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Arithmetical functions commutable with sums of squares

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Abstract: Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$, where $\mathbb{N}_0$, $\mathbb{C}$ denote the set of nonnegative integers and complex numbers, respectively. We give all functions $f, h_1, h_2, h_3, h_4 : \mathbb{N}_0 \to \mathbb{C}$ which satisfy the relation

$$f(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = h_1(x_1) + h_2(x_2) + h_3(x_3) + h_4(x_4) + K$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$. We also give all arithmetical functions $F, H_1, H_2, H_3, H_4 : \mathbb{N} \to \mathbb{C}$ which satisfy the relation

$$F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = H_1(x_1) + H_2(x_2) + H_3(x_3) + H_4(x_4) + K$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers.

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1 Introduction

Let, as usual, $\mathbb{N},\ \mathbb{N}_0,\ \mathbb{C}$ be the set of positive integers, nonnegative integers and complex numbers, respectively.

Recently, in [3] we gave all solutions of the arithmetical functions $f, g, F, G : \mathbb{N}_0 \to \mathbb{C}$, which satisfy the relations

$$f(a^2+b^2+c^2+d^2) = g(a^2)+g(b^2)+g(c^2)+g(d^2)$$

and

$$F(a^2+b^2+c^2+d^2) = G(a^2+b^2)+G(c^2+d^2)$$

for every $a, b, c, d \in \mathbb{N}_0$. We proved that there are numbers $A, B, C, D \in \mathbb{C}$ such that

$$f(n) = An + 4B, \;\; g(m^2) = Am^2 + B$$

and

$$F(n) = Cn + 2D, \;\; G(n^2+m^2) = C(n^2+m^2) + D$$

hold for every $n, m \in \mathbb{N}_0$.

On the other hand, P. V. Chung [2] studied such multiplicative function for which $f(m^2+n^2) = f(m^2)+f(n^2)$ holds for all $m, n \in \mathbb{N}$. B. Bašić [1] characterized all arithmetical functions such that $f(n^2+m^2) = f(n)^2+f(m)^2$, which is slightly different from Chung's condition. Poo-Sung Park in [8] and [9] proved that if a multiplicative function f and $k \in \mathbb{N}$, $k \geq 3$ satisfy one of following two conditions

$$f(x_1^2+\cdots+x_k^2) = f(x_1)^2+\cdots+f(x_k)^2$$

or

$$f(x_1^2+\cdots+x_k^2) = f(x_1^2)+\cdots+f(x_k^2)$$

for all positive integers $x_1, \ldots, x_k$, then f is the identity function.

Recently, B. M. M. Khanh [5–7] gave all solutions of the following equation

$$f(n^2+Dm^2+k) = f(n)^2+Df(m)^2+k,$$

where $k, D \in \mathbb{N}$. She proved that the solution f of this equation is one of the following assertions:

$$a) \quad f(n) = \varepsilon_{D,k}(n)\frac{1-\sqrt{1-4Dk-4k}}{2(D+1)} \;\; \text{for every} \;\; n \in \mathbb{N},$$

$$b) \quad f(n) = \varepsilon_{D,k}(n)\frac{1+\sqrt{1-4Dk-4k}}{2(D+1)} \;\; \text{for every} \;\; n \in \mathbb{N},$$

$$c) \quad f(n) = \varepsilon_{D,k}(n)n \;\; \text{for every} \;\; n \in \mathbb{N},$$

where $\varepsilon_{D,k} : \mathbb{N} \to \{-1, 1\}$ is an arithmetical function satisfying the relation

$$\varepsilon_{D,k}(n^2+Dm^2+k) = 1 \;\; \text{for every} \quad n, m \in \mathbb{N}.$$

In this paper we improve these theorems by proving the following results:

Theorem 1.1. *Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $f, h_1, h_2, h_3, h_4 : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation*

$$f(x_1^2+x_2^2+x_3^2+x_4^2+k) = h_1(x_1)+h_2(x_2)+h_3(x_3)+h_4(x_4)+K$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$. Then there are numbers $A, B_1, B_2, B_3, B_4 \in \mathbb{C}$ such that

$$h_i(m) = Am^2 + B_i \;\; (i = 1, \ldots, 4)$$

and

$$f(n+k) = An + B_1 + B_2 + B_3 + B_4 + K$$

hold for every $n, m \in \mathbb{N}_0$.

Theorem 1.2. *Let $k \in \mathbb{N}$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $F, H_1, H_2, H_3, H_4 : \mathbb{N} \to \mathbb{C}$ satisfy the relation*

$$F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = H_1(x_1) + H_2(x_2) + H_3(x_3) + H_4(x_4) + K$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}$. Then there are numbers $C, D_1, D_2, D_3, D_4 \in \mathbb{C}$ such that

$$H_i(m) = Am^2 + D_i \ \ (i = 1, \dots, 4)$$

and

$$F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = C(x_1^2 + x_2^2 + x_3^2 + x_4^2) + D_1 + D_2 + D_3 + D_4 + K$$

hold for every $m, x_1, x_2, x_3, x_4 \in \mathbb{N}$.

We note that in the proof of Theorem 1.1, we use special values for the variables and the value $x_i = 0$ is of course an interesting choice. This leads to simple proofs that we present. But we also investigate what happens when we discard the possibility $x_i = 0$ and in fact, similar results hold.

We derive the following corollaries from Theorem 1.1 and Theorem 1.2.

Corollary 1.1. *Let $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, $\ell \geq 4$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $f, g_1, \dots, g_\ell : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation*

$$f(x_1^2 + \cdots + x_\ell^2 + k) = g_1(x_1)^2 + \cdots + g_\ell(x_\ell)^2 + K$$

for every $x_1, \dots, x_\ell \in \mathbb{N}_0$. Then there are numbers $A, B_1, \dots, B_\ell \in \mathbb{C}$ such that

$$g_i(m)^2 = Am^2 + B_i \ \ (i = 1, \dots, \ell)$$

and

$$f(n + k) = An + B_1 + \cdots + B_\ell + K$$

hold for every $n, m \in \mathbb{N}_0$.

Corollary 1.2. *Let $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, $\ell \geq 4$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $f, g_1, \dots, g_\ell : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation*

$$f(x_1^2 + \cdots + x_\ell^2 + k) = g_1(x_1^2) + \cdots + g_\ell(x_\ell^2) + K$$

for every $x_1, \dots, x_\ell \in \mathbb{N}_0$. Then there are numbers $A, B_1, \dots, B_\ell \in \mathbb{C}$ such that

$$g_i(m^2) = Am^2 + B_i \ \ (i = 1, \dots, \ell)$$

and

$$f(n + k) = An + B_1 + \cdots + B_\ell + K$$

hold for every $n, m \in \mathbb{N}_0$.

Corollary 1.3. *Let $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, $\ell \geq 4$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $F, G_1, \dots, G_\ell : \mathbb{N} \to \mathbb{C}$ satisfy the relation*

$$F(x_1^2 + \cdots + x_\ell^2 + k) = G_1(x_1)^2 + \cdots + G_\ell(x_\ell)^2 + K$$

for every $x_1, \dots, x_\ell \in \mathbb{N}$. Then there are numbers $C, D_1, \dots, D_\ell \in \mathbb{C}$ such that

$$G_i(m)^2 = Cm^2 + D_i \ \ (i = 1, \dots, \ell)$$

and

$$F(x_1^2+\cdots+x_\ell^2+k)=C(x_1^2+\cdots+x_\ell^2)+D_1+\cdots+D_\ell+K$$

hold for every $x_1,\ldots,x_\ell\in\mathbb{N}$, $m\in\mathbb{N}$.

Corollary 1.4. *Let* $k\in\mathbb{N}_0$, $\ell\in\mathbb{N}$, $\ell\geq 4$ *and* $K\in\mathbb{C}$. *Assume that the arithmetical functions* $F,G_1,\ldots,G_\ell:\mathbb{N}\to\mathbb{C}$ *satisfy the relation*

$$F(x_1^2+\cdots+x_\ell^2+k)=G_1(x_1^2)+\cdots+G_\ell(x_\ell^2)+K$$

for every $x_1,\ldots,x_\ell\in\mathbb{N}$. *Then there are numbers* $C,D_1,\ldots,D_\ell\in\mathbb{C}$ *such that*

$$G_i(m^2)=Cm^2+D_i\ \ (i=1,\ldots,\ell)$$

and

$$F(x_1^2+\cdots+x_\ell^2+k)=C(x_1^2+\cdots+x_\ell^2)+D_1+\cdots+D_\ell+K$$

hold for every $x_1,\ldots,x_\ell\in\mathbb{N}$, $m\in\mathbb{N}$.

Corollary 1.5. *Let* $\ell\geq 4$. *Assume that the arithmetical functions* $F:\mathbb{N}\to\mathbb{C}$ *satisfies the relation*

$$F(x_1^2+\cdots+x_\ell^2)=F(x_1)^2+\cdots+F(x_\ell)^2$$

for every $x_1,\ldots,x_\ell\in\mathbb{N}$. *Then one of the following assertions holds:*

$$\begin{aligned} &a)\quad F(n)=0\ \textit{for every}\ n\in\mathbb{N},\\ &b)\quad F(n)=\frac{\varepsilon(n)}{\ell}\ \textit{for every}\ n\in\mathbb{N},\\ &c)\quad F(n)=\varepsilon(n)n\ \textit{for every}\ n\in\mathbb{N}, \end{aligned}$$

where $\varepsilon(n)\in\{-1,1\}$ *and* $\varepsilon(x_1^2+\cdots+x_\ell^2)=1$.

Corollary 1.6. *Let* $\ell\geq 4$. *Assume that the arithmetical functions* $F:\mathbb{N}\to\mathbb{C}$ *satisfies the relation*

$$F(x_1^2+\cdots+x_\ell^2)=F(x_1^2)+\cdots+F(x_\ell^2)$$

for every $x_1,\ldots,x_\ell\in\mathbb{N}$. *Then there is a complex* C *such that*

$$F(m^2)=Cm^2\ \textit{for every}\ m\in\mathbb{N}$$

and

$$F(x_1^2+\cdots+x_\ell^2)=C(x_1^2+\cdots+x_\ell^2)$$

hold for every $m,x_1,\ldots,x_\ell\in\mathbb{N}$.

2 Proof of Theorem 1.1

First we prove some lemmas.

Lemma 2.1. (Lagrange's Four-Square Theorem) *Every positive integer can be written as the sum of at most four squares.*

Lemma 2.2. *Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $f, h_1, h_2, h_3, h_4 : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation*

$$f(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = h_1(x_1) + h_2(x_2) + h_3(x_3) + h_4(x_4) + K \tag{1}$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$. Then there are numbers $A, B_1, B_2, B_3, B_4 \in \mathbb{C}$ such that

$$h_i(m) = Am^2 + B_i \ \ (i = 1, \dots, 4) \tag{2}$$

Proof. Let $i \in \{2, 3, 4\}$. Then by swapping the positions of x_1 and x_i in (1), we get the following equations

$$f(x_1^2 + \cdots + x_i^2 + \cdots + x_4^2 + k) = h_1(x_1) + \cdots + h_i(x_i) + \cdots + h_4(x_4) + K$$

and

$$\begin{aligned} f(x_1^2 + \cdots + x_i^2 + \cdots + x_4^2 + k) &= f(x_i^2 + \cdots + x_1^2 + \cdots + x_4^2 + k) \\ &= h_1(x_i) + \cdots + h_i(x_1) + \cdots + h_4(x_4) + K, \end{aligned}$$

which implies that

$$h_1(x_1) + h_i(x_i) = h_1(x_i) + h_i(x_1) \ \text{ and } \ h_i(x_1) - h_1(x_1) = h_i(x_i) - h_1(x_i)$$

hold for every $x_1, x_i \in \mathbb{N}_0$. Consequently $h_i(x_1) - h_1(x_1) = h_i(0) - h_1(0)$. Let $H_1 = 0$, $H_i = h_i(0) - h_1(0)$ for $i \in \{2, 3, 4\}$. Then

$$h_i(n) = h_1(n) + H_i \ \text{ for every } \ n \in \mathbb{N}_0 \ \text{ and } \ i \in \{1, 2, 3, 4\}.$$

Let $h = h_1$ and $H = H_1 + H_2 + H_3 + H_4$. Thus

$$h_i(n) = h(n) + H_i \ \text{ for every } \ n \in \mathbb{N}_0 \ \ (i = 1, \dots, 4) \tag{3}$$

and

$$f(x_1^2 + \cdots + x_4^2 + k) = h(x_1) + \cdots + h(x_4) + H + K \tag{4}$$

hold for every $x_1, \dots, x_4 \in \mathbb{N}_0$.

In the following, let $A = h(1) - h(0)$ and $B = h(0)$. We will prove that

$$h(m) = Am^2 + B \ \text{ for every } \ m \in \mathbb{N}. \tag{5}$$

It is obvious from the definitions of A and B that (5) is true for $m = 0$ and $m = 1$.

Assume that (5) is true for every $m \le N$, where $N \ge 1$. Since

$$(N + 1)^2 + (N - 1)^2 + 0^2 + 0^2 + k = N^2 + N^2 + 1^2 + 1^2 + k,$$

we infer from (4) that

$$h(N + 1) + h(N - 1) + 2h(0) + H + K = 2h(N) + 2h(1) + H + K.$$

Thus, from our assumptions, we have

$$\begin{aligned} h(N+1) &= 2h(N) - h(N-1) + 2h(1) - 2h(0) \\ &= 2(AN^2 + B) - (A(N-1)^2 + B) + 2A \\ &= A(N+1)^2 + B, \end{aligned}$$

which proves that (5) holds for $N+1$, and so (5) holds for every $m \in \mathbb{N}$.

Finally, we obtain from (3) and (5) that

$$h_i(m) = h(m) + H_i = Am^2 + (B + H_i) = Am^2 + B_i \text{ for every } m \in \mathbb{N}_0,$$

where $B_i = B + H_i$. Thus, we proved that (2) is true.

Lemma 2.2 is proved. □

Proof of Theorem 1.1

We prove that

$$f(n+k) = An + B_1 + B_2 + B_3 + B_4 + K \text{ for every } n \in \mathbb{N}_0.$$

We infer from Lemma 1 that for every $n \in \mathbb{N}_0$ there exist $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$ such that

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Then we infer from (1) and (2) that

$$\begin{aligned} f(n+k) &= f(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = h_1(x_1^2) + h_2(x_2^2) + h_3(x_3^2) + h_4(x_4^2) + K = \\ &= \Big(Ax_1^2 + B_1\Big) + \Big(Ax_2^2 + B_2\Big) + \Big(Ax_3^2 + B_3\Big) + \Big(Ax_4^2 + B_4\Big) + K = \\ &= An + B_1 + B_2 + B_3 + B_4 + K \end{aligned}$$

is satisfied for every $n \in \mathbb{N}_0$.

The proof of Theorem 1.1 is finished. □

3 Proof of Theorem 1.2

In this section we assume that the numbers $k \in \mathbb{N}$, $K \in \mathbb{C}$ and the arithmetical functions $F, H_1, H_2, H_3, H_4 : \mathbb{N} \to \mathbb{C}$ satisfy the relation

$$F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = H_1(x_1) + H_2(x_2) + H_3(x_3) + H_4(x_4) + K \tag{6}$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}$. We prove the following lemma.

Lemma 3.1. *There are numbers $C, D_1, D_2, D_3, D_4 \in \mathbb{C}$ such that*

$$H_i(m) = Cm^2 + D_i \quad (i = 1, \ldots, 4) \tag{7}$$

is satisfied for every $m \in \mathbb{N}$.

Proof. Similarly as in the proof of Lemma 2.2, let $i \in \{2, 3, 4\}$ and by swapping the positions of x_1 and x_i in (6), we obtain that

$$H_1(x_1) + H_i(x_i) = H_1(x_i) + H_i(x_1) \text{ and } H_i(x_1) - H_1(x_1) = H_i(x_i) - H_1(x_i)$$

hold for every $x_1, x_i \in \mathbb{N}$. Consequently $H_i(x_1) - H_1(x_1) = H_i(1) - H_1(1)$, and so

$$H_i(n) = H_1(n) + \mathcal{H}_i$$

holds for every $n \in \mathbb{N}, i \in \{1, 2, 3, 4\}$, where $\mathcal{H}_1 = 0, \mathcal{H}_i = H_i(1) - H_1(1)$ for every $i \in \{2, 3, 4\}$. Let $H = H_1$ and $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4$. Then

$$H_i(n) = H(n) + \mathcal{H}_i \ \text{ for every } \ n \in \mathbb{N} \ \ (i = 1, \dots, 4) \tag{8}$$

and

$$F(x_1^2 + \cdots + x_4^2 + k) = H(x_1) + \cdots + H(x_4) + \mathcal{H} + K \tag{9}$$

hold for every $x_1, \dots, x_4 \in \mathbb{N}$.

In the following, let

$$C = \frac{H(2) - H(1)}{3} \ \text{ and } \ D = \frac{4H(1) - H(2)}{3}.$$

We will prove that

$$H(m) = Cm^2 + D \ \text{ for every } \ m \in \mathbb{N}. \tag{10}$$

It is obvious from the definitions of C and D that (10) is true for $m = 1$ and $m = 2$.

In the following we write $[a, b, c, d, x, y, z, t] \in E$ if

$$\begin{cases} a, b, c, d, x, y, z, t \in \mathbb{N}, \ \ a \geq b \geq c \geq d \geq 1, \ \ x \geq y \geq z \geq t \geq 1 \\ a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2 + t^2. \end{cases}$$

It is clear from (9) that

$$\begin{cases} \text{If } \ [a, b, c, d, x, y, z, t] \in E, \\ \text{then } \ H(a) = H(x) + H(y) + H(z) + H(t) - H(b) - H(c) - H(d). \end{cases} \tag{11}$$

Since $[4, 2, 2, 2, 3, 3, 3, 1] \in E$ and $[5, 1, 1, 1, 3, 3, 3, 1] \in E$, we have

$$\begin{aligned} H(4) &= H(1) - 3H(2) + 3H(3), \\ H(5) &= -2H(1) + 3H(3). \end{aligned}$$

On the other hand, we infer from the fact $[5, 2, 2, 1, 4, 4, 1, 1] \in E$ that $H(5) + 2H(2) - H(1) - 2H(4) = 0$, which implies that $-5H(1) + 8H(2) - 3H(3) = 0$. Thus, we have

$$\begin{aligned} H(3) &= \frac{-5H(1) + 8H(2)}{3} = 9\frac{H(2) - H(1)}{3} + \frac{4H(1) - H(2)}{3} = C \cdot 3^2 + D, \\ H(4) &= H(1) - 3H(2) + (-5H(1) + 8H(2)) = -4H(1) + 5H(2) = C \cdot 4^2 + D, \\ H(5) &= -2H(1) + (-5H(1) + 8H(2)) = -7H(1) + 8H(2) = C \cdot 5^2 + D. \end{aligned}$$

Thus we have proved that (10) holds for $m \in \{1, 2, 3, 4, 5\}$.

Now assume that (10) holds for every $m \leq N$, where $N \geq 5$. Using the fact that

$$(N+1)^2 + (N-1)^2 + 2^2 + 2^2 = N^2 + N^2 + 3^2 + 1^2,$$

we infer from (11) that

$$H(N+1) = 2H(N) - H(N-1) + H(3) + H(1) - 2H(2),$$

which with our assumptions implies

$$\begin{aligned} H(N+1) &= 2\left(CN^2 + D\right) - \left(C(N-1)^2 + D\right) + (9C + D) + (C + D) - 2\left(4C + D\right) \\ &= C(N+1)^2 + D. \end{aligned}$$

Thus, we have proved that (10) holds for $N+1$, consequently (10) holds for every $m \in \mathbb{N}$.

Finally, we obtain from (8) and (10) that

$$H_i(m) = H(m) + \mathcal{H}_i = Cm^2 + D + \mathcal{H}_i = Cm^2 + D_i \ \ (i = 1, \ldots, 4),$$

where $D_i = D + \mathcal{H}_i$.

The proof of Lemma 3.1 is finished. □

Proof of Theorem 1.2

Assume that the numbers $k \in \mathbb{N}$, $K \in \mathbb{C}$ and the arithmetical functions $F, H_1, H_2, H_3, H_4 : \mathbb{N} \to \mathbb{C}$ satisfy the relation (6). Then, by using Lemma 3.1, we have

$$\begin{aligned} F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) &= H_1(x_1) + H_2(x_2) + H_3(x_3) + H_4(x_4) + K \\ &= \left(Cx_1^2 + D_1\right) + \left(Cx_2^2 + D_2\right) + \left(Cx_3^2 + D_3\right) + \left(Cx_4^2 + D_4\right) + K \\ &= C\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) + D_1 + D_2 + D_3 + D_4 + K \end{aligned}$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}$.

Theorem 1.2 is proved. □

4 Proofs of corollaries

Proof of Corollary 1.1

Assume that $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, $\ell \geq 4$, $K \in \mathbb{C}$ and the arithmetical functions $f, g_1, \ldots, g_\ell : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation

$$f(x_1^2 + \cdots + x_\ell^2 + k) = g_1(x_1)^2 + \cdots + g_k(x_\ell)^2 + K \tag{12}$$

for every $x_1, \ldots, x_\ell \in \mathbb{N}_0$.

Let $g \in \{g_1, \ldots, g_\ell\}$ be arbitrary. Without loss of generality, we may assume that $g \in \{g_1, g_2, g_3, g_4\}$. Since $\ell \geq 4$, putting $x_5 = \cdots = x_\ell = 0$ into (12), we have

$$f(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = \left(g_1(x_1)^2 + g_2(x_2)^2 + g_3(x_3)^2 + g_4(x_4)^2\right) + g_5(0) + \cdots + g_\ell(0) + K.$$

Then, by applying Theorem 1.1, there are $A, B_1, B_2, B_3, B_4 \in \mathbb{C}$ such that

$$g_1(m)^2 = Am^2 + B_1,\ g_2(m)^2 = Am^2 + B_2,\ g_3(m)^2 = Am^2 + B_3,\ g_4(m)^2 = Am^2 + B_4$$

hold for every $m \in \mathbb{N}_0$. Thus, we have

$$g(m)^2 \in \{Am^2 + B_1, Am^2 + B_2,\ Am^2 + B_3,\ Am^2 + B_4\}.$$

Since $g \in \{g_1, \ldots, g_\ell\}$ be arbitrary, it follows from the above result that for each $i \in \{1, \ldots, \ell\}$ there is $B_i \in \mathbb{C}$ such that $g_i(m)^2 = Am^2 + B_i$. Thus, it follows from Lemma 2.1 and (12) that

$$f(n + k) = An + B_1 + \cdots + B_\ell + K$$

hold for every $n, m \in \mathbb{N}_0$. The proof of Corollary 1.1 is completed. □

Proof of Corollary 1.2

Assume that $k \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, $\ell \geq 4$, $K \in \mathbb{C}$ and the arithmetical functions $f, g_1, \ldots, g_\ell : \mathbb{N}_0 \to \mathbb{C}$ satisfy the relation

$$f(x_1^2 + \cdots + x_\ell^2 + k) = g_1(x_1^2) + \cdots + g_k(x_\ell^2) + K \tag{13}$$

for every $x_1, \ldots, x_\ell \in \mathbb{N}_0$.

Let $g \in \{g_1, \ldots, g_\ell\}$ be arbitrary. Without loss of generality, we may assume that $g \in \{g_1, g_2, g_3, g_4\}$.

Similarly as the proof of Corollary 1.2, there are $A, B_1, B_2, B_3, B_4 \in \mathbb{C}$ such that

$$g_1(m^2) = Am^2 + B_1,\ g_2(m^2) = Am^2 + B_2,\ g_3(m^2) = Am^2 + B_3,\ g_4(m^2) = Am^2 + B_4$$

hold for every $m \in \mathbb{N}_0$.

Thus, we have

$$g(m^2) \in \{Am^2 + B_1, Am^2 + B_2,\ Am^2 + B_3,\ Am^2 + B_4\}.$$

Since $g \in \{g_1, \ldots, g_\ell\}$ be arbitrary, it follows from the above result that for each $i \in \{1, \ldots, \ell\}$ there is $B_i \in \mathbb{C}$ such that $g_i(m^2) = Am^2 + B_i$. Thus, it follows from Lemma 2.1 and (12) that

$$f(n + k) = An + B_1 + \cdots + B_\ell + K$$

hold for every $n, m \in \mathbb{N}_0$.

The proof of Corollary 1.2 is completed. □

Proof of Corollary 1.3 and Corollary 1.4

We can prove Corollary 1.3 and Corollary 1.4 similarly as in the proof of Corollary 1.1 and Corollary 1.2. Hence, we omit these proofs.

Proof of Corollary 1.5

Assume that $\ell \geq 4$ and the arithmetical functions $F : \mathbb{N} \to \mathbb{C}$ satisfy the relation

$$F(x_1^2 + \cdots + x_\ell^2) = F(x_1)^2 + \cdots + F(x_\ell)^2$$

for every $x_1, \dots, x_\ell \in \mathbb{N}$. From Corollary 1.3, we have

$$F(m)^2 = Cm^2 + D \quad \text{and} \quad F(x_1^2 + \cdots + x_\ell^2) = C(x_1^2 + \cdots + x_\ell^2) + \ell D.$$

These imply that

$$\Big(C(x_1^2 + \cdots + x_\ell^2) + \ell D\Big)^2 = C(x_1^2 + \cdots + x_\ell^2)^2 + D$$

holds for every $x_1, \dots, x_\ell \in \mathbb{N}$, consequently

$$\begin{cases} C^2 &= C \\ CD &= 0 \\ \Big(\ell D\Big)^2 &= D. \end{cases}$$

The solutions of this system are:

$$(C, D) \in \Big\{\Big(0, 0\Big), \Big(0, \frac{1}{\ell^2}\Big), \Big(1, 0\Big)\Big\}.$$

Therefore, the proof of Corollary 1.5 is completed. □

Proof of Corollary 1.6.
Assume that $\ell \geq 4$ and the arithmetical functions $F : \mathbb{N} \to \mathbb{C}$ satisfy the relation

$$F(x_1^2 + \cdots + x_\ell^2) = F(x_1^2) + \cdots + F(x_\ell^2)$$

for every $x_1, \dots, x_\ell \in \mathbb{N}$. From Corollary 1.4, we have

$$F(m^2) = Cm^2 + D \quad \text{and} \quad F(x_1^2 + \cdots + x_\ell^2) = C(x_1^2 + \cdots + x_\ell^2) + \ell D.$$

Since there are numbers $m, x_1, \dots, x_\ell \in \mathbb{N}$ such that $m^2 = x_1^2 + \cdots + x_\ell^2$, thus

$$Cm^2 + \ell D = Cm^2 + D,$$

consequently $D = 0$. Therefore, the proof of Corollary 1.6 is finished. □

5 Some remarks

Remark 1. Recently in [4], we consider a similar problem for arithmetical functions commutable with sums of three squares. We gave all functions $f, h : \mathbb{N} \to \mathbb{C}$ which satisfy the relation

$$f(a^2 + b^2 + c^2 + k) = h(a) + h(b) + h(c) + K$$

for every $a, b, c \in \mathbb{N}$, where $k \geq 0$ is an integer and K is a complex number. If n cannot be written as $a^2 + b^2 + c^2 + k$ for suitable $a, b, c \in \mathbb{N}$, then $f(n)$ is not determined. This is more complicated if we assume that f and h are multiplicative functions.

Let

$$\mathbb{M} = \{a^2 + b^2 + c^2 \mid a, b, c \in \mathbb{N}\}$$

and for each $k \in \mathbb{N}_0$ let

$$\mathcal{H}_k = \begin{cases} \{1, \dots, e + 2\} & \text{if } k = 2^e k_1, (k_1, 2) = 1, 2 \mid e \text{ and } k_1 \equiv 1 \pmod 8, \\ \mathbb{N} \setminus \{e + 2\} & \text{if } k = 2^e k_1, (k_1, 2) = 1, 2 \mid e \text{ and } k_1 \equiv 5 \pmod 8, \\ \mathbb{N} & \text{in any other cases.} \end{cases}$$

Among the other results, we prove the following theorem.

Theorem A. (Kátai I. and B. M. Phong [4]) *Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. Assume that multiplicative functions F, H satisfy the relation*

$$F(a^2 + b^2 + c^2 + k) = H(a) + H(b) + H(c) + K$$

for every $a, b, c \in \mathbb{N}$.

Then one of the following assertions holds:

1) $H(m) = 1$ *and* $F(\eta + k) = 0$ $(\forall m \in \mathbb{N},\ \forall \eta \in \mathbb{M})$ *if* $K = -3$
2) $H(m) = 1$ *and* $F(2n+1) = 1, F(2^\alpha) = K + 3$ $(\forall m, n \in \mathbb{N},\ \forall \alpha \in \mathcal{H}_k)$ *if* $K \neq -3$
3) $H(m) = \chi_2(m)$ *and* $F(2n+1) = 1$ $(\forall m, n \in \mathbb{N})$ *if* $(\overline{k}, K) \neq (3, -1)$
4) $H(m) = \chi_2(m)$ *and* $F(2n+1) = (-1)^n, F(2) = (-1)^{\frac{k+1}{4}} 2, F(2^\alpha) = 0$ $(\forall m, n \in \mathbb{N},\ \forall \alpha \in \mathcal{H}_k, \alpha \geq 2)$ *if* $(\overline{k}, K) = (3, -1)$
5) $H(m) = m^2, F(2n+1) = 2n+1$ *and* $F(2^\alpha) = 2^\alpha$ $(\forall m, n \in \mathbb{N},\ \forall \alpha \in \mathcal{H}_k)$,

where $\chi_2(m)$ is the Dirichlet character $(\mathrm{mod}\ 2)$.

Remark 2. Let

$\mathcal{R} = \{r \in \mathbb{N} \mid r^2 = a^2+b^2 = u^2+v^2+w^2 = x^2+y^2+z^2+t^2$ for some $a, b, u, v, w, x, y, z, t \in \mathbb{N}.\}$

It is clear to check that

$$\{13, 17, 25, 26, 29, 34, 37, 41, 45, \ldots\} \subset \mathcal{R},$$

for examples

$$13^2 = 12^2 + 5^2 = 12^2 + 4^2 + 3^2 = 8^2 + 8^2 + 5^2 + 4^2,$$
$$45^2 = 36^2 + 27^2 = 29^2 + 28^2 + 20^2 = 26^2 + 24^2 + 22^2 + 17^2.$$

By using Lemma 2.1, we can show that for all $n \in \mathbb{N}$ and $r \in \mathcal{R}$ there are $x_1, x_2, x_3, x_4 \in \mathbb{N}$ such that

$$r^2 n = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

From Theorem 1.2 we obtain the following theorem.

Theorem B. *Let $k \in \mathbb{N}_0$ and $K \in \mathbb{C}$. Assume that the arithmetical functions $F, H_1, H_2, H_3, H_4 : \mathbb{N} \to \mathbb{C}$ satisfy the relation*

$$F(x_1^2 + x_2^2 + x_3^2 + x_4^2 + k) = H_1(x_1) + H_2(x_2) + H_3(x_3) + H_4(x_4) + K$$

for every $x_1, x_2, x_3, x_4 \in \mathbb{N}$. Then there are numbers $C, D_1, D_2, D_3, D_4 \in \mathbb{C}$ such that

$$H_i(m) = Am^2 + D_i \ \ (i = 1, \ldots, 4)$$

and

$$F(r^2 n + k) = Cr^2 n + D_1 + D_2 + D_3 + D_4 + K$$

hold for every $m, n \in \mathbb{N}$.

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