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Determinantal representations for the number of subsequences without isolated odd terms

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Abstract: In this short note we propose two determinantal representations for the number of subsequences without isolated odd terms are presented. One is based on a tridiagonal matrix and other on a Hessenberg matrix. We also establish a new explicit formula for the terms of this sequence based on Chebyshev polynomials of the second kind.
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1 The sequence

In 1996, Guy and Moser considered in [11] the sequence $\{z_n\}_{n\geqslant 0}$ of the number of subsequences of $\{1,\ldots,n\}$ in which each odd number has an even neighbour or, equivalently, every odd member is accompanied by at least one even neighbour. The empty sequence is allowable. It has been coined in *The On-Line Encyclopedia of Integer Sequences* [16] as $A007455$.

The first terms of this sequence are

$$1,\ 1,\ 3,\ 5,\ 11,\ 17,\ 39,\ 61,\ 139,\ 217,\ 495,\ 773,\ 1763,\ 2753,\ \ldots$$

and it satisfies the recurrence relations

$$z_n = \begin{cases} 2z_{n-1} + z_{n-3}, & \text{if } n \text{ is even} \\ 3z_{n-2} + 2z_{n-4}, & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

with initial conditions $z_0 = z_1 = 1, z_2 = 3, z_3 = 5$.

In [11] one can find several tables involving the sequence $\{z_n\}$, as for example the one for the number of subsequences of length $0, \ldots, n$:

n																		z_n
0									1									1
1								1		0								1
2							1		1		1							3
3						1		1		2		1						5
4					1		2		4		3		1					11
5				1		2		5		5		3		1				17
6			1		3		8		11		10		5		1			39
7		1		3		9		14		16		12		5		1		61
8	1		4		13		25		35		33		20		7		1	139

This sequence satisfies also interesting identities, namely $2z_{2k} = z_{2k+1} + z_{2k-1}$, and it is not hard to see that it fulfils the unique recurrence relation

$$z_n = 3z_{n-2} + 2z_{n-4}\,, \tag{2}$$

as asserted in [11].

For explicit formulas, in [11] it is claimed that

$$\begin{aligned} z_{2k} &= \left(\frac{17+3\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-3\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k, \\ z_{2k+1} &= \left(\frac{17+7\sqrt{17}}{34}\right)\left(\frac{3+\sqrt{17}}{2}\right)^k + \left(\frac{17-7\sqrt{17}}{34}\right)\left(\frac{3-\sqrt{17}}{2}\right)^k. \end{aligned}$$

Our aim here is to provide two determinantal interpretations for the recurrence (1) in terms of a tridiagonal 2-Toeplitz matrix and other, perhaps more natural, of a Hessenberg matrix.

2 Tridiagonal 2-Toeplitz matrices

The matrices of the form

$$T_n^{(2)} = \begin{pmatrix} p_1 & 1 & & & & & \\ q_1 & p_2 & 1 & & & & \\ & q_2 & p_1 & 1 & & & \\ & & q_1 & p_2 & 1 & & \\ & & & q_2 & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}_{n\times n}, \tag{3}$$

i.e., tridiagonal matrices (a_{ij}) with entries satisfying the periodic equality

$$a_{i+2,j+2} = a_{ij}\,, \quad \text{for } i,j = 1,2,\dots,n-2\,,$$

are known as *tridiagonal* 2*-Toeplitz* [7, 10]. If instead of a period 2, we have a generic period k, these matrices are called *tridiagonal* k*-Toeplitz* [8] and the determinant is known as *periodic continuant* [14]. The study of these matrices goes back to 1947 Rutherford's seminal paper [15]. Their determinant and spectral properties were studied independently in distinct contexts (cf. [1, 4–8, 10, 13]).

It is known (see, e.g., [9, 14, 15]) that the determinant of these matrices is given by

$$\det T_{2\ell}^{(2)} = (\sqrt{q_1q_2})^{\ell}\left(U_\ell\left(\frac{p_1p_2-q_1-q_2}{2\sqrt{q_1q_2}}\right) + \frac{\sqrt{q_2}}{\sqrt{q_1}}\,U_{\ell-1}\left(\frac{p_1p_2-q_1-q_2}{2\sqrt{q_1q_2}}\right)\right) \tag{4}$$

and

$$\det T_{2\ell+1}^{(2)} = p_1(\sqrt{q_1q_2})^{\ell}\,U_\ell\left(\frac{p_1p_2-q_1-q_2}{2\sqrt{q_1q_2}}\right)\,, \tag{5}$$

where $\{U_n(x)\}_{n\geqslant 0}$ are the Chebyshev polynomials of the second kind of order n, that is, the orthogonal polynomials satisfying the three-term recurrence relations

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) = 0\,, \quad \text{for } \; n = 0,1,2,\dots\;, \tag{6}$$

with initial conditions $U_{-1}(x) = 0$ and $U_0(x) = 1$. One of the most well-known explicit formulas for these polynomials is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

for all $n = 0,1,2\dots.$

Let us consider the sequence $\{f_n\}_{n\geqslant 0}$ satisfying the recurrence relation

$$f_n = \begin{cases} f_{n-1} + 2f_{n-2}, & \text{if } n \text{ is even} \\ 2f_{n-1} - f_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \tag{7}$$

for $n \geqslant 2$, with initial conditions $f_0 = f_1 = 1$. Clearly,

$$f_n = \det \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & -2 & 1 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & -2 & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}_{(n+1)\times(n+1)} . \tag{8}$$

This means that, from (4)–(5), we have

$$f_{2\ell-1} = (-i\sqrt{2})^{\ell} \left(U_\ell \left(\frac{3i\sqrt{2}}{4} \right) - i\sqrt{2}\, U_{\ell-1} \left(\frac{3i\sqrt{2}}{4} \right) \right) \tag{9}$$

and, otherwise,

$$f_{2\ell} = (-i\sqrt{2})^{\ell}\, U_\ell \left(\frac{3i\sqrt{2}}{4} \right) , \tag{10}$$

The interesting fact is that, using elementary operations, we can show that $\{f_n\}$ and $\{z_n\}$ are exactly the same sequence. Therefore, $\{z_n\}$ can be defined in terms of the determinants of the family of tridiagonal 2-Toeplitz matrices (8) as well as in the explicit form (9)–(10). If we take into account [2, 3], from (7) we immediately get (2).

3 Hessenberg matrices

In this brief section, we present a natural family of Hessenberg matrices whose determinants provide the sequence $\{z_n\}$. Since the recurrences (1) can be stated in terms of a single homogeneous linear recurrence relation, we can conclude (cf., e.g., [12, 17]) that

$$z_{n+1} = \det H_n,$$

with

$$H_n = \begin{pmatrix} 1 & 1 & 3 & 5 & & & & & & & \\ -1 & 0 & 0 & 0 & 0 & & & & & & \\ & -1 & 0 & 0 & 1 & 2 & & & & & \\ & & -1 & 0 & 0 & 0 & 0 & & & & \\ & & & -1 & 2 & 3 & 1 & 2 & & & \\ & & & & -1 & 0 & 0 & 0 & 0 & & \\ & & & & & -1 & 2 & 3 & 1 & 2 & \\ & & & & & & -1 & 0 & 0 & 0 & \ddots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & & \ddots & \ddots & \ddots \end{pmatrix}_{n\times n} .$$

To the best of our knowledge, this simple determinantal representation for $\{z_n\}$ is also new.

4 Conclusion

In this note, two determinantal representations for the sequence of the number of subsequences of $\{1, \ldots, n\}$ in which each odd number has an even neighbour. We propose an alternative recurrence relation for such sequence as well as a new explicit formula based on Chebyshev polynomials of the second kind.

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