Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 4, 15–24
DOI: 10.7546/nntdm.2021.27.4.15-24

On Robin's criterion for the Riemann Hypothesis

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Received: 30 November 2020 **Revised:** 15 October 2021 **Accepted:** 8 November 2021

Abstract: Robin's criterion says that the Riemann Hypothesis is equivalent to

$$\forall n \geq 5041, \ \frac{\sigma(n)}{n} \leq e^{\gamma} \log_2 n,$$

where $\sigma(n)$ is the sum of the divisors of n, γ represents the Euler–Mascheroni constant, and $\log_i$ denotes the i-fold iterated logarithm. In this note we get the following better effective estimates:

$$\forall n \geq 3, \ \frac{\sigma(n)}{n} \leq e^{\gamma} \log_2 n + \frac{0.3741}{\log_2^2 n}.$$

The idea employed will lead us to a possible new reformulation of the Riemann Hypothesis in terms of arithmetic functions.
Keywords: Primorial number, Robin's inequality, Riemann Hypothesis.
2020 Mathematics Subject Classification: 11A25, 11N64, 11M26.

1 Introduction and statement of results

As usual, let $(p_k)_{k\geq 1}$ denote the increasing sequence of prime numbers, and let N_k be the primorial integer of index k, the product of its k first terms. The Riemann Hypothesis (RH) claims that the nontrivial zeros of zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$ are located on the critical line $\mathcal{R}(s) = \frac{1}{2}$. Several equivalent formulations of RH appeared, but the one which interests us here is that in terms of arithmetic functions, here we cite the first papers of Gronwall [8], Nicolas [11] and Robin [13], followed by, for instance, Akbary [1], Caveney et al. [6] and Lagarias [10].

Robin in his paper [13] asserted that RH is equivalent to

$$\forall n \geq 5041, \;\; \sigma(n) \leq e^{\gamma} n \log_2 n, \tag{1}$$

with $\sigma(n)$ denotes the sum of divisors function, γ the Euler–Mascheroni constant, and $\log_i$ the i-fold iterated logarithm. This assertion is based on the known following formula (see [9]):

$$\frac{\sigma(n)}{n} = (1 + o(1)) e^{\gamma} \log_2 n. \tag{2}$$

In this note, we intend to join the authors who have attempted to closely determine the o-term in the formula (2). The best upper bound of the normalized of the sum of divisors function is also given by Robin [13] which proved, unconditionally, that

$$\forall n \geq 3, \;\; \frac{\sigma(n)}{n} \leq e^{\gamma} \log_2 n + \frac{0.6483}{\log_2 n}.$$

We propose the following result:

Theorem 1.1. *For every integer $n \geq 3$, we have*

$$\frac{\sigma(n)}{n} \leq e^{\gamma} \log_2 n + \frac{0.3741}{\log_2^2 n}.$$

This improves considerably Robin's upper bound. In parallel, we study another form of upper bound than that exposed in the theorem above, since it is completely expressed in terms of $K(x)$, the primorial counting function which, see Balazard [4], is approximately $\frac{\log x}{\log_2 x}$. We conclude that:

Theorem 1.2. *If $K(n)$ is the number of primorial integers not exceeding n, then*

$$\forall n \geq 30, \;\; \frac{\sigma(n)}{n} \leq e^{\gamma} \left(\log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} + \frac{1}{20 \log_2^2 K(n)} \right).$$

This leads us to examine a conjecture upon which we stumbled:

Conjecture 1. *The Riemann Hypothesis is equivalent to*

$$\forall n \geq 205, \;\; \frac{\sigma(n)}{n} \leq e^{\gamma} \left(\log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right).$$

See Section 4 for more background on this conjecture. The main ingredient of this paper is the recent version of the upper bound of the product over primes $\prod_{p\leq x} \frac{p}{p-1}$, thanks to the paper of the third author in [7], as a consequence of the new estimates of Chebyshev's summatory functions also exposed in [7]. Although there are some updates, such improvements have negligible influence on the final results. Finally, we indicate that e represents Napier's constant, p a prime number, and with this technique, obtaining better approximations is closely linked with progress on extending the known zero-free region of the Riemann zeta-function.

2 Preliminary lemmas

The primorial counting function $K(x)$ is not known in the literature. We begin by showing some basic properties (for a more extended study, see the recent paper of the authors [3]). For each real $x \geq 1$, the integer $K(x)$ can be defined by $\max\{k \in \mathbb{N}^*,\ N_k \leq x\}$. In the following lemma, we prove that for a given $x \geq 1$, the primorial $N_{K(x)}$ represent the smallest integer less than x whose decomposition into prime numbers is the longest. Here $\omega(n)$ denotes the number of prime distinct divisors of n.

Lemma 2.1. *For every real number $x \geq 1$, we have*

$$K(x) = \max_{1 \leq n \leq x} \omega(n).$$

Furthermore, for any integer $n \leq x$ with $\omega(n) = K$, we have $N_K \leq n$.

Proof. As $N_k \leq n \leq x < N_{k+1}$ means that $\omega(n) \leq k$ and $K(n) = k$, hence $\omega(n) \leq K(n)$ in any interval $[N_k, N_{k+1}[$, which implies that

$$\max_{1 \leq n \leq x} \omega(n) = \max_{1 \leq n < N_{K+1}} \omega(n) = K.$$

Let $q_1 q_2 \cdots q_K$ be an integer less than x with $q_1 < q_2 < \cdots < q_K$ prime numbers. For $K = 1$ it is obvious that $q_1 \geq p_1$. Now, assuming $q_i \geq p_i$ for $i < K$, it is necessary that $q_K \geq p_K$, otherwise $q_K < q_{K-1}$. □

Lemma 2.2. *We have, when $x \geq 8$, the following inequalities:*

$$\log_2 x < K(x) \leq \log x.$$

Proof. From the definition of $K(x)$, by taking the logarithm, we can also write the following:

$$K(x) = \max\{k \in \mathbb{N}^*,\ \theta(p_k) \leq \log x\}, \tag{3}$$

where θ denotes the Chebyshev function. So, by recalling the inequality $\theta(p_k) \geq k$ given in Robin [12] valid once $k \geq 3$, one easily deduces that

$$K(x) \leq \max\{k \in \mathbb{N}^*,\ k \leq \log x\} \leq \log x,\ \forall x \geq N_3,$$

which is also valid for $8 \leq x < N_3$. For the second, a short induction on k is necessary. For all $k \geq 1$, we have $N_k < e^{e^{k-1}}$. Indeed, the case $k = 1$ is obvious, and the fact that $\forall k \geq 1,\ p_{k+1} < N_k$ (according to Euclid's proof of the infinity of primes) implies that

$$N_{k+1} = N_k p_{k+1} < e^{e^{k-1}} N_k < e^{2e^{k-1}} < e^{ee^{k-1}} = e^{e^k}.$$

So, by taking the logarithm, one gets that for all $x \geq e$:

$$\log_2 x < \log_2 N_{K+1} < K(x).$$

We conclude the proof using computer verifications for the small values. In relation to $\pi(x)$ the prime counting function, we can also mention that

$$\log_2 x < K(x) \leq \log x < \pi(x).$$ □

Lemma 2.3. *Let* $\delta = 1.000081$. *We have, when* $x \geq 210$*:*

$$K(x) \geq \frac{1}{\delta}\frac{\log x}{\log_2 x}.$$

Proof. Recalling the following estimates given in [14]:

$$\theta(x) < \delta x, \ \forall x > 1 \ \text{ and } \ \pi(x) \geq \frac{x}{\log x}, \ \forall x \geq 17,$$

one reaches successively, for every real $x \geq e^{17\delta}$, that

$$K(x) \geq \max\left\{k \in \mathbb{N}^*, \ \delta p_k \leq \log x\right\} = \pi\left(\frac{\log x}{\delta}\right) \geq \frac{1}{\delta}\frac{\log x}{\log_2 x}.$$

A computer check handles the cases $210 \leq x < e^{17\delta}$. □

Now, for f a decreasing function greater than 1 on $(1, \infty)$, we consider the following sequence

$$\mathfrak{L}(n) = \prod_{p|n} f(p), \ \forall n > 1.$$

The term $\mathfrak{L}(n)$ for the function $f(x) = \frac{x}{x-1}$ is only $\frac{n}{\varphi(n)}$, where $\varphi(n)$ denotes the Euler totient function, and $\mathfrak{L}(n)$ is $\frac{\Psi_t(n)}{n}$ when $f(x) = 1 + 1/x + \cdots + 1/x^{t-1}, \ t \geq 2$, where $\Psi_t(n)$ is the generalized Dedekind psi function. We have the following Lemmas

Lemma 2.4. *For every real number* $x \geq 2$*, the following equality*

$$\max_{1<n\leq x} \mathfrak{L}(n) = \prod_{p \leq p_{K(x)}} f(p)$$

holds.

Proof. To determine the maximum of $\mathfrak{L}(n)$, when n range over all integers less than or equal to x, we first use the fact that f is greater than 1 since this places the maximum at the class of the integers whose number of prime divisors is the largest. Then, as f is also strictly decreasing, the maximum must have the smallest prime numbers in its decomposition. However, according to the previous lemma, we can clearly specify that, it is only true for $N_{K(x)}$, i.e.,

$$\max_{1<n\leq x} \mathfrak{L}(n) = \mathfrak{L}(N_{K(x)}).$$

Finally, as $p|N_k$ is equivalent to $p \leq p_k$, the lemma follows. □

Remark 1. *When* f *is strictly increasing and greater than* 1 *on* $(1, \infty)$*, the maximum of* $\mathfrak{L}(n)$ *is reached at an integer* $q_1 \cdots q_{K(x)}$*, where at least one of* q_i *is a prime number greater than* p_i*.*

In the following lemma, we leave the generalization and show, through a simpler proof, a result concerning the order of the Euler function.

Lemma 2.5. *We have*

$$\limsup_{n\to+\infty} \frac{n}{e^{\gamma}\varphi(n)\log_2 n} = 1.$$

Proof. From the previous lemma and the definition of $K(n)$, we deduce that

$$\frac{\mathfrak{L}(n)}{\log_2 n} \leq \frac{\mathfrak{L}(N_{K(n)})}{\log_2 N_{K(n)}}.$$

So, our limit becomes as follows:

$$\limsup_{n\to+\infty} \frac{\mathfrak{L}(n)}{e^\gamma n \log_2 n} = \lim_{k\to+\infty} \frac{\mathfrak{L}(N_k)}{e^\gamma N_k \log_2 N_k}.$$

In particular, when $f(x) = \frac{x}{x-1}$, one obtains according to Mertens' theorem that

$$\mathfrak{L}(N_k) = \prod_{p \leq p_k} \frac{p}{p-1} \sim e^\gamma \log p_k,$$

as $k \to +\infty$. Thus, the lemma follows by recalling that

$$\log_2 N_k = \log(\theta(p_k)) \sim \log p_k,$$

using the Prime Number Theorem. $\square$

Every proof containing explicit results requires at some point or another a digital verification of the property obtained on the finite number of cases that remain. In our case, we need to compute the values of $\frac{\sigma(n)}{e^\gamma n \log_2 n}$ for fairly large n. We will use the result of Briggs [5], where he checked Robin's inequality up to $10^{10^{10}}$.

Lemma 2.6 (Briggs)**.** *Robin's criterion holds, for* $5040 < n \leq 10^{10^{10}}$.

We end this section by mentioning the following recent explicit bounds of $\theta(x)$ and $\prod_{p\leq x}(1-\frac{1}{p})$.

Lemma 2.7 (Dusart)**.** *The following estimates hold*

$$\theta(x) \geq x\left(1 - \frac{0.01}{\log^3 x}\right), \text{ as soon as } x \geq 7232121212. \quad (4)$$

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1} \leq e^\gamma \log x \left(1 + \frac{0.2}{\log^3 x}\right), \text{ when } x \geq 2278382. \quad (5)$$

$$\theta(p_k) \geq k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 2.050735}{\log k}\right), \text{ when } p_k \geq 10^{11}. \quad (6)$$

$$p_k \leq k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 1.95}{\log k}\right), \text{ when } k \geq 178974. \quad (7)$$

3 Proof of Theorem 1.1

To begin with, for n such that $K := K(n) \geq K_1 = 164607$ we have $p_K \geq 2228382$. This implies by Lemmas [2.4, 2.7] that

$$\frac{n}{\varphi(n)} \leq \prod_{p\leq p_K} \frac{p}{p-1} \leq e^\gamma \log p_K \left(1 + \frac{0.2}{\log^3 p_K}\right). \quad (8)$$

On the other hand, according to inequality (4), once $K \geq K_2 = 7232121212$, it follows that

$$\log_2 N_K = \log\theta(p_K) \geq \log p_K - \frac{0.01}{\log^3 p_K}. \quad (9)$$

Now, with some care, one can write for $K \geq K_2$ the following

$$\begin{aligned} e^\gamma \log p_K \left(1 + \frac{0.2}{\log^3 p_K}\right) &= e^\gamma \log p_K + \frac{0.2e^\gamma}{\log^2 p_K} \\ &= e^\gamma \log p_K \left(1 - \frac{0.01}{\log^2 p_K}\right) + \frac{(0.2+0.01)e^\gamma}{\log^2 p_K} \\ &= e^\gamma \log p_K \left(1 - \frac{0.01}{\log^3 p_K}\right) + \frac{0.3741}{\log^2 p_K}. \end{aligned}$$

Hence, taking into account that the function $e^\gamma t + \frac{0.3741}{t^2}$ is increasing for $t \geq 1$, we easily deduce from inequality (9) that

$$e^\gamma \left(\log p_K - \frac{0.01}{\log^2 p_K}\right) + \frac{0.3741}{\log^2 p_K} < e^\gamma \log_2 N_K + \frac{0.3741}{\log_2^2 N_K},$$

and then

$$\frac{n}{\varphi(n)} \leq e^\gamma \log_2 N_K + \frac{0.3741}{\log_2^2 N_K}, \ \forall K \geq K_2.$$

By computer, the last inequality is shown to be also valid when $2 \leq K < K_2$. Consequently, invoking again the increase of the function $e^\gamma t + \frac{0.3741}{t^2}$, one gets for $n \geq N_2$, and then for $n \geq 3$ that

$$\frac{n}{\varphi(n)} \leq e^\gamma \log_2 n + \frac{0.3741}{\log_2^2 n}.$$

Finally, as the inequality $\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)}$ holds (see [13, page 193]) for $n \geq 1$, the theorem follows. □

The following direct consequence joins the upper bounds of $\frac{\sigma(n)}{n}$ in the form $(1+\epsilon)e^\gamma \log_2 n$ given in [2] for different values of ϵ. The value $\epsilon = 0.0000123$ obtained below, once $n \geq 5041$, remains stable until the best value $\epsilon = 0.005558981\ldots$ obtained in [2], as soon as $n \geq 2521$.

Corollary 3.1. *For every integer $n \geq 5041$, we have*

$$\frac{\sigma(n)}{n} \leq (1.0000123)e^\gamma \log_2 n.$$

Proof. The idea is to take the term $\frac{0.3741}{\log_2^2 n}$ from Theorem 1.1, divide it by $e^\gamma \log_2 n$, then calculate the image of $10^{10^{10}}$. The remainder is guaranteed by Lemma 2.6. □

4 Proof of Theorem 1.2

By inequality (5) we infer that for every $k \geq K_1 = 164607$:

$$\frac{N_k}{\varphi(N_k)} = \prod_{p \leq p_k} \frac{p}{p-1} \leq e^\gamma \log p_k \left(1 + \frac{0.2}{\log^3 p_k}\right).$$

However; see [12], we have

$$k \log k \leq p_k \leq k(\log k + \log_2 k),$$

once $k \geq 6$. So, we obtain the following inequalities:

$$2\log_2 k \leq \log p_k \leq \log k + \log_2 k + \frac{\log_2 k}{\log k}, \forall k \geq 6,$$

which implies successively for $k \geq K_1$:

$$\begin{aligned}\frac{N_k}{\varphi(N_k)} &\leq e^\gamma \left(\log p_k + \frac{0.2}{\log^2 p_k}\right) \\ &\leq e^\gamma \left(\log k + \log_2 k + \frac{\log_2 k}{\log k} + \frac{0.2}{4\log_2^2 k}\right).\end{aligned}$$

Then, it comes by computer that the last upper bound also holds for $k \geq 10$. Hence, one gets for all $n \geq N_{10}$, according to Lemma 2.4, that

$$\frac{n}{\varphi(n)} \leq e^\gamma \left(\log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} + \frac{0.2}{4\log_2^2 K(n)}\right). \tag{10}$$

Now, let us go back to the ratio $\frac{\sigma(n)}{n}$. According to [13], this quantity takes maximal values on so called *colossally abundant* (CA) numbers, and if Robin's inequality is true on consecutive CA numbers CA_i and CA_{i+1}, then it is also true for all integer $n \in [CA_i, CA_{i+1}]$. We say that n is colossally abundant if there exists a positive ϵ for which:

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(k)}{k^{1+\epsilon}}, \ \forall k > 1.$$

Thus, to complete our proof, it suffices to check inequality (10) for $\frac{\sigma(n)}{n}$ only on the CA numbers less than N_{10}, namely: 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, $21621600, 367567200$ and 6983776800. □

Next, this leads us to discuss a possible reformulation of RH in terms of arithmetic functions. First, we observe that the following proposition

Proposition 1. *We have, when $205 \leq n \leq CA_{160}$, the inequality*

$$\frac{\sigma(n)}{n} \leq e^\gamma \left(\log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)}\right),$$

where $CA_{160} > 10^{326}$.

Proof. It suffices to check the list of terms of the sequence registered as A004490 of CA numbers in OEIS [15]. This extends the inequality to all integers between 205 and CA_{160}.

The following table shows part of the calculations, where $e^\gamma A(n)$ is the upper bound of Proposition 1.

n	$\sigma(n)/n$	$K(n)$	$e^\gamma A(n) - \sigma(n)/n$
$CA_{150} = N_{121}N_{11}N_5N_3N_2^3N_1^4$	11.570817	127	0.44727552
$CA_{151} = N_{122}N_{11}N_5N_3N_2^3N_1^4$	11.588010	128	0.44658941
$CA_{152} = N_{123}N_{11}N_5N_3N_2^3N_1^4$	11.605127	129	0.44584657
$CA_{153} = N_{124}N_{11}N_5N_3N_2^3N_1^4$	11.622118	130	0.44509823
$CA_{154} = N_{125}N_{11}N_5N_3N_2^3N_1^4$	11.638937	131	0.44439327
$CA_{155} = N_{126}N_{11}N_5N_3N_2^3N_1^4$	11.655541	132	0.44377752
$CA_{156} = N_{127}N_{11}N_5N_3N_2^3N_1^4$	11.671980	133	0.44320089
$CA_{157} = N_{128}N_{11}N_5N_3N_2^3N_1^4$	11.688214	134	0.44270719
$CA_{158} = N_{129}N_{11}N_5N_3N_2^3N_1^4$	11.704291	135	0.44224879
$CA_{159} = N_{130}N_{11}N_5N_3N_2^3N_1^4$	11.720259	136	0.44178089
$CA_{160} = N_{131}N_{11}N_5N_3N_2^3N_1^4$	11.736118	137	0.44130365

This completes the proof. □

In view of this numerical experiments the natural question is:

Question 1. *Is it true that*

$$\frac{\sigma(n)}{n} \le e^\gamma \left(\log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right),$$

for all $n \ge 205$?

An answer to this question is linked to RH by the following proposition:

Proposition 2. *If the Riemann Hypothesis hold, we have for every integer $n \ge 205$:*

$$\frac{\sigma(n)}{n} \le e^\gamma \left(\log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right).$$

Proof. This is deduced from Robin's criterion and essentially from the fact that $A(n) \ge \log_2 n$, for every $n \ge 10^{322}$. Indeed, one gets from Lemma 2.3 that

$$\log K(x) \ge \log_2 x - \log_3 x - \log \delta, \ \forall x \ge 3, \tag{11}$$

$$\log_2 K(x) \ge \log_3 x + \log \left(1 - \frac{\log_3 x + \log \delta}{\log_2 x} \right), \ \forall x \ge 3, \tag{12}$$

and from Lemma 2.2 the following

$$\frac{\log_2 K(x)}{\log K(x)} \ge \frac{\log_4 x}{\log_3 x}, \ \forall x \ge 15. \tag{13}$$

Thus, inequalities (11), (12) and (13) yield us for $x \ge 15$:

$$A(x) \ge \log_2 x + \frac{\log_4 x}{\log_3 x} + \log \left(1 - \frac{\log_3 x + \log \delta}{\log_2 x} \right) - \log \delta.$$

By setting $\log_2 x = t$, the study of the following function:

$$\frac{\log_4 x}{\log_3 x} + \log\left(1 - \frac{\log_3 x + \log \delta}{\log_2 x}\right) - \log \delta$$

becomes less complicated, and reveals that it is increasing and positive as soon as $x \ge 10^{322}$.

This implies that

$$A(x) \geq \log_2 x, \ \forall x \geq 10^{322}.$$

Finally, if the Riemann Hypothesis holds, first we have from Robin's criterion that $\frac{\sigma(n)}{n} \leq e^\gamma A(n)$ for all $n \geq 10^{322}$, and thanks to the computations of Proposition 1 for the remaining values. $\square$

At this level, part of Conjecture 1 is proven and the persistent question is:

Question 2. *Is it true that if RH is false, the inequality*

$$\frac{\sigma(n)}{n} \leq e^\gamma \left(\log K(n) + \log_2 K(n) + \frac{\log_2 K(n)}{\log K(n)} \right)$$

is violated for infinitely many $n \geq N_3$?

A heuristic motivation runs as follows:

$$\begin{aligned} K(n) \approx \log n / \log_2 n \underset{\log n/\log_2 n \nrightarrow 1}{\Longrightarrow} & \log K(n) \approx \log_2 n - \log_3 n \approx \log_2 n \\ & \Longrightarrow \log K(n) + \log_2 K(n) \approx \log_2 n \\ & \Longrightarrow A(n) \approx \log_2 n. \end{aligned}$$

Hence, according to Robin's criterion, since $\dfrac{\sigma(n)}{n} > e^\gamma \log_2 n$ infinitely often, if the Riemann Hypothesis is false, as $A(n) \approx \log_2 n$, there may exist infinitely many n such that

$$\frac{\sigma(n)}{n} > e^\gamma A(n).$$

Acknowledgements

The authors wish to thank the referees for their constructive comments and suggestions.

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