1}Q_1^2 = y_kQ_2 - y_{k-1}D, \tag{19}$$

where $y_k = Ty_{k-1} - Dy_{k-2},\ y_1 = T,\ y_2 = T^2 - D.$
Thus, the claim is proven. □

For the y_n recurrence relation containing T and D values, algebraic operations are given in the following corollary.

Corollary 2.1. *For two different integers m and n, the following equalities are satisfied:*

$$y_n + y_m = T(y_{n-1} + y_{m-1}) - D(y_{n-2} + y_{m-2}). \tag{20}$$

and

$$y_ny_m = T^2(y_{n-1}y_{m-1}) + D^2(y_{n-2}y_{m-2}) - TD(y_{n-2}y_{m-1} + y_{n-1}y_{m-2}). \tag{21}$$

Proof. The proof of the above equations can be easily seen by direct calculations. For example,

$$\begin{aligned} y_2y_3 &= T^5 - 3T^3D + 2TD^2, \\ y_2 + y_3 &= T^3 + T^2 - 2TD - D. \end{aligned}$$ □

It is well known that Cassini's identity is a special case of Catalan's identity and gives information about the n-th terms of sequences. In the following theorem, we give Cassini's identity for the elements $y_n, n \geq 1$.

Theorem 2.2. *For $n \geq 1$, we have*

$$y_n^2 - y_{n-1}y_{n+1} = (-1)^n(2Q_1 - 3k)^n. \tag{22}$$

Proof. If we use the inductive method and write $n = 1, 2$ then we have

$$\det(\mathbf{Q}) = y_1^2 - y_0y_2 = T^2 - (T^2 - D) = D = -(2Q_1 - 3k) \tag{23}$$

and

$$\det(\mathbf{Q}^2) = y_2^2 - Ty_2y_1 + Dy_1y_1 = D^2 = (2Q_1 - 3k)^2, \tag{24}$$

respectively. And for $n = k$, we assume the claim is true. Then

$$\det(\mathbf{Q}^k) = y_k^2 - y_{k-1}y_{k+1} = (-1)^k(2Q_1 - 3k)^k. \tag{25}$$

Thus, we obtain

$$\det(\mathbf{Q}^{k+1}) = \det(\mathbf{Q}^k)\det(\mathbf{Q}) = D^kD = (-1)^{k+1}(2Q_1 - 3k)^{k+1}. \tag{26}$$

So, the proof is completed. □

We can also give the powers of the **Q** matrix and some equalities by examining the eigenvalues of this matrix. The eigenvalues of **Q** differ from each other and they are as below.

$$\lambda_1 = \frac{T + \sqrt{T^2 - 4D}}{2} \tag{27}$$

and

$$\lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2}. \tag{28}$$

Where $D = Q_0Q_2 - Q_1^2$ and $T = Q_0 + Q_2$ are the determinant and trace of the **Q** matrix, respectively. We would like to point out here that since the quaternion product is not abelian, we multiplied by writing quaternions with small indices first during multiplication throughout the study.

Corollary 2.2. *For $n \geq 1$, the following equality is true:*

$$\mathbf{Q}^n = Q_n\mathbf{Q} - Q_{n-1}DI \tag{29}$$

where $Q_n = \dfrac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$.

Proof. Since the eigenvalues of the $\mathbf{Q}^n$ matrix are λ_1^n and λ_2^n, the trace of the matrix $\mathbf{Q}^n$

$$Tr(\mathbf{Q}^n) = (T^2 - 2D)y_{n-2} - TDy_{n-3}. \tag{30}$$

The powers of matrix **Q** with different eigenvalues are

$$\mathbf{Q}^n = \lambda_1^n\frac{\mathbf{Q} - \lambda_2 I}{\lambda_1 - \lambda_2} + \lambda_2^n\frac{\mathbf{Q} - \lambda_1 I}{\lambda_2 - \lambda_1}. \tag{31}$$

Detailed information on properties of the elements of matrix $\mathbf{Q}$ can be found in [5,6]. If necessary actions are taken for the second side of equation (31) and by using the equalities

$$\lambda_1 + \lambda_2 = T, \ \ \lambda_1\lambda_2 = D$$

and

$$\lambda_1 - \lambda_2 = \sqrt{T^2 - 4D},$$

the following equality is obtained:

$$\mathbf{Q}^n = Q_n\mathbf{Q} - Q_{n-1}DI.$$

Where Q_n in this formula is calculated as

$$Q_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} T^{n-2k-1}(T^2 - 4D)^k. \tag{32}$$

Thus, the proof is completed. □

According to the last result, let us just say that, the calculation of $\mathbf{Q}^n$ depends on the calculation of the Q_n and the determinant of the matrix $\mathbf{Q}$. We can take the Pell matrix as an example. When the n-th power of the well-known Pell matrix P is calculated, the $(1,2)$ element of the matrix found gives the n-th Pell number. And for the Binet formula of Pell sequence, typing $T = 2, \ D = -1$ in the formula (32), the following combinatorial equality is found:

$$P_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 2^{n-2k-1} 8^k. \tag{33}$$

Indeed, for example if we take $n = 4$, then this formula gives the value $P_4 = 12$.

Now, using the result in Corollary 2.2 we give the following Theorem.

Theorem 2.3. *For $n \geq 1$, the following equality is satisfied:*

$$\mathbf{Q}^n = y_{n-1}\mathbf{Q} - y_{n-2}DI. \tag{34}$$

Proof. To see the accuracy of the claim, we first prove that the equation below is correct:

$$Q_n = y_{n-1}. \tag{35}$$

For $n = 1, 2$

$$Q_1 = 1 = y_0, \ \ Q_2 = \lambda_1 + \lambda_2 = T = y_1$$

is obtained, respectively. Also, for $n = 3$, since $\lambda_1 + \lambda_2 = T, \ \lambda_1\lambda_2 = D$

$$Q_3 = \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 = T^2 - D$$

is obtained. Now, assume that this claim is true for n: $Q_n = y_{n-1}$. For $n + 1$, we write

$$Q_{n+1} = \lambda_1^n + \lambda_1^{n-1}\lambda_2 + \lambda_1^{n-2}\lambda_2^2 + \cdots + \lambda_1\lambda_2^{n-1} + \lambda_2^n = Ty_{k-1} - Dy_{k-2} = y_k. \tag{36}$$

Thus, we have

$$Q_n = y_{n-1}.$$

From the equality $\mathbf{Q}^n = Q_n\mathbf{Q} - Q_{n-1}DI$, we get

$$\mathbf{Q}^n = y_{n-1}\mathbf{Q} - y_{n-2}DI.$$

Thus, the correctness of the desired equality is satisfied. Finally, let us check this last equality for $n = 1$ and $n = 2$, respectively. Then

$$\mathbf{Q}^1 = y_0\mathbf{Q} - y_{-1}DI = \mathbf{Q}$$

and

$$\mathbf{Q}^2 = y_1\mathbf{Q} - y_0DI = \begin{pmatrix} TQ_2 - D & TQ_1 \\ TQ_1 & TQ_0 - D \end{pmatrix}$$

is written. If we write matrix elements explicitly then, we get

$$\mathbf{Q}^2 = \begin{pmatrix} TQ_2 - D & TQ_1 \\ TQ_1 & TQ_0 - D \end{pmatrix} = \begin{pmatrix} Q_1^2 + Q_2^2 & Q_1Q_2 + Q_0Q_1 \\ Q_1Q_2 + Q_0Q_1 & Q_0^2 + Q_1^2 \end{pmatrix},$$

where

$$y_1 = T,\ \ y_2 = T^2 - D,\ \ y_n = Ty_{n-1} - Dy_{n-2},\ \ T = Q_0 + Q_2,\ \ D = Q_0Q_2 - Q_1^2.$$

In the multiplication here and throughout the entire article, we just sorted the indices from least to greatest and did the multiplication. □

Now, with the help of this above Theorem, we will give the following Theorem without proof.

Theorem 2.4. *For $n \geq 1$, we have*

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (2Q_0 + Q_1)^{n-2i-1}(5Q_1^2)^i = 2^{n-1} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} (\lambda_1 + \lambda_2)^{n-2i-1}(-\lambda_1\lambda_2)^i \quad (37)$$

where λ_1, λ_2 are the eigenvalues of the matrix $\mathbf{Q}$.

Theorem 2.5. *For the integers $n \geq 1$, we have*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} T^{n-2k}(-D)^k = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} T^{n-2k}(T^2 - 4D)^k, \quad (38)$$

where T and D are the trace and determinant of the matrix $\mathbf{Q}$, respectively.

Proof. Using the equation $\lambda_1^n + \lambda_2^n = 2y_n - Ty_{n-1}$, we get the following identity:

$$2y_n - Ty_{n-1} = 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} T^{n-2k}(-D)^k - T\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n-2k}{n-k} T^{n-2k}(-D)^k, \quad (39)$$

where

$$\binom{n-1-k}{k} = \binom{n-k}{k}\frac{n-2k}{n-k}.$$

If we make the necessary calculations, then we get

$$2y_n - Ty_{n-1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}\frac{n}{n-k}T^{n-2k}(-D)^k, \tag{40}$$

where

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i.$$

Also, we get

$$\lambda_1^n + \lambda_2^n = \frac{1}{2^n}\left(\sum_{k=0}^{n}\binom{n}{k}T^{n-k}(\sqrt{T^2-4D})^k + \sum_{k=0}^{n}\binom{n}{k}T^{n-k}(-\sqrt{T^2-4D})^k\right) \tag{41}$$

and

$$\lambda_1^n + \lambda_2^n = \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}T^{n-2k}(T^2-4D)^k. \tag{42}$$

Thus, with the help of equalities (40) and (42), we have the desired equation. □

For example, when $n = 3$, if calculations are made for the right and left sides of equality (42) then

$$T^3 - 3TD = T^3 - 3TD$$

is obtained.

3 Conclusion and suggestions

Using the equalities and properties given in this work, various and different identities including quaternions can be found more quickly and easily. This method is advantageous to use in future studies due to the use of matrices.

Acknowledgements

This study was supported by Scientific Research Coordination unit of Pamukkale University under the project number 2020KRM005-109.

References

[1] Akyuz, Z., & Halici, S. (2012). Some identities deriving from the *nth* power of a special matrix. *Advances in Difference Equations*, 1, 223.

[2] Cerda-Morales, G. (2013). On generalized Fibonacci and Lucas numbers by matrix methods. *Hacettepe Journal of Mathematics and Statistics*, 42(2), 173–179.

[3] Chen, Y. (2019). Quaternion group algebra and representations of the quaternions. *Communications in Algebra*, 47(2), 749–760.

[4] Deveci, Ö., & Shannon, A. G. (2018). The quaternion-Pell sequence. *Communications in Algebra*, 46(12), 5403–5409.

[5] Flaut, C., & Savin, D. (2019). Some remarks regarding l-elements defined in algebras obtained by the Cayley–Dickson process. *Chaos, Solitons & Fractals*, 118, 112–116.

[6] Halici, S. (2012). On Fibonacci Quaternions. *Advances in Applied Clifford Algebras*, 22(2), 321–327.

[7] Horadam, A. F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. *The American Mathematical Monthly*, 70(3), 289–291.

[8] Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications.* Wiley-Interscience Publications.

[9] Mc Laughlin, J. (2004). Combinatorial identities deriving from the nth power of a 2×2 matrix. *Integers*, 4, 1–15.

[10] Melham, R. S.,& Shannon, A. G. (1995). Some summation identities using generalized Q-matrices. *Fibonacci Quarterly*. 33(1), 64-73.

[11] Shannon, A. G., & Deveci, Ö. (2017). Some variations on Fibonacci matrix graphs. *Notes on Number Theory and Discrete Mathematics*, 23(4), 85–93.

[12] Williams, K. S. (1992). The nth power of a 2×2 matrix. *Mathematics Magazine*, 65(5), 336.