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Binomial formulas via divisors of numbers

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Abstract: The purpose of this note is to prove several binomial-like formulas whose exponents are values of the function $\omega(n)$ counting distinct prime factors of n.
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1 Introduction

Throughout the article, let $n \geq 2$ be an integer with canonical factorization

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

where p_i's are prime numbers and a_i's are positive integers. We define the function $\omega(n)$ (including $n = 1$ as an argument) counting the number of distinct prime factors [1], that is,

$$\omega(n) := \begin{cases} k, & n = \prod_{i=1}^{k} p_i^{a_i}, \\ 0, & n = 1. \end{cases} \tag{1}$$

In the recent paper of Vassilev-Missana [3] the following fact is provided.

Theorem 1.1. *If n is a square-free number, then*

$$(1+x)^{\omega(n)} = \sum_{d|n} x^{\omega(d)}. \tag{2}$$

In particular, after substitution $x \to \frac{b}{a}$ the equation (2) leads to the binomial-like expansion

$$(a+b)^{\omega(n)} = \sum_{d|n} a^{\omega(n)-\omega(d)} b^{\omega(d)}. \tag{3}$$

In the paper we provide several generalizations of formulas (2) and (3). We prove some results for the sum of more than two terms case and also some results for non-square-free numbers.

2 Multinomial theorem for square-free number

In this section, we generalize formula (3) to the power of more than two terms. First, for a given integer $n \geq 1$ and any integer $m \geq 1$ we define the set

$$\mathrm{Div}(n,m) = \{(d_0, d_1, \ldots, d_{m-1}, d_m) \in \mathbb{N}^{m+1} \ : \ d_0 = n,\ d_1|d_0,\ \ldots,\ d_{m-1}|d_{m-2},\ d_m = 1\}.$$

Theorem 2.1. *Suppose n is a square-free number. Then*

$$(x_1 + \cdots + x_m)^{\omega(n)} = \sum_{\mathrm{Div}(n,m)} \prod_{i=1}^{m} x_i^{\omega(d_{i-1})-\omega(d_i)}. \tag{4}$$

Note that (3) is a special case of (4) for $m = 2$.

Proof. The proof goes by induction on m. First, we recall the proof for the case $m = 2$ adapted to our notation.

For arbitrary integer $n \geq 1$ and real x, set $f(n) = x^{\omega(n)}$. Then f is multiplicative and so

$$F(n) = \sum_{d|n} f(d)$$

is multiplicative. Now suppose n is a square-free number, that is $n = \prod_{i=1}^{\omega(n)} p_i$. Then

$$F(n) = \prod_{i=1}^{\omega(n)} F(p_i) = \prod_{i=1}^{\omega(n)} (f(1) + f(p_i)) = \prod_{i=1}^{\omega(n)} (1+x) = (1+x)^{\omega(n)}. \tag{5}$$

On the other hand,

$$F(n) = \sum_{d|n} f(d) = \sum_{d|n} x^{\omega(d)}. \tag{6}$$

Setting $x = \frac{x_2}{x_1}$ yields

$$\left(1 + \frac{x_2}{x_1}\right)^{\omega(n)} = \sum_{d|n} x_2^{\omega(d)} x_1^{-\omega(d)}.$$

Multiplying by $x_1^{\omega(n)}$ we get the formula (4) with $m = 2$. Note that in this case

$$\mathrm{Div}(n,2) = \{(n,d,1) \ : \ d|n\}$$

and the exponents of x_1 and x_2 are $\omega(n) - \omega(d)$ and $\omega(d)$, accordingly.

We now move to the induction step. Suppose (4) holds for $m > 1$. Then

$$
\begin{aligned}
(x_1+\cdots+x_{m+1})^{\omega(n)} &= \sum_{d|n}(x_1+\cdots+x_m)^{\omega(d)}x_{m+1}^{\omega(n)-\omega(d)} && (7)\\
&= \sum_{d|n}\left(\sum_{\mathrm{Div}(d,m)}\prod_{i=1}^{m}x_i^{\omega(d_{i-1})-\omega(d_i)}\right)x_{m+1}^{\omega(n)-\omega(d)} && (8)\\
&= \sum_{d|n}\sum_{\mathrm{Div}(d,m)} x_{m+1}^{\omega(n)-\omega(d)}\prod_{i=1}^{m}x_i^{\omega(d_{i-1})-\omega(d_i)}, && (9)
\end{aligned}
$$

where in (7) we apply (3) for $a = x_{m+1}$ and $b = x_1 + \cdots + x_m$, and in (8) we apply induction hypothesis. Notice that the set of indices of the double sum in (9) and the set $\mathrm{Div}(n, m+1)$ are in one-to-one correspondence, that is

$$\{(n,(d_0,d_1,\ldots,d_m))\ :\ (d_0,\ldots,d_m)\in \mathrm{Div}(d,m),\ d|n\}$$

and

$$\mathrm{Div}(n,m+1)=\{(d'_0,d'_1,\ldots,d'_m,d'_{m+1})\in\mathbb{N}^{m+2}\ :\ d'_0=n,\ d'_1|d'_0,\ \ldots,\ d'_m|d'_{m-1},\ d'_{m+1}=1\}$$

are bijective and the bijection is set by

$$(n,(d_0,d_1,\ldots,d_m))\mapsto(d'_0,d'_1,\ldots,d'_m,d'_{m+1})=(n,d_0,d_1,\ldots,d_m).$$

We use the above reasoning to (9), which leads to the following formula

$$\sum_{d|n}\sum_{\mathrm{Div}(d,m)} x_{m+1}^{\omega(n)-\omega(d)}\prod_{i=1}^{m}x_i^{\omega(d_{i-1})-\omega(d_i)} = \sum_{\mathrm{Div}(n,m+1)}\prod_{i=1}^{m+1}x_i^{\omega(d_{i-1})-\omega(d_i)}$$

and completes the induction. □

Example 2.2. Consider $m = 4$ and $n = 2\cdot 3$ (here $\omega(n) = 2$). Then

$$
\begin{aligned}
\mathrm{Div}(6,4)=\{&(6,6,6,6,1),(6,6,6,3,1),(6,6,6,2,1),(6,6,6,1,1),\\
&(6,6,3,3,1),(6,6,3,1,1),(6,6,2,2,1),(6,6,2,1,1),\\
&(6,6,1,1,1),(6,3,3,3,1),(6,3,3,1,1),(6,3,1,1,1),\\
&(6,2,2,2,1),(6,2,2,1,1),(6,2,1,1,1),(6,1,1,1,1)\}
\end{aligned}
$$

and the corresponding terms of (4) with (for clarity) a, b, c and d instead of x_1, x_2, x_3 and x_4 are:

$$
\begin{array}{cccc}
d^2 & cd & cd & c^2\\
bd & bc & bd & bc\\
b^2 & ad & ac & ab\\
ad & ac & ab & a^2
\end{array}
$$

It is clear that this corresponds to the multinomial expansion of $(a + b + c + d)^2$.

We note two immediate consequences of Theorem 2.1.

Corollary 2.3. *If n is a square-free number, then*

$$\mathrm{card}\,\mathrm{Div}(n,m)=m^{\omega(n)}.$$

Proof. Apply Theorem 2.1 with $x_1 = \cdots = x_m = 1$. □

Corollary 2.4. *The number of non-increasing sequences $d_1, \dots, d_m$ of length m, provided the numbers are from the set of factors of some square-free number n and d_{i+1} is a factor of d_i for $i = 1, \dots, m-1$, is equal to $(m+1)^{\omega(n)}$.*

3 Results for numbers that are not square-free

In Theorem 1.1, we assume that n is a square-free number. It turns out that (2) and (3) are special cases of the following formula (see also [3]).

Theorem 3.1. *For arbitrary integer $n > 0$ and any $x, y \in \mathbb{R}$ we have*

$$\prod_{i=1}^{\omega(n)} (x + a_i y) = \sum_{d|n} x^{\omega(n)-\omega(d)} y^{\omega(d)}. \tag{10}$$

Proof. Notice that for prime p and $a \geq 0$ we have

$$F(p^a) = f(1) + f(p) + \cdots + f(p^a) = 1 + ax,$$

where F and f are defined as in the proof of Theorem 2.1. Hence

$$F(n) = \prod_{i=1}^{\omega(n)} F(p_i^{a_i}) = \prod_{i=1}^{\omega(n)} (1 + a_i x)$$

and equation (6) is valid for arbitrary n. Therefore,

$$\prod_{i=1}^{\omega(n)} (1 + a_i x) = \sum_{d|n} x^{\omega(d)}$$

and substitution $x \to \frac{y}{x}$ leads to (10). □

Example 3.2. Let $n = 360 = 2^3 \cdot 3^2 \cdot 5$. Then the left-hand-side becomes a formula for

$$(x + 3y)(x + 2y)(x + y) = x^3 + 6x^2y + 11xy^2 + 6y^3$$

in terms of the numbers related to divisors of 360. The terms corresponding to given divisor d are gathered in Table 1.

d	360	72	120	24	40	8	180	36	60	12	20	4
$x^{\omega(n)-\omega(d)}y^{\omega(d)}$	y^3	xy^2	y^3	xy^2	xy^2	x^2y	y^3	xy^2	y^3	xy^2	xy^2	x^2y
d	90	18	30	6	10	2	45	9	15	3	5	1
$x^{\omega(n)-\omega(d)}y^{\omega(d)}$	y^3	xy^2	y^3	xy^2	xy^2	x^2y	xy^2	x^2y	xy^2	x^2y	x^2y	x^3

Table 1. Terms corresponding to all divisors d or 360, ordered in decreasing order of the vector of powers of consecutive primes.

Note a trivial observation based on Theorem 3.1. If we substitute $x = y = 1$, then the right-hand side of (10) counts divisors of n, while the left-hand side of that formula is the usual formula for the number of divisors:

$$\prod_{i=1}^{k}(1+a_i).$$

The following results search for the expansion of $(x+y)^{\omega(n)}$ for n's that are not square-free numbers.

The next theorem is a binomial-like expansion for powers of square-free numbers. Here, to compensate changes in the formula, we have to include additional factor to the right-hand side.

Theorem 3.3. *Suppose m is a square-free number and $n = m^\ell$ for some integer $\ell > 1$. Then*

$$(x+y)^{\omega(n)} = \sum_{d|n} \frac{x^{\omega(n)-\omega(d)}y^{\omega(d)}}{\ell^{\omega(d)}}. \tag{11}$$

Proof. We apply previous results to obtain the following equations:

$$\begin{aligned}(x+y)^{\omega(n)} &= \left(x+\ell\cdot\frac{y}{\ell}\right)^{\omega(n)}\\ &= \prod_{i=1}^{\omega(n)}\left(x+\ell\cdot\frac{y}{\ell}\right)\\ &= \sum_{d|n}\frac{x^{\omega(n)-\omega(d)}y^{\omega(d)}}{\ell^{\omega(d)}}, \qquad (12)\end{aligned}$$

where (12) follows from (10). □

Notice that equation (11) can also be written in one of the following fashion, resembling a binomial-like expansion:

$$\begin{aligned}(x+y)^{\omega(n)} &= \ell^{-\omega(n)}\sum_{d|n}(\ell\cdot x)^{\omega(n)-\omega(d)}y^{\omega(d)},\\ (\ell x+\ell y)^{\omega(n)} &= \sum_{d|n}(\ell\cdot x)^{\omega(n)-\omega(d)}y^{\omega(d)}.\end{aligned}$$

Example 3.4. Consider $n = 36 = (2\cdot 3)^2$ (here $\ell = 2$). The terms corresponding to all divisors of n are in Table 2.

d	36	18	12	9	6	4	3	2	1
$x^{\omega(n)-\omega(d)}y^{\omega(d)}$	y^2	y^2	y^2	xy	y^2	xy	xy	xy	x^2
$\ell^{\omega(d)}$	4	4	4	2	4	2	2	2	1

Table 2. Analysis of $n = 36$

Interpreting second and third row of Table 2 as fractions we see that they add up to $x^2+2xy+y^2$, as expected.

We now present a result for arbitrary number n. Recall that if $\mathbb{R}[X_1,\ldots,X_k]$ is a ring of polynomials in k variables over the field of real numbers, then elementary symmetric polynomials $S_m(X_1,\ldots,X_k)$ are defined as the sums of all distinct products of m variables, that is:

$$\begin{aligned}
S_0(X_1,\dots,X_k) &= 1,\\
S_1(X_1,\dots,X_k) &= X_1+\cdots+X_k,\\
S_2(X_1,\dots,X_k) &= \sum_{1\le i<j\le k} X_iX_j,\\
&\ \vdots\\
S_{k-1}(X_1,\dots,X_k) &= \sum_{1\le i_1<i_2<\cdots<i_{k-1}\le k}\ \prod_{j=1}^{k-1} X_{i_j},\\
S_k(X_1,\dots,X_k) &= X_1\cdots X_k.
\end{aligned}$$

See [2] for further details concerning symmetric polynomials.

We now present the binomial-like expansion formula involving the function $\omega(n)$ and symmetric polynomials.

Lemma 3.5. *Suppose $n=\prod_{i=1}^{\omega(n)} p_i^{a_i}$ is a canonical factorization of n and fix $m\ge 0$. Then*

$$\operatorname{card}\{d\in\mathbb{N}\ :\ \omega(d)=m\ \textit{ and }\ d|n\}=S_m(a_1,\dots,a_{\omega(n)}).$$

Proof. Suppose $\omega(d)=m$. Then the number of divisors of n with that many distinct prime factors is, using combinatorial argument, equal to

$$\sum_{1\le i_1<i_2<\cdots<i_m\le\omega(n)}\ \prod_{j=1}^{m} a_{i_j}=S_m(a_1,\dots,a_{\omega(n)}).$$

For example, if $m=2$, then we choose two prime factors p_i and p_j with $i\ne j$ and consider numbers of the form $p_i^{b_i}p_j^{b_j}$, where $b_i\in\{1,\dots,a_i\}$ and $b_j\in\{1,\dots,a_j\}$. There are exactly

$$\sum_{1\le i<j\le\omega(n)} a_ia_j=S_2(a_1,\dots,a_{\omega(n)})$$

many divisors with two distinct prime factors. This generalizes to any number m. □

Theorem 3.6. *Suppose $n=\prod_{i=1}^{\omega(n)} p_i^{a_i}$ is a canonical factorization of n. Then*

$$(x+y)^{\omega(n)}=\sum_{d|n}\frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a_1,\dots,a_{\omega(n)})}x^{\omega(n)-\omega(d)}y^{\omega(d)}.$$

Proof. Let $k=\omega(n)$. Using classic binomial expansion we have

$$(x+y)^k=\sum_{i=0}^{k}\binom{k}{i}x^{k-i}y^i=\sum_{i=0}^{k}\sum_{\substack{d|n\\ \omega(d)=i}} C_i\binom{k}{i}x^{k-i}y^i. \tag{13}$$

Equation (13) includes an additional factor that is a sum over divisors multiplied by a constant C_i, fixed for given i. In particular,

$$\binom{k}{i}=\sum_{\substack{d|n\\ \omega(d)=i}} C_i\binom{k}{i}.$$

To find the constant, notice that for fixed i and by Lemma 3.5 we have

$$C_i = \frac{\binom{k}{i}}{\sum\limits_{\substack{d|n \\ \omega(d)=i}} \binom{k}{i}} = \frac{1}{\text{card}\{d \in \mathbb{N} \,:\, \omega(d) = i \text{ and } d|n\}} = \frac{1}{S_i(a_1, \ldots, a_k)}. \tag{14}$$

Since $k = \omega(n)$ and $i = \omega(d)$, combining (14) with (13) we obtain

$$(x+y)^{\omega(n)} = \sum_{d|n} \frac{\binom{\omega(n)}{\omega(d)}}{S_{\omega(d)}(a_1, \ldots, a_{\omega(n)})} x^{\omega(n)-\omega(d)} y^{\omega(d)}. \qquad \square$$

Example 3.7. To illustrate Theorem 3.6, let $n = 360 = 2^3 \cdot 3^2 \cdot 5^1$. Then

$$S_0(3,2,1) = 1,$$
$$S_1(3,2,1) = 6,$$
$$S_2(3,2,1) = 11,$$
$$S_3(3,2,1) = 6,$$

and using the values in Table 1 in Example 3.2 we see that respective values coincide with the coefficients of the expansion of the polynomial. For example, there are 11 different divisors of n with $\omega(d) = 2$, each of them providing the same term $\frac{\binom{3}{2}}{S_2(3,2,1)} xy^2 = \frac{3}{11} xy^2$.

The above example inspires us to provide one more result. Using Theorem 3.1 and Lemma 3.5, we can easily deduce the following formula.

Corollary 3.8. *Suppose* $n = \prod\limits_{i=1}^{\omega(n)} p_i^{a_i}$ *is a canonical factorization of* n*. Then*

$$\prod_{i=1}^{\omega(n)} (x + a_i y) = \sum_{i=0}^{\omega(n)} S_i(a_1, \ldots, a_{\omega(n)}) x^{\omega(n)-i} y^i.$$

4 Conclusion

We have derived several binomial-like expansions related to the function $\omega(n)$. Our results also cover the cases where n need not be a square-free number. On the other hand, the formula provided in Theorem 3.6 is far from a very elegant formula (11). It would be interesting to find a simplified version of the former, perhaps without using binomial coefficients or symmetric polynomials.

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