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Series expansion of the Gamma function and its reciprocal

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Abstract: In this paper we give representations for the coefficients of the Maclaurin series for $\Gamma(z+1)$ and its reciprocal (where Γ is Euler's Gamma function) with the help of a differential operator $\mathfrak{D}$, the exponential function and a linear functional * (in Theorem 3.1). As a result we obtain the following representations for Γ (in Theorem 3.2):

$$\Gamma(z+1) = \left(e^{-u(x)}e^{-z\mathfrak{D}}[e^{u(x)}]\right)^*,$$
$$\left(\Gamma(z+1)\right)^{-1} = \left(e^{u(x)}e^{-z\mathfrak{D}}[e^{-u(x)}]\right)^*.$$

Theorem 3.1 and Theorem 3.2 are our main results. With the help of the first theorem we give our approach for finding the coefficients of Maclaurin series for $\Gamma(z+1)$ and its reciprocal in an explicit form.
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1 Introduction

Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{C}$ - the complex number field, $\mathbb{E}$ - the set of all entire functions of one complex variable. For $F \in \mathbb{E}$ the operator $D^k : \mathbb{E} \to \mathbb{E}$ is defined by: $D^0[F(z)] := F^{(0)}(z) = F(z)$, $z \in \mathbb{C}$; $D^k[F(z)] := F^{(k)}(z)$, $k \in \mathbb{N}$, $z \in \mathbb{C}$, i.e., $D = \frac{d}{dz}$ and $D^k = \left(\frac{d}{dz}\right)^k$.

If $D^k[F(z)] = H_k(z)$, we define $D^k_s[F(z)] := H_k(s)$.

Further, we shall use the following notation: ζ for Riemann zeta function; γ for Euler's constant, i.e. $\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i} - \ln n\right) = 0.5772156649\ldots$; $\tilde{\zeta}(1) = \gamma$ and $\tilde{\zeta}(k) = \zeta(k)$ for each integer $k > 1$.

The Gamma function admits the following basic representations (see [7], pp. 31, 33, 34):

(i) $\boldsymbol{\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt}$, valid for $z \in \mathbb{C}$, $Rez > 0$ (Euler)
where Γ is a holomorphic function and $D^k[\Gamma(z)] = \int_0^\infty e^{-t}t^{z-1}(\ln t)^k dt$;

(ii) $\boldsymbol{\Gamma(z) = \lim_{m\to\infty}\frac{m!m^z}{z(z+1)\cdots(z+m)}}$, valid for $z \in \mathbb{C}$, $z \neq 0, -1, -2, \ldots$ (Euler–Gauss)

(iii) $\boldsymbol{\Gamma(z) = \frac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty}e^{\frac{z}{n}}\left(1+\frac{z}{n}\right)^{-1}}$, valid for $z \in \mathbb{C}$, $z \neq 0, -1, -2, \ldots$ (Weierstrass)

From (iii) it is seen that $(\Gamma(z))^{-1} \in \mathbb{E}$ and that $\Gamma(z)$ is a meromorphic function without zeroes and with simple poles: $z = 0, -1, -2, \ldots$.

We have $\Gamma(1) = 1$ and for any $z \in \mathbb{C}$ Γ satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z). \tag{1}$$

Hence $\Gamma(z+1)^{-1} \in \mathbb{E}$ and $\Gamma(z+1)$ is a meromoromorphic function without zeroes and with simple poles: $z = -1, -2, \ldots$

Also, from (i), the representations:

$$A(k) = \Gamma^{(k)}(1) = D_0^k[\Gamma(z+1)] = \int_0^\infty e^{-t}(\ln t)^k dt, \quad k = 0, 1, 2, \ldots, \tag{2}$$

hold.

Although the Gamma function was introduced by Euler about two hundred and ninety two years ago it still has its secrets. In the present paper we introduce a linear operator $\mathfrak{D} : \mathbb{E} \to \mathbb{E}$ on which our main results are based – Theorem 3.1 and Theorem 3.2. With the help of Theorem 3.1 we find explicit formulae for the coefficients of Maclaurin series of $\Gamma(z+1)$ and $(\Gamma(z+1))^{-1}$ (Theorem 3.3.). Here we must note that such type of formulae are given by other authors too. For example formulae for these coefficients are contained in: [3, 4, 6, 8].

2 A new operator $\mathfrak{D}$ and its basic properties

Definition 2.1. Let $u(z) \in \mathbb{E}$. The operator $\mathfrak{D}$ is defined by:

- $\mathfrak{D} = \mathfrak{D}^1$; $\quad \mathfrak{D}^0[u(z)] = u(z)$;
- $\mathfrak{D}\left[D^0[u(z)]\right] = D^1[u(z)]$;
- $\mathfrak{D}\left[D^k[u(z)]\right] = kD^{k+1}[u(z)], \quad k \geq 1$;
- $\mathfrak{D}^k[u(z)] = \mathfrak{D}[\mathfrak{D}^{k-1}[u(z)]], \quad k \geq 1$.

Lemma 2.1. *Let $u(z), F(z), G(z) \in \mathbb{E}$. Then*

(j_1) $\mathfrak{D}^k[u(z)] = (k-1)!D^k[u(z)] \quad (\forall k \in \mathbb{N})$

(j_2) $\mathfrak{D}^k$ *is a linear operator* $(\forall k \in \mathbb{N})$

(j_3) $\mathfrak{D}^k[F(z)G(z)] = \sum_{\nu=0}^{k} \binom{k}{\nu} \mathfrak{D}^{k-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)] \quad (\forall k \in \mathbb{N})$

(analogue of Leibnitz formula for $D^k[F(z)G(z)]$)

(j_4) $\mathfrak{D}[F(G(z))] = (DF)(G(z))\mathfrak{D}[G(z)]$

(j_5) *For $f, g \in \mathbb{E}$ and $n \in \mathbb{N}$ the following analogue of Faa di Bruno's formula for $\mathfrak{D}^n[f(g(z))]$:*

$$\mathfrak{D}^n\left[f\big(g(z)\big)\right] = \sum_{m=0}^{n} (D^m f)\big(g(z)\big) \sum_{\alpha} C_n(\alpha) \big(\mathfrak{D}^1[g(z)]\big)^{\alpha_1} \ldots \big(\mathfrak{D}^n[g(z)]\big)^{\alpha_n} \tag{3}$$

is true, where: $C_n(\alpha) = \frac{n!}{(1!)^{\alpha_1}.\alpha_1!(2!)^{\alpha_2}.\alpha_2!\ldots(n!)^{\alpha_n}.\alpha_n!}$ and $\sum_{\alpha}$ means that the sum is over all $\alpha = (\alpha_1, \ldots, \alpha_n)$, for which $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$, $\alpha_1 + \cdots + \alpha_n = m$ and $1.\alpha_1 + \cdots + n.\alpha_n = n$.

Proof. (j_1) follows by induction from the definition of $\mathfrak{D}$.

Let $\lambda, \mu \in \mathbb{C}$. From (j_1) and from the linearity of D^k we obtain:

$$\begin{aligned}\mathfrak{D}^k[\lambda F(z) + \mu G(z)] &= (k-1)!D^k[\lambda F(z) + \mu G(z)] = \lambda(k-1)!D^k[F(z)] + \mu(k-1)!D^k[G(z)] \\ &= \lambda\mathfrak{D}^k[F(z)] + \mu\mathfrak{D}^k[G(z)],\end{aligned}$$

which proves (j_2).

Let us prove (j_3) by induction. In the case $k = 1$, $\mathfrak{D}$ coincides with D and (j_3) is obvious. If for $k \geq 1$ (j_3) is true, then applying $\mathfrak{D}$ to (j_3) and using the linearity of $\mathfrak{D}$ we obtain:

$$\mathfrak{D}^{k+1}[F(z)G(z)] = \sum_{\nu=0}^{k} \binom{k}{\nu} \mathfrak{D}\left[\mathfrak{D}^{k-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)]\right]$$

The right-hand side R in the above equality is $I_1 + I_2$, where:

$$\begin{aligned}I_1 &= \mathfrak{D}^{k+1}[F(z)]\mathfrak{D}^0[G(z)] + \sum_{\nu=1}^{k} \binom{k}{\nu} \mathfrak{D}^{k+1-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)]; \\ I_2 &= \sum_{\nu=1}^{k} \binom{k}{\nu-1} \mathfrak{D}^{k+1-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)] + \mathfrak{D}^0[F(z)]\mathfrak{D}^{k+1}[G(z)].\end{aligned}$$

I_2 is obtained after substitution $\nu + 1 = t$ and replacing t with ν.

Now using that $\binom{k}{\nu} + \binom{k}{\nu-1} = \binom{k+1}{\nu}$, $\nu = 1, \ldots, k$, we obtain

$$R = \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} \mathfrak{D}^{k+1-\nu}[F(z)]\mathfrak{D}^{\nu}[G(z)]$$

and (j_3) is proved.

The proof of (j_4) is obvious since $D[F(G(z))] = (DF)(G(z))D[G(z)]$ and we may replace $D[G(z)]$ with $\mathfrak{D}[G(z)]$ (see Definition 2.1).

Let us prove (j_5). We consider the equalities:

$$D[f(g(z))] = (Df)(g(z))D[g(z)],$$
$$\mathfrak{D}[f(g(z))] = (Df)(g(z))\mathfrak{D}[g(z)]$$

(see (j_4)).

Applying to the left one D^{n-1} and to the right one $\mathfrak{D}^{n-1}$ and after that using Leibnitz formula for the right-hand side of the first equality and the analogue of the Leibnitz formula (see (j_3)) for the right-hand side of the second equality, one may see that $D^n[f(g(z))]$ and $\mathfrak{D}^n[f(g(z))]$ have the same structure with only one difference: $D^m[g(z)]$ for the first expression is replaced with $\mathfrak{D}^m[g(z)]$ for the second expression. But since we have the Faa di Bruno's formula (see [1], p.823):

$$D^n\big[f\big(g(z)\big)\big] = \sum_{m=0}^{n}(D^m f)\big(g(z)\big)\sum_{\alpha} C_n(\alpha)\big(D^1[g(z)]\big)^{\alpha_1}\dots\big(D^n[g(z)]\big)^{\alpha_n},$$

then replacing $D^m(g(z))$ with $\mathfrak{D}^m(g(z))$ in it, we obtain exactly (3). □

Corollary 2.1. *Formula (3) admits the representation*

$$\mathfrak{D}^n\big[f\big(g(z)\big)\big] = \sum_{m=0}^{n}(D^m f)\big(g(z)\big)\sum_{\alpha} C_n(\alpha)\prod_{\nu=1}^{n}((\nu-1)!)^{\alpha_\nu}\big(D^1[g(z)]\big)^{\alpha_1}\cdots\big(D^n[g(z)]\big)^{\alpha_n} \quad (4)$$

Proof. It follows immediately from Lemma 2.1, (j_1). □

Corollary 2.2. *Formula (4) admits the representation*

$$\mathfrak{D}^n\big[f\big(g(z)\big)\big] = \sum_{m=0}^{n}(D^m f)\big(g(z)\big)\sum_{\alpha} A_n(\alpha)\big(D^1[g(z)]\big)^{\alpha_1}\cdots\big(D^n[g(z)]\big)^{\alpha_n}, \quad (5)$$

where

$$A_n(\alpha) = \frac{n!}{(1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\cdots n^{\alpha_n})(\alpha_1!\alpha_2!\alpha_3!\cdots\alpha_n!)} \quad (6)$$

Proof. One may check directly that

$$C_n(\alpha)\prod_{\nu=1}^{n}((\nu-1)!)^{\alpha_\nu} = A_n(\alpha).$$ □

Corollary 2.3. *Let $u \in \mathbb{E}$ and $n \in \mathbb{N}$. Then:*

$$\begin{aligned}\mathfrak{D}^n[e^{u(z)}] &= e^{u(z)}\sum_{m=0}^{n}\sum_{\alpha} A_n(\alpha)\big(D^1[u(z)]\big)^{\alpha_1}\cdots\big(D^n[u(z)]\big)^{\alpha_n}\\ \mathfrak{D}^n[e^{-u(z)}] &= e^{-u(z)}\sum_{m=0}^{n}\sum_{\alpha}(-1)^{\sum_{\nu=1}^{n}\alpha_\nu}A_n(\alpha)\big(D^1[u(z)]\big)^{\alpha_1}\cdots\big(D^n[u(z)]\big)^{\alpha_n}\end{aligned} \quad (7)$$

3 Maclaurin series for $\Gamma(z+1)$ and its reciprocal

If $k \geq 0$ we define a_k, d_k, b_k, ρ_k by:

$$k!a_k = D_0^k[\Gamma(z+1)] \tag{8}$$

$$k!d_k = D_0^k[(\Gamma(z+1))^{-1}] \tag{9}$$

$$b_k = (-1)^k k!a_k \tag{10}$$

$$\rho_k = k!d_k \tag{11}$$

Hence:

$$\Gamma(z+1) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 1, \tag{12}$$

$$\big(\Gamma(z+1)\big)^{-1} = \sum_{k=0}^{\infty} d_k z^k, \quad |z| < \infty. \tag{13}$$

For the sequences $\{a_k\}$ and $\{d_k\}$ recurrence relations are known (see [5], pp. 12, 17; [2], p. 12) that we give below but for the sequences $\{b_k\}$ and $\{\rho_k\}$:

$$b_{k+1} = \sum_{\nu=0}^{k} \binom{k}{\nu} (k-\nu)!\tilde{\zeta}(k-\nu+1) b_\nu, \quad k \geq 0, \ b_0 = 1, \tag{14}$$

$$\rho_{k+1} = \sum_{\nu=0}^{k} \binom{k}{\nu} (k-\nu)!(-1)^{k-\nu}\tilde{\zeta}(k-\nu+1) \rho_\nu, \quad k \geq 0, \ \rho_0 = 1. \tag{15}$$

3.1 Connection between the Gamma function and the exponential function

Definition 3.1. Let $F(z_1, z_2, \ldots, z_k)$ be a polynomial und $u(z) \in \mathbb{E}$. Then we define the mapping * by:

$$\big(F\big(D^1[u(z)], D^2[u(z)], \ldots, D^k[u(z)]\big)\big)^* := F\big(\tilde{\zeta}(1), \tilde{\zeta}(2), \ldots, \tilde{\zeta}(k)\big).$$

Remark. From the above definition it is clear that * is a linear and multiplicative mapping. The multiplicativity of * means that if $F(z_1, \ldots, z_k)$ and $G(z_1, \ldots, z_m)$ are polynomials and

$$H(z_1, \ldots, z_n) = F(z_1, \ldots, z_k) G(z_1, \ldots, z_m),$$

then

$$\big(H\big(D^1[u(z)], \ldots, D^n[u(z)]\big)\big)^* = \big(F\big(D^1[u(z)], \ldots, D^k[u(z)]\big)\big)^* \big(G\big(D^1[u(z)], \ldots, D^m[u(z)]\big)\big)^*.$$

Our first main result in this paper is the following.

Theorem 3.1. $\forall k \in \mathbb{N} \cup \{0\}$ b_k *and* ρ_k *are given by the formulae:*

$$b_k = \big(e^{-u(z)} \mathfrak{D}^k [e^{u(z)}]\big)^*, \tag{16}$$

$$\rho_k = \big((-1)^k e^{u(z)} \mathfrak{D}^k [e^{-u(z)}]\big)^*. \tag{17}$$

Proof. We shall prove only (16) since one may prove (17) in the same way. We prove (16) by induction. For $k = 0$ we have $b_0 = 1$ and $(e^{-u(z)}\mathfrak{D}^k[e^{u(z)}])^* = (e^{-u(z)}\mathfrak{D}^0[e^{u(z)}])^* = (\{1\})^* = 1$. Assume that (16) is true for some $k \geq 0$. We will show that (16) is true for $k+1$ too. Let

$$b'_{k+1} = (e^{-u(z)}\mathfrak{D}^{k+1}[e^{u(z)}])^*.$$

Then

$$\begin{aligned} b'_{k+1} &= (e^{-u(z)}\mathfrak{D}^k[\mathfrak{D}[e^{u(z)}]])^* = (e^{-u(z)}\mathfrak{D}^k[D[e^{u(z)}]])^* \\ &= (e^{-u(z)}\mathfrak{D}^k[e^{u(z)}D[u(z)]])^* = (e^{-u(z)}\mathfrak{D}^k[e^{u(z)}\mathfrak{D}[u(z)]])^* \end{aligned}$$

$\big[$now we apply Lemma 2.1, (j_3) to $\mathfrak{D}^k[e^{u(z)}\mathfrak{D}[u(z)]]\big]$

$$\begin{aligned} &= \big(e^{-u(z)}\sum_{\nu=0}^{k}\binom{k}{\nu}\mathfrak{D}^{k+1-\nu}[u(z)]\mathfrak{D}^{\nu}[e^{u(z)}]\big)^* \\ &= \big(\sum_{\nu=0}^{k}\binom{k}{\nu}\mathfrak{D}^{k+1-\nu}[u(z)]\big(e^{-u(z)}\mathfrak{D}^{\nu}[e^{u(z)}]\big)\big)^* \end{aligned}$$

[now we use $\mathfrak{D}^{k+1-\nu}[u(z)] = (k-\nu)!D^{k+1-\nu}[u(z)]$, see Lemma 2.1, (j_1)]

$$= \left(\sum_{\nu=0}^{k}\binom{k}{\nu}(k-\nu)!D^{k+1-\nu}[u(z)]\big(e^{-u(z)}\mathfrak{D}^{\nu}[e^{u(z)}]\big)\right)^*$$

[from the linearity and multiplicativity of *]

$$= \sum_{\nu=0}^{k}\binom{k}{\nu}(k-\nu)!\big(D^{k+1-\nu}[u(z)]\big)^*(e^{-u(z)}\mathfrak{D}^{\nu}[e^{u(z)}])^*$$

[from the induction hypothesis we have $\big(e^{-u(z)}\mathfrak{D}^{\nu}[e^{u(z)}]\big)^* = b_\nu$, $0 \leq \nu \leq k$]

$$= \sum_{\nu=0}^{k}\binom{k}{\nu}(k-\nu)!\tilde{\zeta}(k-\nu+1)b_\nu = b_{k+1}$$

(the last from (14)). Thus we proved $b'_{k+1} = b_{k+1}$ and (16) is proved. □

From Theorem 3.1, using the fact that the mapping * is linear and that the generating function of the operator $\mathfrak{D}$:

$$\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\mathfrak{D}^k$$

is equal to $e^{-z\mathfrak{D}}$, we obtain our second main result in this paper.

Theorem 3.2. $\Gamma(z+1)$ *and its reciprocal have the following important representations:*

$$\Gamma(z+1) = \left(e^{-u(x)}e^{-z\mathfrak{D}}[e^{u(x)}]\right)^{*}, \tag{18}$$

$$\left(\Gamma(z+1)\right)^{-1} = \left(e^{u(x)}e^{-z\mathfrak{D}}[e^{-u(x)}]\right)^{*}, \tag{19}$$

where * *acts only with respect to the variable* x.

Proof. We prove only (18) since one may prove (19) analogically.

$$\begin{aligned}(e^{-u(x)}e^{-z\mathfrak{D}}[e^{u(x)}])^{*} &= \left(e^{-u(x)}\left(\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\mathfrak{D}^k\right)[e^{u(x)}]\right)^{*}\\ &= \left(e^{-u(x)}\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\mathfrak{D}^k[e^{u(x)}]\right)^{*}\\ &= \left(\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}e^{-u(x)}\mathfrak{D}^k[e^{u(x)}]\right)^{*}\end{aligned}$$

[here we suppose that * is defined not only for polynomials but for power series, too]

$$= \sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\left(e^{-u(x)}\mathfrak{D}^k[e^{u(x)}]\right)^{*}$$

[from Theorem 3.1]

$$= \sum_{k=0}^{\infty}\frac{(-z)^k}{k!}b_k = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}b_k z^k = \sum_{k=0}^{\infty}a_k z^k = \Gamma(z+1). \qquad \square$$

3.2 Explicit formulae for the coefficients

From Theorem 3.1 and Corollary 2.3 we obtain using (16) and (17):

$$b_n = \sum_{m=0}^{n}\sum_{\alpha}A_n(\alpha)\left(\tilde{\zeta}(1)\right)^{\alpha_1}\cdots\left(\tilde{\zeta}(n)\right)^{\alpha_n} \quad \text{(compare with [9], the formula for } \Gamma^{(n)}(1)\text{)}$$

$$\rho_n = (-1)^n\sum_{m=0}^{n}\sum_{\alpha}(-1)^{\sum_{\nu=1}^{n}\alpha_\nu}A_n(\alpha)\left(\tilde{\zeta}(1)\right)^{\alpha_1}\cdots\left(\tilde{\zeta}(n)\right)^{\alpha_n}$$

In [6], p. 43, (10) and (11), explicit formulae for a_n and d_n are given.

Further we shall find formulae for b_n and ρ_n in a polynomial form, of one variable, using instead of this variable Euler's constant γ. For this purpose we need two lemmas.

Lemma 3.1. *For* $k \geq 2$ *the following identities are valid*

$$e^{-u(z)}\mathfrak{D}^k[e^{u(z)}] = (D^1[u(z)])^k + \sum_{p=2}^{k}\binom{k}{p}.\Delta_p.(D^1[u(z)])^{k-p}, \tag{20}$$

$$(-1)^k e^{u(z)}\mathfrak{D}^k[e^{-u(z)}] = (D^1[u(z)])^k + \sum_{p=2}^{k}\binom{k}{p}.\sigma_p.(D^1[u(z)])^{k-p}, \tag{21}$$

where:

$$\Delta_2 = D^2[u(z)], \quad \Delta_3 = \mathfrak{D}[D^2[u(z)]], \quad \Delta_{p+1} = p.(D^2[u(z)]).\Delta_{p-1} + \mathfrak{D}[\Delta_p], \quad p > 3, \tag{22}$$

$$\sigma_2 = -D^2[u(z)], \quad \sigma_3 = \mathfrak{D}[D^2[u(z)]], \quad \sigma_{p+1} = -p.(D^2[u(z)]).\sigma_{p-1} - \mathfrak{D}[\sigma_p], \quad p > 3. \tag{23}$$

Proof. We prove (20) by induction. For $k = 2$ and $k = 3$ we check directly the validity of (20). Let $k \geq 3$. We denote by R_k the right-hand side of (20) and let (20) be true for some $k \geq 3$. For $k+1$ we must verify that $e^{-u}\mathfrak{D}^{k+1}[e^u] = R_{k+1}$, i.e., $R_{k+1} = e^{-u}\mathfrak{D}\big[\mathfrak{D}^k[e^u]\big]$ $= e^{-u}\mathfrak{D}\big[e^u e^{-u}\mathfrak{D}^k[e^u]\big]$ = (from our assumption for k) $= e^{-u}\mathfrak{D}[e^u R_k]$ = (after computation) $= D[u]R_k + \mathfrak{D}[R_k]$. Thus, to prove (20), it remains to prove that $R_{k+1} = D[u]R_k + \mathfrak{D}[R_k]$.

It is easy to see that the coefficient in front of $(D^1[u(z)])^{k+1-p}$ in the left-hand side of the last equality equals to $\binom{k+1}{p}\Delta_p$ and in the right-hand side equals to $(k+2-p)\binom{k}{p-2}\Delta_{p-2}.D^2[u(z)] + \binom{k}{p}\Delta_p + \binom{k}{p-1}\mathfrak{D}[\Delta_{p-1}]$. So, to prove (20), we must check that

$$\binom{k+1}{p}\Delta_p = (k+2-p)\binom{k}{p-2}\Delta_{p-2}.D^2[u(z)] + \binom{k}{p}\Delta_p + \binom{k}{p-1}\mathfrak{D}[\Delta_{p-1}].$$

One may easily check the above equality using (22) and the well-known relations:

$$\binom{k+1}{p} = \binom{k}{p} + \binom{k}{p-1}; \quad t\binom{k}{t} = k\binom{k-1}{t-1}.$$

In the same way, one may prove (21) (using (23)) and Lemma 3.1 is proved. □

Lemma 3.2. *For* $p \geq 2$ *the following representations hold:*

$$\Delta_p = \sum_{\tilde{\alpha}} A_p(\tilde{\alpha})\big(D^2[u(z)]\big)^{\alpha_2} \cdots \big(D^p[u(z)]\big)^{\alpha_p}, \tag{24}$$

$$\sigma_p = \sum_{\tilde{\alpha}} (-1)^{p+\sum_{\nu=2}^{p}\alpha_\nu} A_p(\tilde{\alpha})\big(D^2[u(z)]\big)^{\alpha_2} \cdots \big(D^p[u(z)]\big)^{\alpha_p}, \tag{25}$$

where, for $\tilde{\alpha} := (\alpha_2, \ldots, \alpha_p)$, $A_p(\tilde{\alpha})$ *is given by:*

$$A_p(\tilde{\alpha}) = p!\prod_{\nu=2}^{p}\big((\nu-1)!\big)^{\alpha_\nu}\prod_{\nu=2}^{p}\big(\alpha_\nu!(\nu!)^{\alpha_\nu}\big)^{-1} = \frac{p!}{(2^{\alpha_2}3^{\alpha_3}\cdots p^{\alpha_p})(\alpha_2!\alpha_3!\cdots\alpha_p!)}$$

and $\sum_{\tilde{\alpha}}$ *means that the sum is over all nonnegative integers* α_ν *such that* $\sum_{\nu=2}^{p}\nu\alpha_\nu = p$.

Proof. We shall prove only (24) since (25) may be proved in the same way. First in (22) we write $\Delta_t^{\mathfrak{D}}$ instead of Δ_t, $\forall t \geq 2$. Second we replace in (22) $\mathfrak{D}$ with D. As a result we obtain:

$$\Delta_2^D = D^2[u(z)], \quad \Delta_3^D = D[D^2[u(z)]], \quad \Delta_{p+1}^D = p.(D^2[u(z)]).\Delta_{p-1}^D + D[\Delta_p^D], \quad p > 3.$$

Now we shall prove that

$$\Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2}.\alpha_2!\cdots(p!)^{\alpha_p}.\alpha_p!}\big(D^2[u(z)]\big)^{\alpha_2}\cdots\big(D^p[u(z)]\big)^{\alpha_p}$$

using induction with respect to p. For $p = 2, 3$ the validity of the above equality is a matter of direct check. Let $\tilde{\alpha} = \tilde{\alpha}(p)$, $\tilde{C}_p(\tilde{\alpha}(p)) = \frac{p!}{(2!)^{\alpha_2}.\alpha_2!\cdots(p!)^{\alpha_p}.\alpha_p!}$ and the equality is checked for $\nu = 2, 3, \ldots, p$. For the sake of brevity, let $\tilde{C}_p := \tilde{C}_p(\tilde{\alpha}(p))$.

We must prove that

$$\sum_{\tilde{\alpha}(p+1)} \tilde{C}_{p+1} \prod_{\nu=2}^{p+1} (D^\nu[u(z)])^{\alpha_\nu} = \Delta_{p+1}^D = p.(D^2[u(z)]).\Delta_{p-1}^D + D[\Delta_p^D],$$

i.e. that the equality (further denoted by **H**):

$$\sum_{\tilde{\alpha}(p+1)} \tilde{C}_{p+1} \prod_{\nu=2}^{p+1} (D^\nu[u(z)])^{\alpha_\nu} = \sum_{\tilde{\alpha}(p-1)} p\tilde{C}_{p-1} (D^2[u(z)])^{\alpha_2+1} \prod_{\nu=3}^{p-1} (D^\nu[u(z)])^{\alpha_\nu}$$
$$+ \sum_{j=2}^{p-1} \sum_{\tilde{\alpha}(p)} \alpha_j \tilde{C}_p.P.\big(D^j[u(z)]\big)^{\alpha_j-1} (D^{j+1}[u(z)])^{\alpha_{j+1}+1}.Q + D^{p+1}[u(z)],$$

holds, where P and Q are given by:

$$P = \prod_{\nu=2}^{j-1} \big(D^\nu[u(z)]\big)^{\alpha_\nu}; \quad Q = \prod_{\nu=j+2}^{p-1} \big(D^\nu[u(z)]\big)^{\alpha_\nu}.$$

Let $\prod_{\nu=2}^{p-1}(D^\nu[u(z)])^{\gamma_\nu}$ be an arbitrary monomial from the right-hand side of **H**. Below we calculate the coefficient C in front of this monomial:

$$C = p\tilde{C}_{p-1}(\gamma_2 - 1, \gamma_3, \ldots, \gamma_{p+1}) + \sum_{j=2}^{p-1} (\gamma_j + 1) \tilde{C}_p(\gamma_2, \ldots, \gamma_j + 1, \gamma_{j+1} - 1, \ldots, \gamma_{p+1})$$
$$= p! \frac{\sum_{\nu=2}^{p+1} \nu\gamma_\nu}{\prod_{\nu=2}^{p+1} (\nu!)^{\gamma_\nu} \gamma_\nu!} = \frac{(p+1)!}{\prod_{\nu=2}^{p+1} (\nu!)^{\gamma_\nu} \gamma_\nu!} = \tilde{C}_{p+1}(\gamma_2, \ldots, \gamma_{p+1}).$$

Hence, the right-hand side of **H** is a sum of the terms $\tilde{C}_{p+1}(\gamma_2, ..., \gamma_{p+1}) \prod_{\nu=2}^{p+1}(D^\nu[u(z)])^{\gamma_\nu}$, for which $\sum_{\nu=2}^{p+1} \nu\gamma_\nu = p + 1$. Then, to prove **H**, it remains to show that in the right-hand side of **H** all partitions $\sum_{\nu=2}^{p+1} \nu\gamma_\nu = p + 1$ are met.

Indeed, let $\prod_{\nu=2}^{p+1}(D^\nu[u(z)])^{\gamma_\nu}$ be an arbitrary monomial from the left-hand side of **H**. If $\gamma_{p+1} = 1$, then $\gamma_2 = \gamma_3 = \cdots = \gamma_p = 0$, so this monomial is $D^{p+1}[u(z)]$ and it is contained in the right-hand side of **H**, too. If $\gamma_{p+1} = 0$ but for some j, such that $2 < j \le p$, $\gamma_j \ne 0$, we consider in the right-hand side of **H** the sum:

$$\sum_{\tilde{\alpha}(p)} \alpha_{j-1} \tilde{C}_p(\alpha_2, \ldots, \alpha_p) \left(\prod_{\nu=2}^{j-2} \big(D^\nu[u(z)]\big)^{\alpha_\nu} \right) \big(D^{j-1}[u(z)]\big)^{\alpha_{j-1}-1} \big(D^j[u(z)]\big)^{\alpha_j+1} \prod_{\nu=j+1}^{p} \big(D^\nu[u(z)]\big)^{\alpha_\nu}.$$

Now we set: $\alpha_2 = \gamma_2, \ldots, \alpha_{j-2} = \gamma_{j-2}; \alpha_{j-1} = \gamma_{j-1}+1; \alpha_j = \gamma_j - 1; \alpha_{j+1} = \gamma_{j+1}, \ldots, \alpha_p = \gamma_p$. Then $\sum_{\nu=2}^{p} \nu\alpha_\nu = (\sum_{\nu=2}^{p+1} \nu\gamma_\nu) - 1 = (p + 1) - 1 = p$. Hence, according to the given definition of $\sum_{\tilde{\alpha}(p)}$ the monomial $\prod_{\nu=2}^{p+1}(D^\nu[u(z)])^{\gamma_\nu}$ is contained in the right-hand side of **H**.

If $\gamma_2 \ne 0$, then we consider in the right-hand side of **H** the sum:

$$\sum_{\tilde{\alpha}(p-1)} p\tilde{C}_{p-1}(\alpha_2, \ldots, \alpha_{p-1}) (D^2[u(z)])^{\alpha_2+1} \prod_{\nu=3}^{p-1} (D^\nu[u(z)])^{\alpha_\nu}.$$

Letting $\alpha_2 = \gamma_2 - 1, \alpha_3 = \gamma_3, \ldots, \alpha_{p-1} = \gamma_{p-1}$, we obtain $\sum_{\nu=2}^{p-1} \nu\alpha_\nu = \left(\sum_{\nu=2}^{p+1} \nu\gamma_\nu\right) - 2 = p-1$. Hence according to the definition of $\sum_{\tilde{\alpha}(p-1)}$ the monomial $\prod_{\nu=2}^{p+1} \left(D^\nu[u(z)]\right)^{\gamma_\nu}$ is contained in the right-hand side of **H**.

Thus we proved by induction that

$$\Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2}.\alpha_2! \cdots (p!)^{\alpha_p}.\alpha_p!} \left(D^2[u(z)]\right)^{\alpha_2} \cdots \left(D^p[u(z)]\right)^{\alpha_p}.$$

In the above equality we replace D with $\mathfrak{D}$ (using the fact that Lemma 3.1 remains valid if we replace in (22) and (23) $\mathfrak{D}$ with D) and obtain

$$\Delta_p^{\mathfrak{D}} = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2}.\alpha_2! \cdots (p!)^{\alpha_p}.\alpha_p!} \left(\mathfrak{D}^2[u(z)]\right)^{\alpha_2} \cdots \left(\mathfrak{D}^p[u(z)]\right)^{\alpha_p}.$$

Now, using the fact that $\mathfrak{D}^k[u(z)] = (k-1)!D^k[u(z)]$ (see Lemma 2.1, (j_1)), we obtain

$$\Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2!)^{\alpha_2}.\alpha_2! \cdots (p!)^{\alpha_p}.\alpha_p!} \prod_{\nu=2}^{p} ((\nu-1)!)^{\alpha_\nu} \left(D^2[u(z)]\right)^{\alpha_2} \cdots \left(D^p[u(z)]\right)^{\alpha_p}.$$

Hence

$$\Delta_p = \Delta_p^D = \sum_{\tilde{\alpha}} \frac{p!}{(2^{\alpha_2}3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2!\alpha_3! \cdots \alpha_p!)} \left(D^2[u(z)]\right)^{\alpha_2} \cdots \left(D^p[u(z)]\right)^{\alpha_p}.$$

Thus (24) and therefore Lemma 3.2 are proved. □

From Theorem 3.1, Lemma 3.1 and Lemma 3.2 we obtain the following explicit formulae for the coefficients.

Theorem 3.3. *The following representations are valid:*

$$b_k = \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\tilde{\alpha}} A_p(\tilde{\alpha}) \prod_{\nu=2}^{p} \left(\zeta(\nu)\right)^{\alpha_\nu}; \tag{26}$$

$$\rho_k = \gamma^k + \sum_{p=2}^{k} \binom{k}{p} \gamma^{k-p} \sum_{\tilde{\alpha}} (-1)^{p+\sum_{\nu=2}^{p}\alpha_\nu} A_p(\tilde{\alpha}) \prod_{\nu=2}^{p} \left(\zeta(\nu)\right)^{\alpha_\nu}, \tag{27}$$

where $\sum_{\tilde{\alpha}}$ *means that we sum over all nonnegative integers* α_ν *such that* $\sum_{\nu=2}^{p} \nu\alpha_\nu = p$.

Corollary 3.1. *From (10)–(13) and from (26), (27) we obtain:*

$$\Gamma(z+1) = 1 - \gamma z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < 1 \tag{28}$$

where

$$a_k = \frac{(-1)^k}{k!} \left(\gamma^k + \sum_{p=2}^{k} \binom{k}{p} p! \gamma^{k-p} \sum_{\tilde{\alpha}} \frac{1}{(2^{\alpha_2}3^{\alpha_3} \cdots p^{\alpha_p})(\alpha_2!\alpha_3! \cdots \alpha_p!)} \prod_{\nu=2}^{p} \zeta(\nu)^{\alpha(\nu)}\right); \tag{29}$$

$$(\Gamma(z+1))^{-1} = 1 + \gamma z + \sum_{k=2}^{\infty} d_k z^k, \quad |z| < \infty \tag{30}$$

where

$$d_k = \frac{\gamma^k}{k!} + \sum_{p=2}^{k} \frac{\gamma^{k-p}}{(k-p)!} \sum_{\tilde{\alpha}} (-1)^{p+\sum_{\nu=2}^{p}\alpha_\nu} \frac{1}{(2^{\alpha_2}3^{\alpha_3}\cdots p^{\alpha_p})(\alpha_2!\alpha_3!\cdots\alpha_p!)} \prod_{\nu=2}^{p} (\zeta(\nu))^{\alpha(\nu)} \tag{31}$$

and $\sum_{\tilde{\alpha}}$ *means that we take the sum over all nonnegative integers* α_ν *such that* $\sum_{\nu=2}^{p} \nu\alpha_\nu = p$.

In particular (see (2)):

$$A(k) = \int_0^{\infty} e^{-t}(\ln t)^k dt = (-1)^k \left(\gamma^k + \sum_{p=2}^{k} \binom{k}{p} p! \gamma^{k-p} \sum_{\tilde{\alpha}} \frac{1}{(2^{\alpha_2}3^{\alpha_3}...p^{\alpha_p})(\alpha_2!\alpha_3!...\alpha_p!)} \prod_{\nu=2}^{p} \zeta(\nu)^{\alpha(\nu)}\right)$$

and: $a_0 = 1$, $a_1 = -\gamma$, $a_2 = \frac{1}{2}(\gamma^2 + \zeta(2)) = \frac{1}{2}(\gamma^2 + \frac{\pi^2}{6})$, $a_3 = -\frac{1}{6}(\gamma^3 + 3\zeta(2)\gamma + 2\zeta(3)) = -\frac{1}{6}(\gamma^3 + \frac{\pi^2}{2}\gamma + 2\zeta(3))$; $d_0 = 1$, $d_1 = \gamma$, $d_2 = \frac{1}{2}(\gamma^2 - \zeta(2)) = \frac{1}{2}(\gamma^2 - \frac{\pi^2}{6})$, $d_3 = \frac{1}{6}\gamma^3 - \frac{1}{2}\zeta(2)\gamma + \frac{1}{3}\zeta(3) = \frac{1}{6}\gamma^3 - \frac{\pi^2}{12}\gamma + \frac{1}{3}\zeta(3)$.

Remark. Using (1) and (28), one may observe that the Laurent series of $\Gamma(z)$ around $z = 0$ is:

$$\Gamma(z) = \frac{1}{z} - \gamma + \sum_{k=2}^{\infty} a_k z^{k-1}, \quad |z| < 1,$$

where a_k are given by (29). Also, using (1) and (30), one may observe the Maclaurin series of $(\Gamma(z))^{-1}$ is

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + \sum_{k=2}^{\infty} d_k z^{k+1},$$

where d_k are given by (31).

We must note that the representation (30)–(31) for $(\Gamma(z+1))^{-1}$ is given in another form by Mika Sakata's formula for $(\Gamma(z))^{-1}$ in [8], where Sakata uses multiple zeta values.

4 Conclusion

In the paper two important results are found in Theorem 3.1 and Theorem 3.2. With the help of Theorem 3.1 we obtain explicit formulae for the coefficients of $\Gamma(z+1)$ and $(\Gamma(z+1))^{-1}$ in their Maclaurin series.

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