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Explicit formulas for Euler polynomials and Bernoulli numbers

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Abstract: In this paper, we give several explicit formulas involving the n-th Euler polynomial $E_n(x)$. For any fixed integer $m \geq n$, the obtained formulas follow by proving that $E_n(x)$ can be written as a linear combination of the polynomials x^n, $(x+r)^n, \ldots, (x+rm)^n$, with $r \in \left\{1, -1, \frac{1}{2}\right\}$. As consequence, some explicit formulas for Bernoulli numbers may be deduced.
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1 Introduction

A polynomial sequence $A = (A_n(x))_{n \geq 0}$ is called an Appell sequence [2] if one of the following equivalent conditions is satisfied

$$A'_n(x) = nA_{n-1}(x), \quad n \geq 1 \text{ and } A_0(x) \text{ is a non-zero constant,} \tag{1}$$

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = S_A(t)\, e^{xt}, \text{ where } S_A(t) \text{ is a formal power series such } S_A(0) \neq 0, \tag{2}$$

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} A_k(0)\, x^{n-k} \text{ with } A_0(0) \neq 0. \tag{3}$$

Let $r \neq 0$ be a complex number and $m \geq 0$ be an integer. It is easy to see that the family of polynomials $\{x^m, (x+r)^m, (x+2r)^m, \ldots, (x+mr)^m\}$ forms a base of the $\mathbb{C}$-vectorial space $\mathbb{C}_m[x] := \{P(x) \in \mathbb{C}[x] : \deg P(x) \leq m\}$. For any Appell sequence A, $A_m(x)$ is a polynomial of degree m which we want to decompose on this basis. Therefore, there exists a unique sequence of complex numbers $\mu_j = \mu_j(A, r, m)$ such that

$$A_m(x) = \sum_{j=0}^{m} \mu_j (x + rj)^m. \tag{4}$$

Note that for $0 \leq n \leq m$, by (1), we have

$$A_m^{(m-n)}(x) = \frac{m!}{n!} A_n(x).$$

Then, by differentiating $m-n$ times the two sides of (4) and dividing by $\frac{m!}{n!}$, we deduce that we have more generally

$$A_n(x) = \sum_{j=0}^{m} \mu_j (x + rj)^n, 0 \leq n \leq m. \tag{5}$$

The aim of this article is to determine simple expressions of $\mu_j(E, r, m)$ for $r \in \left\{1, -1, \frac{1}{2}\right\}$ and to deduce explicit formulas for Euler polynomials and Bernoulli numbers, where $E = (E_n(x))_{n \geq 0}$ is the Appell sequence of Euler polynomials defined by their exponential generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \qquad (|t| < \pi). \tag{6}$$

2 Lemmas

To obtain the desired expressions and explicit formulas, we give some lemmas which will be used later.

Lemma 2.1. *For all integers j and m such that $0 \leq j \leq m$, we have*

$$\sum_{k=j}^{m} \frac{(-1)^j}{2^k} \binom{k}{j} = \frac{(-1)^j}{2^m} \sum_{k=j}^{m} \binom{m+1}{k-j} \tag{7}$$

and

$$\sum_{k=j}^{m} \frac{(-1)^j}{2^k} \binom{k}{j} = \sum_{k=j}^{m} \frac{(-1)^k}{2^k} \binom{m+1}{k+1} \binom{k}{j}. \tag{8}$$

Proof. We have [7]

$$\sum_{k=j}^{m} \binom{k}{j} x^k = \sum_{k=j}^{m} \binom{m+1}{k-j} x^k (1-x)^{m-k}. \tag{9}$$

In particular, for $x = \frac{1}{2}$, we get (7) after multiplying by $(-1)^j$.
By deriving j times and dividing by $j!$ the two sides of the identity

$$\sum_{k=0}^{m} x^k = \sum_{k=0}^{m} \binom{m+1}{k+1} (x-1)^k ,$$

we obtain

$$\sum_{k=j}^{m} \binom{k}{j} x^{k-j} = \sum_{k=j}^{m} \binom{m+1}{k+1} \binom{k}{j} (x-1)^{k-j} .$$

The identity (8) follows by substituting $x = \frac{1}{2}$ in this identity and multiplying by $(-2)^{-j}$. $\square$

Lemma 2.2. *We have*

$$\frac{1}{1+t+\frac{1}{2}t^2} = \sum_{k=0}^{\infty} a_k t^k \qquad \left(|t| < \sqrt{2}\right),$$

where a_k is

$$a_k = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+k}}{2^s} \binom{k-s}{s}, \tag{10}$$

$$a_k = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+k}}{2^k} \binom{k+1}{2s+1}, \tag{11}$$

$$a_k = \frac{(-1)^k}{2^{\frac{k-1}{2}}} \sin\left((k+1)\frac{\pi}{4}\right), \tag{12}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Proof. We have

$$\frac{1}{1+t+\frac{1}{2}t^2} = \sum_{j=0}^{\infty} (-1)^j \sum_{s=0}^{j} \binom{j}{s} \frac{t^{j+s}}{2^s} = \sum_{k=0}^{\infty} a_k t^k,$$

which gives (10). To prove (11) and (12), we use the well-known identity [6, 1.60. pp. 8]

$$\sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^s \binom{k-s}{s} (xy)^s (x+y)^{k-2s} = \frac{x^{k+1}-y^{k+1}}{x-y},$$

with

$$x = \frac{1}{2}(1+i) = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \text{ and } y = \frac{1}{2}(1-i) = \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}. \tag{13}$$

With this we get the desired result. $\square$

3 Explicit formulas for Euler polynomials

We will begin by giving some operators properties defined on the vector space $\mathbb{C}[x]$. Since we have

$$\frac{2}{e^t+1} = \sum_{k=0}^{\infty} E_k(0) \frac{t^k}{k!} \qquad (|t| < \pi),$$

we denote by Ω_E for the operator

$$\Omega_E = \frac{2}{e^D + 1}, \tag{14}$$

defined by

$$\Omega_E = \sum_{k=0}^{\infty} E_k(0) \frac{D^k}{k!},$$

where D is the usual differential operator. The operator Ω_E verifies the relation $E_n(x) = \Omega_E(x^n)$.

If we consider the translations τ_r $(r \in \mathbb{C})$ of $\mathbb{C}[x]$ which are the operators defined by Robert [9, pp. 195] as

$$\tau_r(x^n) = (x+r)^n, n \geq 0,$$

and the operators Δ_r defined for $r \neq 0$ as

$$\Delta_r(x^n) = (x+r)^n - x^n, \tag{15}$$

we get

$$\Delta_r = \tau_r - 1 = e^{rD} - 1.$$

We denote by Δ for the operator Δ_1.
For $k \geq 0$, we have

$$\Delta_r^k = (\tau_r - 1)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{rj}$$

and

$$\Delta_r^k(x^n) = (\tau_r - 1)^k (x^n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+rj)^n. \tag{16}$$

We want express Ω_E as [7]

$$\Omega_E = \sum_{k=0}^{\infty} b_k \frac{\Delta_r^k}{k!},$$

where b_k depends of r. In the following lemmas, we study cases where $r \in \left\{1, -1, \frac{1}{2}\right\}$.

Lemma 3.1. *We have*

$$\Omega_E = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \Delta^k \tag{17}$$

and

$$\Omega_E = 1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \Delta_{-1}^k. \tag{18}$$

Proof. It is easy to see that we have

$$\Omega_E = \frac{2}{e^D + 1} = \frac{2}{2 + \Delta} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \Delta^k$$

and

$$\Omega_E = \frac{1 + \Delta_{-1}}{1 + \frac{1}{2}\Delta_{-1}} = 1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \Delta_{-1}^k. \qquad \square$$

Lemma 3.2. *We have*

$$\Omega_E = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+k}}{2^s} \binom{k-s}{s} \right) \Delta_{\frac{1}{2}}^{k}, \tag{19}$$

$$\Omega_E = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+k}}{2^k} \binom{k+1}{2s+1} \right) \Delta_{\frac{1}{2}}^{k}, \tag{20}$$

$$\Omega_E = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2^{\frac{k-1}{2}}} \sin\left((+1) \frac{\pi}{4} \right) \right) \Delta_{\frac{1}{2}}^{k}. \tag{21}$$

Proof. By the expression

$$\Omega_E = \frac{1}{1 + \Delta_{\frac{1}{2}} + \frac{1}{2}\Delta_{\frac{1}{2}}^{2}}, \tag{22}$$

and with the help of Lemma 2.2, we have

$$\Omega_E = \sum_{k=0}^{\infty} a_k \Delta_{\frac{1}{2}}^{k}.$$

The relations (19), (20) and (21) result from expressions (10), (11) and (12) of a_k given in Lemma 2.2. □

Theorem 3.3 (Case $r = \pm 1$)**.** *For all integers m, n such that $0 \leq n \leq m$, we have*

$$E_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^j}{2^k} \binom{k}{j} \right) (x+j)^n, \tag{23}$$

$$E_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^j}{2^m} \binom{m+1}{k-j} \right) (x+j)^n, \tag{24}$$

$$E_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^k}{2^k} \binom{m+1}{k+1} \binom{k}{j} \right) (x+j)^n, \tag{25}$$

$$E_n(x) = \frac{x^n}{2^m} + \sum_{j=1}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^{j+1}}{2^k} \binom{k}{j} \right) (x-j)^n. \tag{26}$$

Proof. In what follows, we suppose that $0 \leq n \leq m$. From (17) of Lemma 3.1, we have

$$E_n(x) = \Omega_E(x^n) = \sum_{k=0}^{m} \frac{(-1)^k}{2^k} \Delta^k(x^n), \tag{27}$$

and with the help of (16), we have

$$\Delta^k(x^n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n. \tag{28}$$

By (27) and (28), we have

$$\begin{aligned} E_n(x) &= \sum_{k=0}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{2^k} \binom{k}{j} (x+j)^n \\ &= \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^j}{2^k} \binom{k}{j} \right) (x+j)^n . \end{aligned}$$

The relation (23) is thus proved and we have

$$\mu_j(E,1,m) = \sum_{k=j}^{m} \frac{(-1)^j}{2^k} \binom{k}{j}. \tag{29}$$

By relation (7) of Lemma 2.1 and (29), we deduce that we have also

$$\mu_j(E,1,m) = \sum_{k=j}^{m} \frac{(-1)^j}{2^m} \binom{m+1}{k-j},$$

and (24) follows. By relation (8) of Lemma 2.1 and (29), we deduce that we have also

$$\mu_j(E,1,m) = \sum_{k=j}^{m} \frac{(-1)^k}{2^k} \binom{m+1}{k+1} \binom{k}{j},$$

and (25) follows. By (18) of Lemma 2.1, we have

$$E_n(x) = \Omega_E(x^n) = x^n - \sum_{k=1}^{m} \frac{(-1)^k}{2^k} \Delta_{-1}^k (x^n) . \tag{30}$$

With the help of (16), we have

$$\Delta_{-1}^k (x^n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x-j)^n . \tag{31}$$

Relation (26) follows from (30) and (31). □

Remark 3.4. *For $m = n$ in (23) we obtain*

$$E_n(x) = \sum_{k=0}^{n} \frac{1}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^n .$$

Theorem 3.5 (Case $r = 1/2$)**.** *For all integers m, n such that $0 \leq n \leq m$, we have*

$$E_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+j}}{2^s} \binom{k-s}{s} \binom{k}{j} \right) \left(x + \frac{j}{2} \right)^n, \tag{32}$$

$$E_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+j}}{2^k} \binom{k+1}{2s+1} \binom{k}{j} \right) \left(x + \frac{j}{2} \right)^n, \tag{33}$$

$$E_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^j}{2^{\frac{k-1}{2}}} \binom{k}{j} \sin\left((k+1) \frac{\pi}{4} \right) \right) \left(x + \frac{j}{2} \right)^n . \tag{34}$$

Proof. From (19), we have for $m \geq n$

$$\Omega_E(x^n) = \sum_{k=0}^{m} \left(\sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+k}}{2^s} \binom{k-s}{s} \right) \Delta_{\frac{1}{2}}^{k}(x^n). \tag{35}$$

With the help of (16), we have

$$\Delta_{\frac{1}{2}}^{k}(x^n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(x - \frac{j}{2} \right)^n. \tag{36}$$

By (35) and (36), we deduce (32) and we have

$$\mu_j\left(E, \frac{1}{2}, m\right) = \sum_{k=j}^{m} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{s+j}}{2^s} \binom{k-s}{s} \binom{k}{j}.$$

In the same way, we obtain (33) and (34) thanks to relations(20) and (21) of Lemma 3.2. □

4 Explicit formulas for Bernoulli numbers

The Bernoulli polynomials $B_n(x)$ are defined by the following exponential generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{37}$$

The Bernoulli numbers are then $B_n = B_n(0)$. From the definitions (6) and (37), we can easily deduce the following well-know properties [8, pp. 218 and 222]

$$E_{n-1}(x) = \frac{2}{n} \left(B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right), \quad n \geq 1, \tag{38}$$

$$E_n(1-x) = (-1)^n E_n(x),$$

$$E_n(0) = (-1)^n E_n(1). \tag{39}$$

From (38) and (39), we deduce that

$$B_{n+1} = \frac{n+1}{2(1-2^{n+1})} E_n(0) \tag{40}$$

and

$$B_{n+1} = \frac{(-1)^n (n+1)}{2(1-2^{n+1})} E_n(1). \tag{41}$$

From (38), we also deduce

$$E_{2n+1}\left(\frac{1}{3}\right) = \frac{1}{n+1} \left(B_{2n+2}\left(\frac{1}{3}\right) - 2^{2n+2} B_{2n+2}\left(\frac{1}{6}\right) \right).$$

Using the following relations [1, pp. 806]

$$B_{2n+2}\left(\frac{1}{3}\right) = \left(1 - 3^{2n+1}\right) \frac{B_{2n+2}}{2.3^{2n+1}}$$

and

$$B_{2n+2}\left(\frac{1}{6}\right) = \left(1 - 2^{2n+1}\right)\left(1 - 3^{2n+1}\right) \frac{B_{2n+2}}{2^{2n+2}.3^{2n+1}},$$

we deduce that

$$B_{2n+2}=\frac{2\left(n+1\right).3^{2n+1}}{\left(4^{n+1}-1\right)\left(1-3^{2n+1}\right)}E_{2n+1}\left(\frac{1}{3}\right). \tag{42}$$

These relations that we have just proved will be useful for us to establish the following theorem.

Theorem 4.1. *For all integers m, n such that $0\leq n\leq m$, we have*

$$B_{n+1}=\frac{\left(-1\right)^{n+1}\left(n+1\right)}{2^{n+1}-1}\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{\left(-1\right)^{j}}{2^{k+1}}\binom{k}{j}\left(j+1\right)^{n}, \tag{43}$$

$$B_{n+1}=\frac{n+1}{1-2^{n+1}}\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{\left(-1\right)^{k-j}}{2^{k+1}}\binom{k}{j}\left(k-j\right)^{n}, \tag{44}$$

$$B_{n+1}=\frac{n+1}{2^{m+1}\left(1-2^{n+1}\right)}\sum_{k=0}^{m}\sum_{j=0}^{k}\left(-1\right)^{k-j}\binom{m+1}{j}\left(k-j\right)^{n}, \tag{45}$$

$$B_{2n+2}=\frac{n+1}{\left(4^{n+1}-1\right)\left(3^{2n+1}-1\right)}\sum_{k=0}^{2m+1}\sum_{j=0}^{k}\frac{\left(-1\right)^{j+1}}{2^{k-1}}\binom{k}{j}\left(3j+1\right)^{2n+1}. \tag{46}$$

Proof. In all that follows, we suppose that $0\leq n\leq m$. From relation (23), we deduce for $x=1$

$$E_{n}\left(1\right)=\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{\left(-1\right)^{j}}{2^{k}}\binom{k}{j}\left(j+1\right)^{n}. \tag{47}$$

Using (41) and (47), we get (43). From relation (23), we deduce for $x=0$

$$\begin{aligned}E_{n}\left(0\right)&=\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{\left(-1\right)^{j}}{2^{k}}\binom{k}{j}j^{n}\\&=\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{\left(-1\right)^{k-j}}{2^{k}}\binom{k}{j}\left(k-j\right)^{n}.\end{aligned} \tag{48}$$

Using (40) and (48), we get (44).

From relation (24), we deduce for $x=0$

$$\begin{aligned}E_{n}\left(0\right)&=\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{\left(-1\right)^{j}}{2^{m}}\binom{m+1}{k-j}j^{n}\\&=\frac{1}{2^{m}}\sum_{k=0}^{m}\sum_{j=0}^{k}\left(-1\right)^{k-j}\binom{m+1}{j}\left(k-j\right)^{n}.\end{aligned} \tag{49}$$

Using (40) and (49), we get (45).

From relation (23), we deduce for $x=\frac{1}{3}$

$$E_{2n+1}\left(\frac{1}{3}\right)=\frac{1}{3^{2n+1}}\sum_{j=0}^{2m+1}\sum_{k=j}^{2m+1}\frac{\left(-1\right)^{j}}{2^{k}}\binom{k}{j}\left(3j+1\right)^{2n+1}. \tag{50}$$

Using (42) and (50), we get (46). □

Theorem 4.1 generalizes many explicit formulas given in [5]. Indeed for $m = n$, the identity (43) becomes

$$B_{n+1} = \frac{(-1)^{n+1}(n+1)}{2^{n+1}-1} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^j}{2^{k+1}} \binom{k}{j} (j+1)^n .$$

This last identity is exactly the identity (2) given in [5]. This formula has been proven in 1940 by Garabedian [4] by use of divergent power series. In 1953, Carlitz [3] also gave a short proof while pointing out that formula was a very old formula proved in 1883 by Worpitzky [11, pp. 224]. This same identity was again proved in 2004 by Rzadkowski [10].

Using the following Gould notation [5, pp. 48]:

$$B_{r,q}^{n} := \sum_{j=0}^{r} (-1)^j \binom{q}{j} (r-j)^n , \tag{51}$$

relations (44) and (45) can be written

$$B_{n+1} = \frac{n+1}{2\left(1-2^{n+1}\right)} \sum_{k=0}^{m} \frac{(-1)^k}{2^k} B_{k,k}^{n} \tag{52}$$

and

$$B_{n+1} = \frac{n+1}{2^{m+1}\left(2^{n+1}-1\right)} \sum_{k=0}^{m} (-1)^{k+1} B_{k,m+1}^{n}. \tag{53}$$

Gould's identities (18) and (19) of [5] are obtained when $m = n$ in (52) and (53) respectively.

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