Series expansion of the Gamma function and its reciprocal

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Abstract: In this paper we give representations for the coefficients of the Maclaurin series for \( \Gamma(z + 1) \) and its reciprocal (where \( \Gamma \) is Euler’s Gamma function) with the help of a differential operator \( \mathcal{D} \), the exponential function and a linear functional \( * \) (in Theorem 3.1). As a result we obtain the following representations for \( \Gamma \) (in Theorem 3.2):

\[
\Gamma(z + 1) = \left( e^{-u(x)} e^{-z \mathcal{D}[e^{u(x)}]} \right)^{*},
\]

\[
(\Gamma(z + 1))^{-1} = \left( e^{u(x)} e^{-z \mathcal{D}[e^{-u(x)}]} \right)^{*}.
\]

Theorem 3.1 and Theorem 3.2 are our main results. With the help of the first theorem we give our approach for finding the coefficients of Maclaurin series for \( \Gamma(z + 1) \) and its reciprocal in an explicit form.

Keywords: Gamma function, Zeta function, Euler’s constant, Maclaurin series.

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1 Introduction

Let \( \mathbb{N} \) be the set of all positive integers, \( \mathbb{C} \) - the complex number field, \( \mathbb{E} \) - the set of all entire functions of one complex variable. For \( F \in \mathbb{E} \) the operator \( D^k : \mathbb{E} \to \mathbb{E} \) is defined by:

\[
D^0[F(z)] := F^{(0)}(z) = F(z), ~ z \in \mathbb{C}; ~ D^k[F(z)] := F^{(k)}(z), \quad k \in \mathbb{N}, ~ z \in \mathbb{C}, \quad \text{i.e.,} \quad D = \frac{d}{dz}
\]
and \( D^k = \left( \frac{d}{dz} \right)^k \).

If \( D^k[F(z)] = H_k(z) \), we define \( D^k_s[F(z)] := H_k(s) \).
Further, we shall use the following notation: \( \zeta \) for Riemann zeta function; \( \gamma \) for Euler's constant, i.e. \( \gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \ln n \right) = 0.5772156649 \ldots \); \( \zeta(1) = \gamma \) and \( \tilde{\zeta}(k) = \zeta(k) \) for each integer \( k > 1 \).

The Gamma function admits the following basic representations (see [7], pp. 31, 33, 34):

(i) \( \Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} \, dt \), valid for \( z \in \mathbb{C}, \Re z > 0 \) (Euler)

where \( \Gamma \) is a holomorphic function and \( D^k[\Gamma(z)] = \int_{0}^{\infty} e^{-t} t^{z-1} (\ln t)^k \, dt \);

(ii) \( \Gamma(z) = \lim_{m \to \infty} \frac{m! m^z}{z(z+1) \cdots (z+m)}, \) valid for \( z \in \mathbb{C}, z \neq 0, -1, -2, \ldots \) (Euler–Gauss)

(iii) \( \Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} e^{\frac{z}{n}} (1 + \frac{z}{n})^{-1} \), valid for \( z \in \mathbb{C}, z \neq 0, -1, -2, \ldots \) (Weierstrass)

From (iii) it is seen that \( (\Gamma(z))^{-1} \in \mathbb{E} \) and that \( \Gamma(z) \) is a meromorphic function without zeroes and with simple poles: \( z = 0, -1, -2, \ldots \)

We have \( \Gamma(1) = 1 \) and for any \( z \in \mathbb{C} \) \( \Gamma \) satisfies the functional equation

\[
\Gamma(z+1) = z\Gamma(z).
\] (1)

Hence \( \Gamma(z+1)^{-1} \in \mathbb{E} \) and \( \Gamma(z+1) \) is a meromoromorphic function without zeroes and with simple poles: \( z = -1, -2, \ldots \)

Also, from (i), the representations:

\[
A(k) = \Gamma^{(k)}(1) = D_0^k[\Gamma(z+1)] = \int_{0}^{\infty} e^{-t} (\ln t)^k \, dt, \quad k = 0, 1, 2, \ldots,
\] (2)

hold.

Although the Gamma function was introduced by Euler about two hundred and ninety two years ago it still has its secrets. In the present paper we introduce a linear operator \( \mathcal{D} : \mathbb{E} \to \mathbb{E} \) on which our main results are based – Theorem 3.1 and Theorem 3.2. With the help of Theorem 3.1 we find explicit formulae for the coefficients of Maclaurin series of \( \Gamma(z+1) \) and \( (\Gamma(z+1))^{-1} \) (Theorem 3.3.). Here we must note that such type of formulae are given by other authors too. For example formulae for these coefficients are contained in: [3, 4, 6, 8].

2 A new operator \( \mathcal{D} \) and its basic properties

**Definition 2.1.** Let \( u(z) \in \mathbb{E} \). The operator \( \mathcal{D} \) is defined by:

- \( \mathcal{D} = \mathcal{D}^1; \quad \mathcal{D}^0[u(z)] = u(z); \)
- \( \mathcal{D}[D^0[u(z)]] = D^1[u(z)]; \)
- \( \mathcal{D}[D^k[u(z)]] = kD^{k+1}[u(z)], \quad k \geq 1; \)
- \( \mathcal{D}^k[u(z)] = \mathcal{D}[\mathcal{D}^{k-1}[u(z)]], \quad k \geq 1. \)

**Lemma 2.1.** Let \( u(z), F(z), G(z) \in \mathbb{E} \). Then

\( (j_1) \quad \mathcal{D}^k[u(z)] = (k - 1)! D^k[u(z)] \quad (\forall k \in \mathbb{N}) \)

\( (j_2) \quad \mathcal{D}^k \) is a linear operator \( (\forall k \in \mathbb{N}) \)
Proof. (j1) follows by induction from the definition of $\mathfrak{D}$.

Let $\lambda, \mu \in \mathbb{C}$. From (j1) and from the linearity of $D^k$ we obtain:

$$
\mathfrak{D}^k[\lambda F(z) + \mu G(z)] = (k - 1)!D^k[\lambda F(z) + \mu G(z)] = \lambda(k - 1)!D^k[F(z)] + \mu(k - 1)!D^k[G(z)]
$$

which proves (j2).

Let us prove (j3) by induction. In the case $k = 1$, $\mathfrak{D}$ coincides with $D$ and (j3) is obvious. If for $k \geq 1$ (j3) is true, then applying $\mathfrak{D}$ to (j3) and using the linearity of $\mathfrak{D}$ we obtain:

$$
\mathfrak{D}^{k+1}[F(z)G(z)] = \sum_{\nu=0}^{k} \binom{k}{\nu} \mathfrak{D}^{k-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)]
$$

The right-hand side $R$ in the above equality is $I_1 + I_2$, where:

$$
I_1 = \mathfrak{D}^{k+1}[F(z)]\mathfrak{D}^{0}[G(z)] + \sum_{\nu=1}^{k} \binom{k}{\nu} \mathfrak{D}^{k+1-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)];
$$

$$
I_2 = \sum_{\nu=1}^{k} \binom{k}{\nu-1} \mathfrak{D}^{k+1-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)] + \mathfrak{D}^{0}[F(z)]\mathfrak{D}^{k+1}[G(z)].
$$

$I_2$ is obtained after substitution $\nu + 1 = t$ and replacing $t$ with $\nu$.

Now using that $\binom{k}{\nu} + \binom{k}{\nu-1} = \binom{k+1}{\nu}, \nu = 1, \ldots, k$, we obtain

$$
R = \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} \mathfrak{D}^{k+1-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)]
$$

and (j3) is proved.

The proof of (j4) is obvious since $D[F(G(z))] = (DF)(G(z))D[G(z)]$ and we may replace $D[G(z)]$ with $\mathfrak{D}[G(z)]$ (see Definition 2.1).
Let us prove \((j_3)\). We consider the equalities:

\[
D[f(g(z))] = (Df)(g(z))D[g(z)],
\]

\[
\mathcal{D}[f(g(z))] = (Df)(g(z))\mathcal{D}[g(z)]
\]

(see \((j_4)\)).

Applying to the left one \(D^{n-1}\) and to the right one \(\mathcal{D}^{n-1}\) and after that using Leibnitz formula for the right-hand side of the first equality and the analogue of the Leibnitz formula (see \((Corollary 2.3.\) Let

Formula (4) admits the representation

\[
\text{Corollary 2.2. Formula (4) admits the representation}
\]

\[
\mathcal{D}^n[f(g(z))] = \sum_{m=0}^{n} (D^m f)(g(z)) \sum_{\alpha} C_n(\alpha)(D^1[g(z)])^{\alpha_1} \cdots (D^n[g(z)])^{\alpha_n},
\]

then replacing \(D^m(g(z))\) with \(\mathcal{D}^m(g(z))\) in it, we obtain exactly (3).

**Proof.** It follows immediately from Lemma 2.1, \((j_1)\).

**Corollary 2.2. Formula (4) admits the representation**

\[
\mathcal{D}^n[f(g(z))] = \sum_{m=0}^{n} (D^m f)(g(z)) \sum_{\alpha} A_n(\alpha)(D^1[g(z)])^{\alpha_1} \cdots (D^n[g(z)])^{\alpha_n},
\]

where

\[
A_n(\alpha) = \frac{n!}{(1! \cdot 2! \cdot 3! \cdots n!)(\alpha_1! \alpha_2! \cdots \alpha_n!)}
\]

**Proof.** One may check directly that

\[
C_n(\alpha) \prod_{\nu=1}^{n} ((\nu - 1)!)^{\alpha_{\nu}} = A_n(\alpha).
\]

**Corollary 2.3. Let \(u \in E\) and \(n \in \mathbb{N}\). Then:**

\[
\mathcal{D}^n[e^u(z)] = e^u(z) \sum_{m=0}^{n} \sum_{\alpha} A_n(\alpha)(D^1[u(z)])^{\alpha_1} \cdots (D^n[u(z)])^{\alpha_n}
\]

\[
\mathcal{D}^n[e^{-u(z)}] = e^{-u(z)} \sum_{m=0}^{n} \sum_{\alpha} (-1)^{\sum_{\nu=1}^{n} \alpha_{\nu}} A_n(\alpha)(D^1[u(z)])^{\alpha_1} \cdots (D^n[u(z)])^{\alpha_n}
\]
3 Maclaurin series for $\Gamma(z + 1)$ and its reciprocal

If $k \geq 0$ we define $a_k$, $d_k$, $b_k$, $\rho_k$ by:

\begin{align*}
  k!a_k &= D_0^k[\Gamma(z + 1)] \quad (8) \\
  k!d_k &= D_0^k[(\Gamma(z + 1))^{-1}] \quad (9) \\
  b_k &= (-1)^kk!a_k \quad (10) \\
  \rho_k &= k!d_k \quad (11)
\end{align*}

Hence:

\begin{align*}
  \Gamma(z + 1) &= \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 1, \quad (12) \\
  (\Gamma(z + 1))^{-1} &= \sum_{k=0}^{\infty} d_k z^k, \quad |z| < \infty. \quad (13)
\end{align*}

For the sequences $\{a_k\}$ and $\{d_k\}$ recurrence relations are known (see [5], pp. 12, 17; [2], p. 12) that we give below but for the sequences $\{b_k\}$ and $\{\rho_k\}$:

\begin{align*}
  b_{k+1} &= \sum_{\nu=0}^{k} \binom{k}{\nu} (k - \nu)!\tilde{\zeta}(k - \nu + 1)b_{\nu}, \quad k \geq 0, \quad b_0 = 1, \quad (14) \\
  \rho_{k+1} &= \sum_{\nu=0}^{k} \binom{k}{\nu} (k - \nu)!(-1)^{k-\nu}\tilde{\zeta}(k - \nu + 1)\rho_{\nu}, \quad k \geq 0, \quad \rho_0 = 1. \quad (15)
\end{align*}

3.1 Connection between the Gamma function and the exponential function

**Definition 3.1.** Let $F(z_1, z_2, \ldots, z_k)$ be a polynomial and $u(z) \in \mathbb{E}$. Then we define the mapping $*$ by:

\[(F(D^1[u(z)], D^2[u(z)], \ldots, D^k[u(z)]))^* := F(\tilde{\zeta}(1), \tilde{\zeta}(2), \ldots, \tilde{\zeta}(k)).\]

**Remark.** From the above definition it is clear that $*$ is a linear and multiplicative mapping. The multiplicativity of $*$ means that if $F(z_1, \ldots, z_k)$ and $G(z_1, \ldots, z_m)$ are polynomials and

\[H(z_1, \ldots, z_n) = F(z_1, \ldots, z_k)G(z_1, \ldots, z_m),\]

then

\[(H(D^1[u(z)], \ldots, D^n[u(z)]))^* = (F(D^1[u(z)], \ldots, D^k[u(z)]))^* (G(D^1[u(z)], \ldots, D^m[u(z)]))^*.\]

Our first main result in this paper is the following.

**Theorem 3.1.** \(\forall k \in \mathbb{N} \cup \{0\} \) $b_k$ and $\rho_k$ are given by the formulae:

\begin{align*}
  b_k &= (e^{-u(z)}D^k[e^u(z)])^*, \quad (16) \\
  \rho_k &= ((-1)^k e^u(z)D^k[e^{-u(z)}])^*. \quad (17)
\end{align*}
Proof. We shall prove only (16) since one may prove (17) in the same way. We prove (16) by induction. For \( k = 0 \) we have \( b_0 = 1 \) and \((e^{-u(z)}\mathcal{D}^k[e^u(z)])^* = (e^{-u(z)}\mathcal{D}^0[e^u(z)])^* = (\{1\})^* = 1\).

Assume that (16) is true for some \( k \geq 0 \). We will show that (16) is true for \( k + 1 \) too. Let

\[
b'_{k+1} = (e^{-u(z)}\mathcal{D}^{k+1}[e^u(z)])^*.
\]

Then

\[
b'_{k+1} = (e^{-u(z)}\mathcal{D}^k[e^u(z)])^* = (e^{-u(z)}\mathcal{D}^k[D[e^u(z)]])^* = (e^{-u(z)}\mathcal{D}^k[e^u(z)\mathcal{D}[u(z)]])^*\]

[now we apply Lemma 2.1, \((j_3)\) to \(\mathcal{D}^k[e^u(z)\mathcal{D}[u(z)]]\)]

\[
= (e^{-u(z)} \sum_{\nu=0}^{k} \left( \binom{k}{\nu} \mathcal{D}^{k+1-\nu}[u(z)]\mathcal{D}^{\nu}[e^u(z)] \right))^* \]

\[
= (\sum_{\nu=0}^{k} \left( \binom{k}{\nu} \mathcal{D}^{k+1-\nu}[u(z)](e^{-u(z)}\mathcal{D}^{\nu}[e^u(z)]) \right))^* \]

[now we use \(\mathcal{D}^{k+1-\nu}[u(z)] = (k - \nu)!D^{k+1-\nu}[u(z)], \) see Lemma 2.1, \((j_1)\)]

\[
= (\sum_{\nu=0}^{k} \left( \binom{k}{\nu} (k - \nu)!D^{k+1-\nu}[u(z)](e^{-u(z)}\mathcal{D}^{\nu}[e^u(z)]) \right))^* \]

[from the linearity and multiplicativity of *]

\[
= \sum_{\nu=0}^{k} \left( \binom{k}{\nu} (k - \nu)!D^{k+1-\nu}[u(z)](e^{-u(z)}\mathcal{D}^{\nu}[e^u(z)]) \right)^* \]

[from the induction hypothesis we have \((e^{-u(z)}\mathcal{D}^{\nu}[e^u(z)])^* = b_\nu, \ 0 \leq \nu \leq k\)]

\[
= \sum_{\nu=0}^{k} \left( \binom{k}{\nu} (k - \nu)!\zeta(k - \nu + 1)b_\nu = b_{k+1}\right) \]

(the last from (14)). Thus we proved \(b'_{k+1} = b_{k+1}\) and (16) is proved.

From Theorem 3.1, using the fact that the mapping * is linear and that the generating function of the operator \(\mathcal{D}\):

\[
\sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \mathcal{D}^k
\]

is equal to \(e^{-z\mathcal{D}}\), we obtain our second main result in this paper.
Theorem 3.2. \( \Gamma(z + 1) \) and its reciprocal have the following important representations:

\[
\Gamma(z + 1) = (e^{-u(x)}e^{-zD[e^{u(x)}]})^*, \quad \text{(18)}
\]
\[
(\Gamma(z + 1))^{-1} = (e^{u(x)}e^{-zD[e^{-u(x)}]})^*, \quad \text{(19)}
\]

where * acts only with respect to the variable \( x \).

Proof. We prove only (18) since one may prove (19) analogically.

\[
(e^{-u(x)}e^{-zD[e^{u(x)}]})^* = \left( e^{-u(x)} \left( \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} D^k \right)[e^{u(x)}] \right)^*
\]
\[
= \left( e^{-u(x)} \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} D^k[e^{u(x)}] \right)^*
\]
\[
= \left( \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} e^{-u(x)} D^k[e^{u(x)}] \right)^*
\]

[here we suppose that * is defined not only for polynomials but for power series, too]

\[
= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \left( e^{-u(x)} D^k[e^{u(x)}] \right)^*
\]

[from Theorem 3.1]

\[
= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} b_k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} b_k z^k = \sum_{k=0}^{\infty} a_k z^k = \Gamma(z + 1). \quad \square
\]

3.2 Explicit formulae for the coefficients

From Theorem 3.1 and Corollary 2.3 we obtain using (16) and (17):

\[
b_n = \sum_{m=0}^{n} \sum_{\alpha} A_n(\alpha)(\tilde{\zeta}(1))^\alpha_1 \cdots (\tilde{\zeta}(n))^\alpha_n \quad \text{(compare with [9], the formula for } \Gamma(n)(1))
\]
\[
\rho_n = (-1)^n \sum_{m=0}^{n} \sum_{\alpha} (-1)^{\sum_{\nu=1}^{n} \alpha_\nu} A_n(\alpha)(\tilde{\zeta}(1))^\alpha_1 \cdots (\tilde{\zeta}(n))^\alpha_n
\]

In [6], p. 43, (10) and (11), explicit formulae for \( a_n \) and \( d_n \) are given.

Further we shall find formulae for \( b_n \) and \( \rho_n \) in a polynomial form, of one variable, using instead of this variable Euler’s constant \( \gamma \). For this purpose we need two lemmas.

Lemma 3.1. For \( k \geq 2 \) the following identities are valid

\[
e^{-u(z)} D^k[e^{u(z)}] = (D^1[u(z)])^k + \sum_{p=2}^{k} \binom{k}{p} \Delta_p.(D^1[u(z)])^{k-p}, \quad \text{(20)}
\]
\[
(-1)^k e^{u(z)} D^k[e^{-u(z)}] = (D^1[u(z)])^k + \sum_{p=2}^{k} \binom{k}{p} \sigma_p.(D^1[u(z)])^{k-p}, \quad \text{(21)}
\]
where:
\[
\Delta_2 = D^2[u(z)], \quad \Delta_3 = D[D^2[u(z)]], \quad \Delta_{p+1} = p.(D^2[u(z)])\Delta_{p-1} + D[\Delta_p], \quad p > 3, \quad (22)
\]
\[
\sigma_2 = -D^2[u(z)], \quad \sigma_3 = D[D^2[u(z)]], \quad \sigma_{p+1} = -p.(D^2[u(z)])\sigma_{p-1} - D[\sigma_p], \quad p > 3. \quad (23)
\]

Proof. We prove (20) by induction. For \(k = 2\) and \(k = 3\) we check directly the validity of (20). Let \(R_k\) the right-hand side of (20) and let (20) be true for some \(k \geq 3\). For \(k + 1\) we must verify that \(e^{-u} D^{k+1}[e^u] = R_{k+1}\), i.e., \(R_{k+1} = e^{-u} D[D^k[e^u]] = e^{-u} D[e^u e^{-u} D^k[e^u]] = (\text{from our assumption for } k) = e^{-u} D[e^u R_k] = (\text{after computation}) = D[u] R_k + D[R_k].\) Thus, to prove (20), it remains to prove that \(R_{k+1} = D[u] R_k + D[R_k].\)

It is easy to see that the coefficient in front of \((D^1[u(z)])^{k+1-p}\) in the left-hand side of the last equality equals to \(\binom{k+1}{p}\) \(\Delta_p\) and in the right-hand side equals to \((k + 2 - p) (k - p - 1) D^2[u(z)] + \sum_{p} \binom{k}{p} \sigma_p\). So, to prove (20), we must check that
\[
\binom{k+1}{p} \Delta_p = (k + 2 - p) \binom{k}{p-2} \Delta_{p-2} D^2[u(z)] + \binom{k}{p} \Delta_p + \binom{k}{p-1} D[\Delta_{p-1}].
\]

One may easily check the above equality using (22) and the well-known relations:
\[
\binom{k+1}{p} = \binom{k}{p} + \binom{k}{p-1}; \quad t\binom{k}{t} = \binom{k-1}{t-1}.
\]

In the same way, one may prove (21) (using (23)) and Lemma 3.1 is proved.

Lemma 3.2. For \(p \geq 2\) the following representations hold:
\[
\Delta_p = \sum_{\hat{\alpha}} A_p(\hat{\alpha}) \left( D^2[u(z)] \right)^{\hat{\alpha}_2} \cdots \left( D^p[u(z)] \right)^{\hat{\alpha}_p},
\]
\[
\sigma_p = \sum_{\hat{\alpha}} (-1)^p \prod_{\nu=2}^p \alpha_\nu \left( D^2[u(z)] \right)^{\hat{\alpha}_2} \cdots \left( D^p[u(z)] \right)^{\hat{\alpha}_p},
\]
where, for \(\hat{\alpha} := (\alpha_2, \ldots, \alpha_p)\), \(A_p(\hat{\alpha})\) is given by:
\[
A_p(\hat{\alpha}) = p! \prod_{\nu=2}^p \left( (\nu - 1)! \right)^{\alpha_\nu} \prod_{\nu=2}^p \left( \alpha_\nu!(\nu!)^{\alpha_\nu} \right)^{-1} = \frac{p!}{(2^{\alpha_2}3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2!\alpha_3! \cdots \alpha_p)!}
\]

and \(\sum_{\hat{\alpha}}\) means that the sum is over all nonnegative integers \(\alpha_\nu\) such that \(\sum_{\nu=2}^p \nu \alpha_\nu = p\).

Proof. We shall prove only (24) since (25) may be proved in the same way. First in (22) \(D\) with \(D\). As a result we obtain:
\[
\Delta_2^D = D^2[u(z)], \quad \Delta_3^D = D[D^2[u(z)]], \quad \Delta_{p+1}^D = p.(D^2[u(z)])\Delta_{p-1}^D + D[\Delta_p^D], \quad p > 3.
\]
Now we shall prove that
\[
\Delta_p^D = \sum_{\hat{\alpha}} \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} \left( D^2[u(z)] \right)^{\hat{\alpha}_2} \cdots \left( D^p[u(z)] \right)^{\hat{\alpha}_p}
\]

using induction with respect to \(p\). For \(p = 2, 3\) the validity of the above equality is a matter of direct check. Let \(\hat{\alpha} = \hat{\alpha}(p), \hat{C}_p(\hat{\alpha}(p)) = \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} \) and the equality is checked for \(\nu = 2, 3, \ldots, p\). For the sake of brevity, let \(\hat{C}_p := \hat{C}_p(\hat{\alpha}(p))\).
We must prove that

$$\sum_{\tilde{\alpha}(p+1)} \tilde{C}_{p+1}^{\alpha} \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\alpha_\nu} = \Delta_{p+1}^D = p \cdot (D^2 [u(z)]) \cdot \Delta_{p-1}^D + D[\Delta_p^D],$$

i.e. that the equality (further denoted by $\mathbf{H}$):

$$\sum_{\tilde{\alpha}(p+1)} \tilde{C}_{p+1}^{\alpha} \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\alpha_\nu} = \sum_{\tilde{\alpha}(p-1)} p \tilde{C}_{p-1}^{\alpha} (D^2[u(z)])^{\alpha_2+1} \prod_{\nu=3}^{p-1} (D^\nu [u(z)])^{\alpha_\nu}$$

$$+ \sum_{j=2}^{p-1} \sum_{\tilde{\alpha}(p)} \alpha_j \tilde{C}_p \cdot P. (D^j[u(z)])^{\alpha_j-1} (D^{j+1}[u(z)])^{\alpha_{j+1}+1}.Q + D^{p+1}[u(z)],$$

holds, where $P$ and $Q$ are given by:

$$P = \prod_{\nu=2}^{j-1} (D^\nu [u(z)])^{\alpha_\nu}; \quad Q = \prod_{\nu=j+2}^{p-1} (D^\nu [u(z)])^{\alpha_\nu}.$$  

Let $\prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_\nu}$ be an arbitrary monomial from the right-hand side of $\mathbf{H}$. Below we calculate the coefficient $C$ in front of this monomial:

$$C = p \tilde{C}_{p-1}(\gamma_2 - 1, \gamma_3, \ldots, \gamma_{p+1}) + \sum_{j=2}^{p-1} (\gamma_j + 1) \tilde{C}_{p}(\gamma_2, \ldots, \gamma_{j} + 1, \gamma_{j+1} - 1, \ldots, \gamma_{p+1})$$

$$= p! \frac{\sum_{\nu=2}^{p+1} \nu \gamma_\nu}{\prod_{\nu=2}^{p+1}(\nu)!} \cdot \frac{(p+1)!}{\prod_{\nu=2}^{p+1}(\nu)!^{\gamma_\nu}} = \tilde{C}_{p+1}(\gamma_2, \ldots, \gamma_{p+1}).$$

Hence, the right-hand side of $\mathbf{H}$ is a sum of the terms $\tilde{C}_{p+1}(\gamma_2, \ldots, \gamma_{p+1}) \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_\nu}$, for which $\sum_{\nu=2}^{p+1} \nu \gamma_\nu = p + 1$. Then, to prove $\mathbf{H}$, it remains to show that in the right-hand side of $\mathbf{H}$ all partitions $\sum_{\nu=2}^{p+1} \nu \gamma_\nu = p + 1$ are met.

Indeed, let $\prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_\nu}$ be an arbitrary monomial from the left-hand side of $\mathbf{H}$. If $\gamma_{p+1} = 1$, then $\gamma_2 = \gamma_3 = \ldots = \gamma_p = 0$, so this monomial is $D^{p+1}[u(z)]$ and it is contained in the right-hand side of $\mathbf{H}$, too. If $\gamma_{p+1} = 0$ but for some $j$, such that $2 < j \leq p$, $\gamma_j \neq 0$, we consider in the right-hand side of $\mathbf{H}$ the sum:

$$\sum_{\tilde{\alpha}(p)} \alpha_{j-1} \tilde{C}_p (\alpha_2, \ldots, \alpha_p) \left( \prod_{\nu=2}^{j-2} (D^\nu [u(z)])^{\alpha_\nu} \right) (D^{j-1}[u(z)])^{\alpha_{j-1}-1} (D^j[u(z)])^{\alpha_j+1} \prod_{\nu=j+1}^{p} (D^\nu [u(z)])^{\alpha_\nu}.$$

Now we set: $\alpha_2 = \gamma_2, \ldots, \alpha_{j-2} = \gamma_{j-2}; \alpha_{j-1} = \gamma_{j-1} + 1; \alpha_j = \gamma_j - 1; \alpha_{j+1} = \gamma_{j+1}, \ldots, \alpha_p = \gamma_p$. Then $\sum_{\nu=2}^{p+1} \nu \alpha_\nu = (\sum_{\nu=2}^{p+1} \nu \gamma_\nu) - 1 = (p+1) - 1 = p$. Hence, according to the given definition of $\gamma_\nu$ the monomial $\prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_\nu}$ is contained in the right-hand side of $\mathbf{H}$.

If $\gamma_2 \neq 0$, then we consider in the right-hand side of $\mathbf{H}$ the sum:

$$\sum_{\tilde{\alpha}(p-1)} p \tilde{C}_{p-1} (\alpha_2, \ldots, \alpha_{p-1}) (D^2[u(z)])^{\alpha_2+1} \prod_{\nu=3}^{p-1} (D^\nu [u(z)])^{\alpha_\nu}.$$
Letting \( \alpha_2 = \gamma_2 - 1, \alpha_3 = \gamma_3, \ldots, \alpha_{p-1} = \gamma_{p-1} \), we obtain \( \sum_{\nu=2}^{p-1} \nu \alpha_\nu = \left( \sum_{\nu=2}^{p+1} \nu \gamma_\nu \right) - 2 = p - 1. \)

Hence according to the definition of \( \sum_{\tilde{\alpha}(p-1)} \) the monomial \( \prod_{\nu=2}^{p+1} (D^\nu [u(z)])^{\gamma_\nu \nu} \) is contained in the right-hand side of \( H. \)

Thus we proved by induction that

\[
\Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} \left( D^2[u(z)] \right)^{\alpha_2} \cdots \left( D^p[u(z)] \right)^{\alpha_p}.
\]

In the above equality we replace \( D \) with \( D \) (using the fact that Lemma 3.1 remains valid if we replace in (22) and (23) \( D \) with \( D \)) and obtain

\[
\Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} \left( \mathcal{D}^2[u(z)] \right)^{\alpha_2} \cdots \left( \mathcal{D}^p[u(z)] \right)^{\alpha_p}.
\]

Now, using the fact that \( \mathcal{D}^k[u(z)] = (k - 1)! D^k[u(z)] \) (see Lemma 2.1, (j1)), we obtain

\[
\Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} \prod_{\nu=2}^{p} ((\nu - 1)! \nu^\alpha (D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p}.
\]

Hence

\[
\Delta_p = \Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2} \alpha_2! \cdots (p!)^{\alpha_p} \alpha_p!} (D^2[u(z)])^{\alpha_2} \cdots (D^p[u(z)])^{\alpha_p}.
\]

Thus (24) and therefore Lemma 3.2 are proved. \( \square \)

From Theorem 3.1, Lemma 3.1 and Lemma 3.2 we obtain the following explicit formulae for the coefficients.

**Theorem 3.3.** The following representations are valid:

\[
b_k = \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\tilde{\alpha}} A_p(\tilde{\alpha}) \prod_{\nu=2}^{p} (\zeta(\nu))^{\alpha_\nu}; \quad (26)
\]

\[
\rho_k = \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\tilde{\alpha}} (-1)^{p+\sum_{\nu=2}^{p} \alpha_\nu} A_p(\tilde{\alpha}) \prod_{\nu=2}^{p} (\zeta(\nu))^{\alpha_\nu}, \quad (27)
\]

where \( \sum_{\tilde{\alpha}} \) means that we sum over all nonnegative integers \( \alpha_\nu \) such that \( \sum_{\nu=2}^{p} \nu \alpha_\nu = p. \)

**Corollary 3.1.** From (10)–(13) and from (26), (27) we obtain:

\[
 \Gamma(z + 1) = 1 - \gamma z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < 1 \quad (28)
\]

where

\[
a_k = \frac{(-1)^k}{k!} \left( \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\tilde{\alpha}} \frac{1}{(2^{\alpha_2} \cdots \alpha_3 \cdots \alpha_p)!} \prod_{\nu=2}^{p} (\zeta(\nu))^{\alpha_\nu} \right); \quad (29)
\]
(Γ(z + 1))^{-1} = 1 + \gamma z + \sum_{k=2}^{\infty} d_k z^k, \quad |z| < \infty \tag{30}

where

\[ d_k = \frac{\gamma^k}{k!} + \sum_{p=2}^{k} \frac{\gamma^{k-p}}{(k-p)!} \sum_{\alpha} (-1)^{\sum_{\nu=2}^{\infty} \alpha_{\nu}} \frac{1}{(2^{\alpha_2} 3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2! \alpha_3! \cdots \alpha_p!)} \prod_{\nu=2}^{p} (\zeta(\nu)^{\alpha(\nu)}) \tag{31} \]

and \( \sum_{\alpha} \) means that we take the sum over all nonnegative integers \( \alpha_\nu \) such that \( \sum_{\nu=2}^{p} \nu \alpha_{\nu} = p. \)

In particular (see (2)),

\[ A(k) = \int_{0}^{\infty} e^{-t}(\ln t)^k dt = (-1)^k \left( \gamma^k + \sum_{p=2}^{k} \binom{k}{p} p! \gamma^{k-p} \sum_{\alpha} \frac{1}{(2^{\alpha_2} 3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2! \alpha_3! \cdots \alpha_p!)} \prod_{\nu=2}^{p} (\zeta(\nu)^{\alpha(\nu)}) \right) \]

and: \( a_0 = 1, a_1 = -\gamma, a_2 = \frac{1}{2}(\gamma^2 + \zeta(2)) = \frac{1}{2}(\gamma^2 + \frac{\pi^2}{6}), a_3 = -\frac{1}{6}(\gamma^3 + 3\zeta(2)\gamma + 2\zeta(3)) = -\frac{1}{6}(\gamma^3 + \frac{\pi^2}{2} \gamma + 2\zeta(3)); d_0 = 1, d_1 = \gamma, d_2 = \frac{1}{2}(\gamma^2 - \zeta(2)) = \frac{1}{2}(\gamma^2 - \frac{\pi^2}{6}), d_3 = \frac{1}{6}\gamma^3 - \frac{1}{2}\zeta(2)\gamma + \frac{1}{3}\zeta(3) = \frac{1}{6}\gamma^3 - \frac{\pi^2}{12} \gamma + \frac{1}{3}\zeta(3). \)

**Remark.** Using (1) and (28), one may observe that the Laurent series of \( \Gamma(z) \) around \( z = 0 \) is:

\[ \Gamma(z) = \frac{1}{z} - \gamma + \sum_{k=2}^{\infty} a_k z^{k-1}, \quad |z| < 1, \]

where \( a_k \) are given by (29). Also, using (1) and (30), one may observe the Maclaurin series of \( (\Gamma(z))^{-1} \) is

\[ \frac{1}{\Gamma(z)} = z + \gamma z^2 + \sum_{k=2}^{\infty} d_k z^{k+1}, \]

where \( d_k \) are given by (31).

We must note that the representation (30)–(31) for \( (\Gamma(z + 1))^{-1} \) is given in another form by Mika Sakata’s formula for \( (\Gamma(z))^{-1} \) in [8], where Sakata uses multiple zeta values.

### 4 Conclusion

In the paper two important results are found in Theorem 3.1 and Theorem 3.2. With the help of Theorem 3.1 we obtain explicit formulae for the coefficients of \( \Gamma(z + 1) \) and \( (\Gamma(z + 1))^{-1} \) in their Maclaurin series.

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