Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 3, 119–122
DOI: 10.7546/nntdm.2021.27.3.119-122

Perron numbers that satisfy Fermat's equation

Pietro Paparella

Division of Engineering and Mathematics, University of Washington Bothell
18115 Campus Way NE, Bothell, WA 98011, United States
e-mail: pietrop@uw.edu

Received: 22 December 2020 **Revised:** 16 July 2021 **Accepted:** 13 August 2021

Abstract: In this note, it is shown that if ℓ and m are positive integers such that $\ell > m$, then there is a *Perron number* ρ such that $\rho^n + (\rho + m)^n = (\rho + \ell)^n$. It is also shown that there is an aperiodic integer matrix C such that $C^n + (C + mI_n)^n = (C + \ell I_n)^n$.
Keywords: Perron number, Fermat equation, Integer matrix, Aperiodic matrix.
2020 Mathematics Subject Classification: 11D41, 15B36.

1 Introduction

As is well-known, *Fermat's last theorem* guarantees that the Diophantine equation

$$x^n + y^n = z^n, \tag{1}$$

called the *Fermat equation*, has no nontrivial solutions. Although the case concerning integer solutions is settled, effort has been spent in identifying solutions of (1) with respect to other rings: e.g., Arnold and Eydelzon [1] presented a parameterization for *Pythagorean matrices*, which is an ordered triple (A, B, C) of integral matrices such that $A^2 + B^2 = C^2$; and Brenner and de Pillis [2] investigated when the existence of nonsingular matrices $A, B, C \in \mathsf{M}_n(\mathbb{Z})$ to the Fermat matrix equation $A^p + B^p = C^p$, $p > 2$ guaranteed the existence of a nontrivial triple a, b, and c of algebraic integers to the corresponding Fermat equation $a^p + b^p = c^p$, and vice-versa.

In this work, we extend the work of Brenner and de Pillis and show that if ℓ and m are positive integers such that $\ell > m$, then there is a *Perron number* ρ such that $(\rho, \rho + m, \rho + \ell)$ satisfies (1). Additionally, it is also shown that there is an aperiodic integer matrix C such that $(C, C + mI_n, C + \ell I_n)$ satisfies (1).

2 Background

If A is a nonnegative matrix, then A is called *aperiodic* if there is a positive integer k such that A^k is entrywise positive. The Perron–Frobenius theorem for positive matrices (see, e.g., [4, Theorem 8.2.10]) asserts that the spectral radius

$$\rho = \rho(A) := \max(\{|\lambda| : \lambda \in \sigma(A)\})$$

is a simple eigenvalue of A. If, in addition, A has integer entries, then ρ is a positive algebraic integer that dominates its algebraic conjugates in modulus. Such a number is called a *Perron number* [5]. Conversely, if ρ is an algebraic integer that dominates its algebraic conjugates in modulus, then there is an aperiodic integer matrix with spectral radius ρ [5, Theorem 1]. The set of Perron numbers $\mathbb{P}$ is closed with respect to addition and multiplication [6, Section 5] and represents the closure of $\mathbb{N}$ with respect to taking the spectral radius of aperiodic integer matrices.

If $A = [a_{ij}] \in \mathsf{M}_n(\mathbb{C})$, then the *digraph of* A, denoted by $\Gamma(A)$, is the directed graph with vertices $V = \{1, \ldots, n\}$ and edges $E = \{(i,j) \in V^2 \mid a_{ij} \neq 0\}$. For $n \geq 2$, an $n \times n$ matrix A is called *reducible* if there is a permutation matrix P such that

$$P^\top AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

in which A_{11} and A_{22} are nonempty square matrices. If A is not reducible, then A is called *irreducible*, and A is irreducible if and only if $\Gamma(A)$ is strongly connected [3, Theorem 3.2.1].

An irreducible nonnegative matrix is called *primitive* if its digraph is primitive (i.e., the greatest common divisor of the lengths of the closed directed walks is one); otherwise it is *imprimitive*. If A is nonnegative, then A is aperiodic if and only if A is primitive [3, Theorem 3.4.4].

Given an $n \times n$ matrix A, the *characteristic polynomial of* A, denoted by χ_A, is defined by $\chi_A(t) = \det(tI - A)$. The *companion matrix* $C = C_p$ of a monic polynomial

$$p(t) = t^n + \sum_{k=1}^{n} c_{n-k} t^{n-k}$$

is the $n \times n$ matrix

$$\begin{bmatrix} 0 & I \\ -c_0 & -c \end{bmatrix},$$

where $c = \begin{bmatrix} c_1 & \cdots & c_{n-1} \end{bmatrix}$. It is well-known that $\chi_C = p$. Notice that if $c_i \neq 0$, then $\Gamma(C)$ contains a cycle of length $n - i$.

3 Perron numbers that satisfy the Fermat equation

Given positive integers $\ell > m$ and $\rho \in \mathbb{C}$, note that, as a consequence of the binomial theorem, $\rho^n + (\rho + m)^n = (\rho + \ell)^n$ if and only if ρ is a zero of the polynomial

$$p(t) = p_n(t) = t^n - \sum_{k=1}^{n} \binom{n}{k} t^{n-k} (\ell^k - m^k). \tag{2}$$

The fundamental theorem of algebra ensures that p has, counting multiplicities, n zeros. However, more can be said about these zeros.

Theorem 3.1. *Let $n \in \mathbb{N}$. If ℓ and m are positive integers such that $\ell > m$, then there is a Perron number ρ such that $(\rho, \rho + m, \rho + \ell)$ satisfies* (1).

Proof. Let p be the monic polynomial defined as in (2). For $i \in \{0, 1, \ldots, n-1\}$, let

$$c_i := \binom{n}{i} \left(\ell^{n-i} - m^{n-i}\right).$$

The companion matrix

$$C = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix} \in \mathsf{M}_n(\mathbb{Z}) \tag{3}$$

is nonnegative because $c_i > 0$, $0 \le i \le n-1$. Since the digraph $\Gamma(C)$ has a cycle of length i, for every $i \in \{1, \ldots, n\}$, it follows that $\Gamma(C)$ is primitive, i.e., the matrix C is aperiodic. Thus, the spectral radius $\rho = \rho(C)$ is a Perron number that is a zero of p.

Lastly, notice that $\rho + m$ and $\rho + \ell$ are the spectral radii of the aperiodic integral matrices $C + mI_n$ and $C + \ell I_n$, respectively. □

Remark 3.2. Since

$$p_n'(t) = nt^{n-1} - \sum_{k=1}^{n-1} \binom{n}{k} (n-k) t^{n-k-1} (\ell^k - m^k)$$

and

$$\binom{n}{k}(n-k) = \frac{n!(n-k)}{k!(n-k)!} = n\frac{(n-1)!}{k!(n-1-k)!} = n\binom{n-1}{k},$$

it follows that

$$\frac{p_n'(t)}{n} = t^{n-1} - \sum_{k=1}^{n-1} \binom{n-1}{k} t^{n-k-1} (\ell^k - m^k) = p_{n-1}(t).$$

By the Gauss–Lucas theorem, which asserts that the critical points of a polynomial lie in the convex hull of its zeros, it follows that the zeros of p_{n-1} are in the convex hull of the zeros of p_n.

4 Aperiodic matrices that satisfy the Fermat equation

Brenner and de Pillis [2] showed that if

$$A := \begin{bmatrix} & a & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{bmatrix} \in \mathsf{M}_n(\mathbb{R}),$$

with $a \in \mathbb{N}$, then $A^n = aI$. As such, identifying solutions to (1) with respect to irreducible nonnegative integer matrices is somewhat trivial. However, requiring that the matrices be aperiodic is a natural and interesting restriction.

Theorem 4.1. *Let $n \in \mathbb{N}$. If ℓ and m are positive integers such that $\ell > m$, then there is an aperiodic integer matrix C such that $(C, C + mI_n, C + \ell I_n)$ satisfies* (1).

Proof. Let p be the monic polynomial defined as in (2) and C be the nonnegative companion matrix defined as in (3). By the Cayley–Hamilton theorem, $p(C) = 0$, i.e.,

$$0 = C^n - \sum_{k=1}^{n} \binom{n}{k} C^{n-k}(\ell^k - m^k). \tag{4}$$

Since C commutes with aI_n, $\forall a \in \mathbb{N}$, it follows from the binomial theorem that

$$(C + aI_n)^n = \sum_{k=0}^{n} \binom{n}{k} C^{n-k}(aI_n)^k = \sum_{k=0}^{n} \binom{n}{k} a^k C^{n-k}. \tag{5}$$

Adding C^n to both sides of (4) and applying (5) yields

$$C^n + (C + mI_n)^n = (C + \ell I_n)^n. \qquad \square$$

Remark 4.2. We conclude by noting that if Z is an invertible integer matrix such that $\det Z = \pm 1$ and $A = ZCZ^{-1}$, then $(A, A + mI_n, A + \ell I_n)$ is an ordered triple of integer matrices that satisfies (1).

References

[1] Arnold, M., & Eydelzon, A. (2019). On matrix Pythagorean triples. *The American Mathematical Monthly*, 126(2), 158–160.

[2] Brenner, J. L., & de Pillis, J. (1972). Fermat's equation $A^p + B^p = C^p$ for matrices of integers. *Mathematics Magazine*, 45, 12–15.

[3] Brualdi, R. A., & Ryser, H. J. (1991). *Combinatorial Matrix Theory*. Cambridge: Cambridge University Press.

[4] Horn, R. A., & Johnson, C. R. (1990). *Matrix Analysis*. Cambridge: Cambridge University Press.

[5] Lind, D. A. (1983). Entropies and factorizations of topological Markov shifts. *Bulletin of the American Mathematical Society*, 9(2), 219–222.

[6] Lind, D. A. (1984). The entropies of topological Markov shifts and a related class of algebraic integers. *Ergodic Theory and Dynamical Systems*, 4, 283–300.