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New consequences of prime-counting function

Sadani Idir

Department of Mathematics, University of Mouloud Mammeri
15000 Tizi-Ouzou, Algeria
e-mail: sadani.idir@yahoo.fr

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Abstract: Our objective in this paper is to study a particular set of prime numbers, namely $\{p \in \mathbb{P}$ and $\pi(p) \notin \mathbb{P}\}$. As a consequence, estimations of the form $\sum f(p)$ with p being prime belonging to this set are derived.
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1 Introduction

As usual, let $\mathbb{P}$ be the set of all primes, $\pi(x) = \#\mathbb{P} \cap [2, x]$ and

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt = \frac{x}{\log x} \left(1 + \sum_{k=1}^{N} \frac{k!}{\log^k x} + O\left(\frac{1}{\log^{N+1} x}\right)\right), \ (x \to +\infty). \tag{1.1}$$

The Prime Number Theorem states that

$$\pi(x) \sim \mathrm{Li}(x), (x \to +\infty). \tag{1.2}$$

The theorem was proved, independently, by Hadamard [1] and de la Vallée-Poussin [2] in 1896. Another paper of de la Vallée-Poussin is [3], where he estimated the error term in the Prime Number Theorem by showing existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\Re(s) = 1$. The error is given by

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right) \text{ as } x \to \infty, \tag{1.3}$$

where a is a positive absolute constant.

The aim of this paper is to use the Prime Number Theorem to give some estimations related to the following subset of primes

$$\{p \in \mathbb{P} \text{ and } \pi(p) \notin \mathbb{P}\}.$$

2 Preparatory lemmas

We will need several preparatory lemmas. The first one is a new version and extension of the result obtained in [4]. Let us use the denotations $\pi_2(x)$ for $\pi(\pi(x))$, $\mathrm{Li}_2(x)$ for $\mathrm{Li}(\mathrm{Li}(x))$ and $\mathrm{Li}_c(x) = \mathrm{Li}(x) - \mathrm{Li}_2(x)$.

Lemma 2.1. *Let x be a positive real number. Let us denote by $\pi_c(x)$ (respectively, $\overline{\pi}_c(x)$) the number of primes $p \leq x$ such as $\pi(p)$ is not a prime (respectively, $\pi(p)$ is prime). Precisely,*

$$\pi_c(x) := \#\{p \leq x | \pi(p) \text{ is not prime}\} = \sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} 1,$$

and

$$\overline{\pi}_c(x) := \#\{p \leq x | \pi(p) \text{ is prime}\} = \sum_{\substack{p \leq x \\ \pi(p) \in \mathbb{P}}} 1.$$

Then,

1. $\pi(x) = \pi_c(x) + \overline{\pi}_c(x)$.
2. $\pi_c(x) = \pi(x) - \pi(\pi(x))$.
3. $\overline{\pi}_c(x) = \pi(\pi(x))$.

Proof. 1. It is straightforward to see that the set of prime numbers less than or equal to x can be partitioned into two subsets as follows

$$\{p \leq x | p \text{ is prime}\} = \{p \leq x | p \text{ is prime and } \pi(p) \text{ is prime}\} \cup \{p \leq x | p \text{ is prime and } \pi(p) \text{ is not prime}\}. \tag{2.1}$$

By passage to cardinality, we get

$$\#\{p \leq x | p \text{ is prime}\} = \#\{p \leq x | p \text{ is prime and } \pi(p) \text{ is prime}\} + \#\{p \leq x | p \text{ is prime and } \pi(p) \text{ is not prime}\}$$

or

$$\pi(x) = \pi_c(x) + \overline{\pi}_c(x). \tag{2.2}$$

2. It is not difficult to see that $\#\{p \leq x | p \text{ is prime and } \pi(p) \text{ is not prime}\}$ is equal to the number of different equivalence classes $\dot{p}$ which was denoted in [4] by $\pi_c(x)$ (for more details, see [4]).

3. From the equation (2.2), we obtain

$$\overline{\pi}_c(x) = \pi(x) - \pi_c(x) = \pi(x) - (\pi(x) - \pi(\pi(x))) = \pi(\pi(x)). \qquad \square$$

Lemma 2.2. *We have the following estimations*

$$\frac{1}{\log \mathrm{Li}(x)} = \frac{1}{\log x} + O\left(\frac{\log\log x}{\log^2 x}\right) \ (x \to \infty). \tag{2.3}$$

Proof. Using formula (1.1), we get

$$\frac{1}{\log \mathrm{Li}(x)} = \frac{1}{\log x - \log\log x + \log\left(1 + \sum_{k=1}^{N} \frac{k!}{\log^k x} + O\left(\frac{1}{\log^{N+1} x}\right)\right)}, \tag{2.4}$$

and by using Taylor's expansion, we acquire

$$\log\left(1 + \sum_{k=1}^{N} \frac{k!}{\log^k x} + O\left(\frac{1}{\log^{N+1} x}\right)\right) = \frac{1}{\log x} + O\left(\frac{1}{\log^2 x}\right) \ (x \to \infty). \tag{2.5}$$

Next, we replace (2.5) in (2.4), we get

$$\begin{aligned}\frac{1}{\log \mathrm{Li}(x)} &= \frac{1}{\log x\left(1 - \frac{\log\log x}{\log x} + \frac{1}{\log^2 x} + O\left(\frac{1}{\log^3 x}\right)\right)}\\ &= \frac{1}{\log x}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right) \ (x \to \infty).\end{aligned}$$

□

Lemma 2.3. *We have*

$$\pi_2(x) - \mathrm{Li}_2(x) = O\left(\frac{x}{\log x} e^{-a\sqrt{\log \frac{x}{\log x}}}\right) \ (x \to \infty), \tag{2.6}$$

$$\pi_c(x) - \mathrm{Li}_c(x) = O\left(\frac{x}{\log x} e^{-a\sqrt{\log \frac{x}{\log x}}}\right) \ (x \to \infty). \tag{2.7}$$

Proof. From (1.3), we have on the one hand

$$\pi(\pi(x)) = \mathrm{Li}(\pi(x)) + O\left(\pi(x) e^{-a\sqrt{\log \pi(x)}}\right) = \mathrm{Li}(\pi(x)) + O\left(\frac{x}{\log x} e^{-a\sqrt{\log \frac{x}{\log x}}}\right). \tag{2.8}$$

And, on the other hand, by Taylor's series, we acquire

$$\begin{aligned}\mathrm{Li}(\pi(x)) - \mathrm{Li}_2(x) &= \mathrm{Li}\left(\mathrm{Li}(x) + O(x e^{-a\sqrt{\log x}})\right) - \mathrm{Li}(\mathrm{Li}(x))\\ &= \frac{1}{\log \mathrm{Li}(x)} O\left(x e^{-a\sqrt{\log x}}\right)\\ &= O\left(\frac{x}{\log x} e^{-a\sqrt{\log x}}\right).\end{aligned} \tag{2.9}$$

Now, we can estimate $\pi_2(x) - \mathrm{Li}_2(x)$:

$$\pi_2(x) - \mathrm{Li}_2(x) = \pi_2(x) - \mathrm{Li}(\pi(x)) + \mathrm{Li}(\pi(x)) - \mathrm{Li}_2(x).$$

Using (2.8) and (2.9), we get

$$\pi_2(x) - \mathrm{Li}_2(x) = O\left(\frac{x}{\log x} e^{-a\sqrt{\log x}}\right) + O\left(\frac{x}{\log x} e^{-a\sqrt{\log \frac{x}{\log x}}}\right) = O\left(\frac{x}{\log x} e^{-a\sqrt{\log \frac{x}{\log x}}}\right).$$

For the formula (2.7), we have

$$\begin{aligned}
\pi_c(x) - \mathrm{Li}_c(x) &= \pi(x) - \pi_2(x) - (\mathrm{Li}(x) - \mathrm{Li}_2(x)) \\
&= \pi(x) - \mathrm{Li}(x) - (\pi_2(x) - \mathrm{Li}_2(x)) \\
&= O\left(x e^{-a\sqrt{\log x}}\right) - O\left(\frac{x}{\log x} e^{-a\sqrt{\log \frac{x}{\log x}}}\right) \\
&= O\left(\frac{x}{\log x} e^{-a\sqrt{\log \frac{x}{\log x}}}\right).
\end{aligned}$$

□

3 Main results

Our main result may be stated as follows.

Theorem 3.1. *Let us have* $f(x) = Cx^{-b} \log^w x$ *with* $C, b \geq 0$ *and* $w \geq 1$. *Then*

$$\sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} f(p) = \int_2^x \frac{f(y)}{\log y}\left(1 - \frac{1}{\log y}\right) dy + O\left(x^{1-b} (\log x)^{w-3} \log\log x\right). \tag{3.1}$$

Proof. Using Stieltjes integral, we obtain

$$\sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} f(p) = \int_2^x f(y) d\pi_c(y).$$

Integration by parts gives

$$\begin{aligned}
\sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} f(p) &= f(x)\pi_c(x) - f(2) - \int_2^x f'(y)\pi_c(y) dy \\
&= f(x)\pi_c(x) - f(2) - \int_2^x f'(y)\mathrm{Li}_c(y) dy - \int_2^x f'(y)(\pi_c(y) - \mathrm{Li}_c(y)) dy.
\end{aligned}$$

Then integration by parts gives

$$\begin{aligned}
\sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} f(p) = \int_2^x \frac{f(y)}{\log y}\left(1 - \frac{1}{\log \mathrm{Li}(y)}\right) dy + f(2)\mathrm{Li}_c(2) - f(2) + f(x)(\pi_c(x) - \mathrm{Li}_c(x)) \\
- \int_2^x f'(y)(\pi_c(y) - \mathrm{Li}_c(y)) dy
\end{aligned} \tag{3.2}$$

Now, using estimation (2.3) of Lemma 2.2, we get

$$\sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} f(p) = \int_2^x \frac{f(y)}{\log y}\left(1-\frac{1}{\log y}-O\left(\frac{\log\log y}{\log^2 y}\right)\right)dy$$

$$+ f(2)\mathrm{Li}_c(2) + f(x)(\pi_c(x)-\mathrm{Li}_c(x)) - \int_2^x f'(y)(\pi_c(y)-\mathrm{Li}_c(y))dy. \quad (3.3)$$

We have, on the one hand, by (2.7) of Lemma 2.3

$$f(x)(\pi_c(x)-\mathrm{Li}_c(x)) = O\left(x^{1-b}\log^w e^{-a\sqrt{\frac{x}{\log x}}}\right) \quad (3.4)$$

and

$$\int_2^x \frac{f(y)}{\log y} O\left(\frac{\log\log y}{\log^2 y}\right)dy = O\left(x^{1-b}\log^{w-3} x\log\log x\right). \quad (3.5)$$

On the other hand, we have

$$f'(x) = Cx^{-b-1}\log^{w-1} x(-b\log x + w).$$

Then, again by (2.7) of Lemma 2.3

$$f'(y)(\pi_c(y)-\mathrm{Li}_c(y)) = O\left(x^{-b}(\log x)^w e^{-a\sqrt{\frac{x}{\log x}}}\right).$$

Consequently,

$$\int_2^x f'(y)(\pi_c(y)-\mathrm{Li}_c(y))dy = \int_2^x O\left(y^{-b}(\log y)^w e^{-a\sqrt{\frac{y}{\log y}}}\right)dy$$

$$= O\left(x^{1-b}(\log x)^w e^{-a\sqrt{\frac{x}{\log x}}}\right) \quad (3.6)$$

Finally, by replacing estimations (3.4), (3.5) and (3.6) in (3.3), we find that

$$\sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} f(p) = \int_2^x \frac{f(y)}{\log y}\left(1-\frac{1}{\log y}\right)dy + O\left(x^{1-b}(\log x)^{w-3}\log\log x\right). \qquad \square$$

We now present applications of Theorem 3.1.

Corollary 1. We have

$$\sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \log p = x - \mathrm{Li}(x) - 2 + O\left(x\log^{-2} x\log\log x\right) \ (x\to\infty), \quad (3.7)$$

$$\sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{\log p}{p} = \log x - \log\log x + \log\log\sqrt{2} + O\left(\log^{-2} x\log\log x\right) \ (x\to\infty), \quad (3.8)$$

$$\sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{1}{p} = \log\log x - (\log\log 2 + (\log 2)^{-1}) + O\left(\log^{-3} x \log\log x\right) \ (x\to\infty), \tag{3.9}$$

$$\sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{\log^n p}{p} = \frac{\log^n x}{n} - \frac{\log^{n-1} x}{n-1} + c_n(2) + O\left(\log^{n-3} x \log\log x\right) \ (x\to\infty), \tag{3.10}$$

with $c_n(2) = -\left(\frac{\log^n 2}{n} - \frac{\log^{n-1} 2}{n-1}\right)$ and $n \ge 2$.

$$\sum_{i=2}^{n} \sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{\log^i p}{p} = \frac{\log^n x}{n} - \log x + C_n(2) + O\left(\log^{n-3} x \log\log x\right) (x\to\infty), \tag{3.11}$$

with $C_n(2) = \sum_{i=2}^{n} c_i(2)$.

Proof. The first four estimations are immediate from formula (3.1). Now, for the latest, we have

$$k=2,\ \sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{\log^2 p}{p} = \frac{\log^2 x}{2} - \log x + c_2(2) + O\left(\log^{-1} x \log\log x\right),$$

$$k=3,\ \sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{\log^3 p}{p} = \frac{\log^3 x}{3} - \frac{\log^2 x}{2} + +c_3(2) + O\left(\log\log x\right),$$

$$\vdots$$

$$k=n,\ \sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{\log^n p}{p} = \frac{\log^n x}{n} - \frac{\log^{n-1} x}{n-1} + c_n(2) + O\left(\log^{n-3} x \log\log x\right).$$

These equations can be added to yield the desired formula. □

Remark. The absolute error in (3.9) tends to zero as x tends to infinity, then

$$\lambda_0 = \lim_{x\to\infty}\left(\sum_{\substack{p\le x\\ \pi(p)\notin\mathbb{P}}} \frac{1}{p} - \log\log x\right)$$

exists and has a finite value.

Theorem 3.2. *1. We have*

$$\pi_2(x) \le k\frac{x}{\log^2 x}, k \backsimeq 2.4919 \text{ and } x \ge 2. \tag{3.12}$$

2. The following sum is convergent

$$\sum_{p,\pi(p) \text{ are primes}} \frac{1}{p}. \tag{3.13}$$

Proof. 1. As it is well-known

$$\pi(x) < c\,\frac{x}{\ln x}, \quad x \geq 1$$

with $c = 1.25506$. Then

$$\pi(\pi(x)) < c\frac{\pi(x)}{\ln \pi(x)} < c^2\frac{x}{\ln x \ln \pi(x)} < c^2\frac{x}{\ln x(\ln x - \ln\ln x)} = c^2\frac{x}{\ln^2 x\left(1 - \frac{\ln\ln x}{\ln x}\right)}, \quad x \geq 2$$

The function $\frac{1}{1-\frac{\ln\ln x}{\ln x}}$ has its maximum value $\frac{e}{e-1}$ at e^e, then we get

$$\pi(\pi(x)) < \frac{c^2 e}{e-1}\frac{x}{\ln^2 x} \approx 2.4919\frac{x}{\ln^2 x}, \quad x \geq 2.$$

2. Using Abel's summation formula and since inequality (3.12) holds, we get

$$\begin{aligned}\sum_{\substack{p\leq x\\ p,\pi(p) \text{ are primes}}} \frac{1}{p} &= \overline{\pi_c(x)}\frac{1}{x} + \int_{\lambda_1}^{x} \overline{\pi_c(t)}\frac{1}{t^2}dt \\ &\leq \frac{1}{\log^2 x} + k\int_{\lambda_1}^{x}\frac{1}{t\log^2 t}dt = \frac{k}{\log^2 x} + \frac{k}{\log x} + \frac{k}{\log\lambda_1} \\ &\leq k\left(\frac{1}{\log^2 x} + \frac{1}{\log x} + \frac{1}{\log\lambda_1}\right) \leq k\left(\frac{1}{\log^2 3} + \frac{2}{\log 3}\right).\end{aligned}$$

This implies that $\exists M > 0$ such that for all sufficiently large x,

$$\sum_{p,\pi(p) \text{ are primes}} \frac{1}{p} \leq M.$$

This implies that the sum is convergent. □

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