Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 3, 113–118
DOI: 10.7546/nntdm.2021.27.3.113-118

A Diophantine equation about polygonal numbers

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Received: 8 July 2020 **Revised:** 15 August 2021 **Accepted:** 27 August 2021

Abstract: It is well known that the number $P_k(x) = \frac{x((k-2)(x-1)+2)}{2}$ is called the x-th k-gonal number, where $x \geq 1, k \geq 3$. Many Diophantine equations about polygonal numbers have been studied. By the theory of Pell equation, we show that if $G(k-2)(A(p-2)a^2+2Cab+B(q-2)b^2)$ is a positive integer but not a perfect square, $(2A(p-2)\alpha-(p-4)A+2C\beta+2D)a+(2B(q-2)\beta-(q-4)B+2C\alpha+2E)b > 0$, $2G(k-2)\gamma-(k-4)G+2H > 0$ and the Diophantine equation

$$AP_p(x) + BP_q(y) + Cxy + Dx + Ey + F = GP_k(z) + Hz$$

has a nonnegative integer solution (α, β, γ), then it has infinitely many positive integer solutions of the form $(at+\alpha, bt+\beta, z)$, where $p,q,k \geq 3$ and $p, q, k, a, b, t, A, B, G \in \mathbb{Z}^+$,$C, D, E, F, H \in \mathbb{Z}$.
Keywords: Polygonal number, Diophantine equation, Pell equation, Positive integer solution.
2020 Mathematics Subject Classification: 11D09, 11D72.

1 Introduction

A polygonal number [3] is a positive number, corresponding to an arrangement of points on the plane, which forms a regular polygon. The x-th k-gonal number [3, p. 5] is

$$P_k(x) = \frac{x\big((k-2)(x-1)+2\big)}{2},$$

where $x \geq 1, k \geq 3$. There are many papers about the polygonal numbers and many properties of them have been studied, we can refer to the first chapter of [4] and D3 of [6].

Several authors investigated the question of when a linear combination of two polygonal numbers is a perfect square. Such as M. H. Le [10] ($1+9P_3(n) = z^2$, proposed by M. Bencze [1]);

X. G. Guan [5] ($1+\frac{8s^2}{s^2-1}P_3(n) = z^2$); M. J. Hu [8] ($1 + nP_3(y-1) = z^2$); J. Y. Peng [12] ($1 + nP_3(y-1) = z^2$ and $mP_3(x-1) + nP_3(y-1) = z^2$); M. Jiang and Y. C. Li [9] ($1 + nP_k(y) = z^2$ and $mP_k(x) + nP_k(y) = z^2$); Y. C. Li [11] ($mP_p(x) + nP_q(y) = z^2$).

For the Diophantine equation

$$P_3(a) + P_3(b) = P_3(c), \tag{1.1}$$

K. R. S. Sastry [14] studied the problem: given a natural number N, determine the number $T(N)$ of Eq. (1.1) such that $a = N$.

A. Hamtat and D. Behloul [7] proved that all nonnegative integer solutions of Eq. (1.1) are given by

$$a = mn, \quad b = \frac{nq - mp - 1}{2}, \quad c = \frac{nq + mp - 1}{2},$$

where $pq - mn = 1$, $nq - mp - 1 \in 2\mathbb{N}$ and $m, n, p, q \in \mathbb{N}$.

K. R. S. Sastry [15] investigated the positive integer solutions of the Diophantine equation

$$P_n(a) + P_n(b) = P_n(c), \quad n \geq 3. \tag{1.2}$$

E. Scheffold [16] gave a parametric representation for Eq. (1.2), i.e.,

$$a = (n-2)r + t, \quad b = (n-2)r + s, \quad c = (n-2)r + s + t,$$

where $n \geq 3$, r, s, t are natural numbers such that

$$r((n-2)^2 r - (n-4)) = 2st.$$

H. Cohen [2, Corollary 6.3.6.] introduced the general solutions of the Diophantine equation

$$Ax^2 + By^2 = Cz^2,$$

i.e., "assume that $ABC \neq 0$, let (x_0,y_0,z_0) be a particular nontrivial solution of $Ax^2+By^2=Cz^2$, and assume that $z_0 \neq 0$. The general solution in rational numbers to the equation is given by

$$\begin{cases} x = d\big(x_0(As^2 - Bt^2) + 2y_0 Bst\big), \\ y = d\big(2x_0 Ast - y_0(As^2 - Bt^2)\big), \\ z = dz_0(As^2 + Bt^2), \end{cases}$$

where $d \in \mathbb{Q}$, $s, t \in \mathbb{Z}$, and $\gcd(s,t) = 1$."

J. Pla [13] investigated some subsets of the rational solutions of the Diophantine equation

$$aX^2 + bXY + cY^2 = dZ^2,$$

where $a, b, c, d \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$.

2 Main result

We consider the polygonal numbers satisfying the Diophantine equation

$$AP_p(x) + BP_q(y) + Cxy + Dx + Ey + F = GP_k(z) + Hz, \tag{2.1}$$

where $p, q, k \geq 3$, $p, q, k, A, B, G \in \mathbb{Z}^+$, $C, D, E, F, H \in \mathbb{Z}$. By the theory of Pell equation, we have the following theorem.

Theorem 2.1. *If $G(k-2)(A(p-2)a^2+2Cab+B(q-2)b^2)$ is a positive integer but not a perfect square, $(2A(p-2)\alpha-(p-4)A+2C\beta+2D)a+(2B(q-2)\beta-(q-4)B+2C\alpha+2E)b>0$, $2G(k-2)\gamma-(k-4)G+2H>0$ and Eq. (2.1) has a nonnegative integer solution (α,β,γ), then it has infinitely many positive integer solutions of the form $(at+\alpha, bt+\beta, z)$, where t, z are given by the recurrence relation (3.7) and $p,q,k\geq 3$, $p,q,k,a,b,A,B,G\in\mathbb{Z}^+$, $C,D,E,F,H\in\mathbb{Z}$.*

3 Proof of the Theorem

Proof of Theorem 2.1. Suppose (α,β,γ) is a nonnegative integer solution of Eq. (2.1), i.e.,

$$AP_p(\alpha)+BP_q(\beta)+C\alpha\beta+D\alpha+E\beta+F=GP_k(\gamma)+H\gamma.$$

Let $x=at+\alpha,\ y=bt+\beta$, then Eq. (2.1) is equivalent to

$$\varphi(a,b)t^2+\phi(a,b)t+\psi(a,b)=Gz((z-1)(k-2)+2)+2Hz,$$

where

$$\begin{aligned}\varphi(a,b)&=A(p-2)a^2+2Cab+B(q-2)b^2,\\ \phi(a,b)&=(2A(p-2)\alpha-(p-4)A+2C\beta+2D)a\\ &\quad+(2B(q-2)\beta-(q-4)B+2C\alpha+2E)b,\\ \psi(a,b)&=\alpha((p-2)\alpha-(p-4))A+\beta((q-2)\beta-(q-4))B\\ &\quad+2(C\alpha\beta+D\alpha+E\beta+F).\end{aligned}$$

Solving it for z, we have

$$z=\frac{G(k-4)-2H+\sqrt{\Delta}}{2G(k-2)},\tag{3.1}$$

where

$$\Delta=4G(k-2)(\varphi(a,b)t^2+\phi(a,b)t+\psi(a,b))+((k-4)G-2H)^2.$$

It is necessary to take $\Delta=w^2$. Let

$$X=G(k-2)(2\varphi(a,b)t+\phi(a,b)),\quad Z=w,\tag{3.2}$$

we obtain the Pell equation

$$X^2-G(k-2)\varphi(a,b)Z^2=(G(k-2)\phi(a,b))^2-G(k-2)\varphi(a,b)L^2,\tag{3.3}$$

where

$$L=2G(k-2)\gamma-(k-4)G+2H.$$

If $\phi(a,b)>0$ and $L>0$, Eq. (3.3) has a positive integer solution

$$(X_0,Z_0)=(G(k-2)\phi(a,b),L).$$

By the theory of Pell equation, if $G(k-2)\varphi(a,b)$ is a positive integer but not a perfect square, the Pell equation

$$X^2-G(k-2)\varphi(a,b)Z^2=1\tag{3.4}$$

has infinitely many positive integer solutions. Let (u,v) be the least positive integer solution of Eq. (3.4). It is easy to provide infinitely many positive integer solutions of Eq. (3.3) by the formula

$$
\begin{aligned}
&X_s + Z_s\sqrt{G(k-2)\varphi(a,b)} \\
&= \left(X_0 + Z_0\sqrt{G(k-2)\varphi(a,b)}\right)\left(u + v\sqrt{G(k-2)\varphi(a,b)}\right)^s, s \geq 0.
\end{aligned}
$$

Thus,

$$
\begin{cases}
X_{s+1} = 2uX_s - X_{s-1}, \\
Z_{s+1} = 2uZ_s - Z_{s-1},
\end{cases}
$$

where

$$
\begin{aligned}
&X_0 = G(k-2)\phi(a,b), \quad && X_1 = G(k-2)(Lv\varphi(a,b) + \phi(a,b)u), \\
&Z_0 = L, && Z_1 = Lu + Gv(k-2)\phi(a,b).
\end{aligned}
$$

Using the recurrence relations of X_s and Z_s twice, we get

$$
\begin{cases}
X_{2s+2} = 2(2u^2-1)X_{2s} - X_{2s-2}, \\
Z_{2s+2} = 2(2u^2-1)Z_{2s} - Z_{2s-2},
\end{cases} \tag{3.5}
$$

where

$$
\begin{aligned}
&X_0 = G(k-2)\phi(a,b), \\
&X_2 = G(k-2)(\phi(a,b)u^2 + 2L\varphi(a,b)uv + G(k-2)\varphi(a,b)\phi(a,b)v^2), \\
&Z_0 = L, \\
&Z_2 = Lu^2 + 2G(k-2)\phi(a,b)uv + GL(k-2)\varphi(a,b)v^2.
\end{aligned}
$$

It is easy to prove that

$$
\begin{aligned}
X_{2s} &\equiv G(k-2)\phi(a,b) \pmod{2G(k-2)\varphi(a,b)}, \\
Z_{2s} &\equiv 2H - G(k-4) \pmod{2G(k-2)},
\end{aligned}
$$

where $s \geq 0$.

By (3.1) and (3.2), we have

$$
t = \frac{X - G(k-2)\phi(a,b)}{2G(k-2)\varphi(a,b)}, \quad z = \frac{G(k-4) - 2H + Z}{2G(k-2)}. \tag{3.6}
$$

Substituting (3.6) into (3.5), we obtain

$$
\begin{cases}
t_{2s+2} = 2(2u^2-1)t_{2s} - t_{2s-2} + 2G(k-2)\phi(a,b)v^2, \\
z_{2s+2} = 2(2u^2-1)z_{2s} - z_{2s-2} + 2((k-2)G - 2H)\varphi(a,b)v^2,
\end{cases} \tag{3.7}
$$

where

$$
\begin{aligned}
&t_0 = 0, \quad t_2 = (Lu + Gv(k-2)\phi(a,b))v, \\
&z_0 = \gamma, \quad z_2 = L\varphi(a,b)v^2 + \phi(a,b)uv + \gamma.
\end{aligned}
$$

It follows that $t_{2s}, z_{2s} \in \mathbb{Z}^+$ for all $s > 0$.

Therefore, if $X_0 > 0,\ Z_0 > 0$, $G(k-2)\varphi(a,b)$ is a positive integer but not a perfect square and Eq. (2.1) has a nonnegative integer solution (α, β, γ), then we get infinitely many positive integer solutions of the form $(at + \alpha, bt + \beta, z)$, where t, z given by the recurrence relation (3.7). □

Example 3.1. *When $A = B = C = D = E = F = G = H = 1,\ p = 3,\ q = 4,\ k = 5$, then Eq. (2.1) becomes*

$$P_3(x) + P_4(y) + xy + x + y + 1 = P_5(z) + z,$$

it has a positive integer solution $(6, 7, 9)$. Let $a = b = 1$, then $u = 4, v = 1$. Hence, Eq. (2.1) has infinitely many positive integer solutions $(t_{2s} + 7, t_{2s} + 9, z_{2s})$, where

$$\begin{cases} t_{2s+2} = 62_{2s} - t_{2s-2} + 426, & t_0 = 0,\ \ t_2 = 433, \\ z_{2s+2} = 62_{2s} - z_{2s-2} + 10, & z_0 = 9,\ \ t_2 = 568, \end{cases}$$

where s is a nonnegative integer.

Remark 3.2. *For $D = E = F = H = 0$, Eq. (2.1) becomes*

$$AP_p(x) + BP_q(y) + Cxy = GP_k(z).$$

When $p = q = k = 4$, this was investigated by J. Pla [13].

Further, if $C = 0$, we have

$$AP_p(x) + BP_q(y) = GP_k(z). \tag{3.8}$$

<u>Case 1:</u> When $p = q = k = 4$, Eq. (3.8) becomes

$$Ax^2 + By^2 = Gz^2.$$

This is the case studied by H. Cohen [2, Corollary 6.3.6.].

<u>Case 2:</u> When $A = B = G = 1,\ p = q = k$, we have

$$P_k(x) + P_k(y) = P_k(z),$$

this case was investigated by K. R. S. Sastry [15] and E. Scheffold [16]. In particular, when $p = q = k = 3$, we get

$$P_3(x) + P_3(y) = P_3(z),$$

this case was studied by K. R. S. Sastry [14] and A. Hamtat and D. Behloul [7].

<u>Case 3:</u> When $G = 1,\ k = 4$, the conclusion becomes a linear combination of two polygonal numbers is a perfect square (see [1, 5, 8–12]).

Remark 3.3. *Using the undetermined coefficient method, we obtain some parametric solutions of Eq. (3.8).*

<u>Case 1:</u> $(p, q, k, A, B, G, x, y, z) = (3, 3, 3, 1, 1, 1, (2r + 1)tus^2, 2r(r + 1)tus^2 + r, (2r^2 + 2r + 1)tus^2 + r).$

<u>Case 2:</u> $(p, q, k, A, B, G, x, y, z) = \big(3, 3, 3, \frac{(2b+1)^2-(2c+1)^2}{4}, \frac{(2a+1)^2-(2b+1)^2}{4}, \frac{(2a+1)^2-(2c+1)^2}{4}, (2a+1)t+a, (2c+1)t+c, (2b+1)t+b\big)$, *where $a > b > c$.*

<u>Case 3:</u> $(p, q, k, A, B, G, x, y, z) = (k, k, k, 4(k-2)^2t^2 - 1, 1, 1, as + (k-4)t,\ as, t(2(k-2)(k-4)t + 2a(k-2)s - (k-4))).$

<u>Case 4:</u> $(p, q, k, A, B, G, x, y, z) = (k - 1, k, k + 1, 1, 1, 1, 1, 2, 2),$

$(p, q, k, A, B, G, x, y, z) = (3, 4, 5, 1, 1, 1, t - 1, t, t).$

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