Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 3, 73–78
DOI: 10.7546/nntdm.2021.27.3.73-78

A note on ratios of Fibonacci hybrid and Lucas hybrid numbers

Emrah Polatlı

Department of Mathematics, University of Bulent Ecevit
Zonguldak, Turkey
e-mail: emrah.polatli@beun.edu.tr

Received: 1 February 2021 **Revised:** 13 August 2021 **Accepted:** 16 August 2021

Abstract: Irmak recently asked an open question related to divisibility properties of Fibonacci and Lucas quaternions [4, p. 374]. In this paper, we give an answer to Fibonacci and Lucas hybrid number version of this question.
Keywords: Fibonacci hybrid numbers, Lucas hybrid numbers, Divisibility.
2020 Mathematics Subject Classification: 11B39, 11K31.

1 Introduction

Recently, in [9], Özdemir introduced the set of hybrid numbers which contains complex, dual and hyberbolic numbers, as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\epsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \epsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \epsilon + \mathbf{i} \right\}.$$

This number system is a generalization of complex $(\mathbf{i}^2 = -1)$, hyperbolic $(\mathbf{h}^2 = 1)$ and dual number $(\epsilon^2 = 0)$ systems where $\mathbf{i}$ is a complex unit, ϵ is a dual unit and $\mathbf{h}$ is a hyperbolic unit. We call these hybrid units. From the definition of hybrid numbers, the multiplication of the hybrid units is given by the following table:

$\bullet$	1	$\mathbf{i}$	ϵ	$\mathbf{h}$
1	1	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\epsilon+\mathbf{i}$
ϵ	ϵ	$\mathbf{h}+\mathbf{1}$	0	$-\epsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\epsilon-\mathbf{i}$	ϵ	1

Table 1. The multiplication table for hybrid units

This table shows that the multiplication of hybrid numbers is not commutative. Based on this definition and table, Özdemir [9] examined many algebraic and geometric properties of hybrid numbers. For example, he defined a ring isomorphism between the hybrid number ring $\mathbb{K}$ and the ring of real 2×2 matrices $\mathbb{M}_{2\times 2}$. This map is $\varphi : \mathbb{K} \longrightarrow \mathbb{M}_{2\times 2}$ where

$$\varphi\left(a + b\mathbf{i} + c\epsilon + d\mathbf{h}\right) = \begin{bmatrix} a+c & b-c+d \\ c-b+d & a-c \end{bmatrix}. \tag{1}$$

For more details and properties related to hybrid numbers, we refer to [9].

The well-known Fibonacci and Lucas sequences are defined by the following recurrence relations:

$$F_{n+2} = F_{n+1} + F_n, \quad (n \geq 0)$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad (n \geq 0),$$

respectively, where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$. Note that $F_n + F_{n+2} = L_{n+1}$.

Recently, in [16], Szynal-Liana and Włoch studied the Fibonacci hybrid numbers and obtained some combinatorial properties of these numbers. For $n \geq 0$, they defined the n-th Fibonacci hybrid and n-th Lucas hybrid numbers as

$$FH_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\epsilon + F_{n+3}\mathbf{h}$$

and

$$LH_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\epsilon + L_{n+3}\mathbf{h}$$

where $FH_0 = \mathbf{i}+\epsilon+2\mathbf{h}$, $FH_1 = 1+\mathbf{i}+2\epsilon+3\mathbf{h}$, $LH_0 = 2+\mathbf{i}+3\epsilon+4\mathbf{h}$ and $LH_1 = 1+3\mathbf{i}+4\epsilon+7\mathbf{h}$.

The Fibonacci and Lucas hybrid numbers have been studied in various papers. For more details and properties related to the Fibonacci and Lucas hybrid numbers, see [1–3, 5–8, 10–22].

Recently, in [4], Irmak obtained various identities about Fibonacci and Lucas quaternions by matrix methods. At the end of his paper, he asked an open question about divisibility identities of Fibonacci and Lucas quaternions.

In this paper, we consider the Fibonacci and Lucas hybrid number version of this open question and then we give an answer to this question.

2 Main result

Theorem 2.1. *There are no triples (m, n, s) of integers satisfying the following equations:*

$$(i)\ \frac{FH_m}{FH_n} = FH_s, \tag{2}$$

$$(ii)\ \frac{LH_m}{LH_n} = LH_s, \tag{3}$$

$$(iii)\ \frac{FH_m}{FH_n} = LH_s, \tag{4}$$

$$(iv)\ \frac{LH_m}{LH_n} = FH_s, \tag{5}$$

$$(v)\ \frac{FH_m}{LH_n} = LH_s, \tag{6}$$

$$(vi)\ \frac{LH_m}{FH_n} = FH_s. \tag{7}$$

Proof. We only give a proof for (2) and (3), respectively. The rest of our assertions can be established in similar manner.

(i) Now, we know that from (1) that

$$\begin{aligned}
\varphi(FH_n) &= \varphi(F_n + F_{n+1}\mathbf{i} + F_{n+2}\epsilon + F_{n+3}\mathbf{h}) \\
&= \begin{bmatrix} F_n + F_{n+2} & F_{n+1} - F_{n+2} + F_{n+3} \\ F_{n+2} - F_{n+1} + F_{n+3} & F_n - F_{n+2} \end{bmatrix} \\
&= \begin{bmatrix} L_{n+1} & 2F_{n+1} \\ 2F_{n+2} & -F_{n+1} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(LH_n) &= \varphi(L_n + L_{n+1}\mathbf{i} + L_{n+2}\epsilon + L_{n+3}\mathbf{h}) \\
&= \begin{bmatrix} L_n + L_{n+2} & L_{n+1} - L_{n+2} + L_{n+3} \\ L_{n+2} - L_{n+1} + L_{n+3} & L_n - L_{n+2} \end{bmatrix} \\
&= \begin{bmatrix} 5F_{n+1} & 2L_{n+1} \\ 2L_{n+2} & -L_{n+1} \end{bmatrix}.
\end{aligned}$$

Assume that there exists at least one solution satisfying Eq. (2). By the hybrid isomorphism (1) and $\varphi(FH_m) = \varphi(FH_s)\,\varphi(FH_n)$, we get the following matrix equation:

$$\begin{aligned}
\begin{bmatrix} L_{m+1} & 2F_{m+1} \\ 2F_{m+2} & -F_{m+1} \end{bmatrix} &= \begin{bmatrix} L_{s+1} & 2F_{s+1} \\ 2F_{s+2} & -F_{s+1} \end{bmatrix} \begin{bmatrix} L_{n+1} & 2F_{n+1} \\ 2F_{n+2} & -F_{n+1} \end{bmatrix} \\
&= \begin{bmatrix} L_{s+1}L_{n+1} + 4F_{s+1}F_{n+2} & 2L_{s+1}F_{n+1} - 2F_{s+1}F_{n+1} \\ 2F_{s+2}L_{n+1} - 2F_{s+1}F_{n+2} & 4F_{s+2}F_{n+1} + F_{s+1}F_{n+1} \end{bmatrix}.
\end{aligned}$$

Then we have the following system of equations:

$$\begin{aligned} L_{s+1}L_{n+1}+4F_{s+1}F_{n+2} &= L_{m+1} &(8)\\ L_{s+1}F_{n+1}-F_{s+1}F_{n+1} &= F_{m+1} &(9)\\ F_{s+2}L_{n+1}-F_{s+1}F_{n+2} &= F_{m+2} &(10)\\ 4F_{s+2}F_{n+1}+F_{s+1}F_{n+1} &= -F_{m+1} &(11)\end{aligned}$$

If we sum the left-hand sides and the right-hand sides of Eq. (9) and Eq. (11), then we obtain

$$F_{n+1}\left(L_{s+1}+4F_{s+2}\right)=0.$$

Since $L_{s+1}+4F_{s+2}=F_s+5F_{s+2}\neq 0$ for $s\in\mathbb{Z}$, F_{n+1} must be equal to 0. So $n=-1$. If we take $n=-1$ in Eq. (9) and Eq. (11), we get $m=-1$. If we take $n=-1$ and $m=-1$ in Eq. (8) and Eq. (10), we get the following equations:

$$\begin{aligned} 1 &= L_{s+1}+2F_{s+1}=F_{s+4} &(12)\\ 1 &= 2F_{s+2}-F_{s+1}=L_{s+1} &(13)\end{aligned}$$

From the Eq. (13), s must be equal to 0. But $s=0$ is not a solution of Eq. (12). Thus

$$\varphi\left(FH_m\right)\neq\varphi\left(FH_s\right)\varphi\left(FH_n\right).$$

So there are no triples (m,n,s) of integers satisfying Eq. (2).

(ii) Assume that there exists at least one solution satisfying Eq. (3). By the hybrid isomorphism (1) and $\varphi\left(LH_m\right)=\varphi\left(LH_s\right)\varphi\left(LH_n\right)$, we get the following matrix equation:

$$\begin{aligned}\begin{bmatrix}5F_{m+1} & 2L_{m+1}\\ 2L_{m+2} & -L_{m+1}\end{bmatrix} &= \begin{bmatrix}5F_{s+1} & 2L_{s+1}\\ 2L_{s+2} & -L_{s+1}\end{bmatrix}\begin{bmatrix}5F_{n+1} & 2L_{n+1}\\ 2L_{n+2} & -L_{n+1}\end{bmatrix}\\ &= \begin{bmatrix}25F_{s+1}F_{n+1}+4L_{s+1}L_{n+2} & 10F_{s+1}L_{n+1}-2L_{s+1}L_{n+1}\\ 10L_{s+2}F_{n+1}-2L_{s+1}L_{n+2} & 4L_{s+2}L_{n+1}+L_{s+1}L_{n+1}\end{bmatrix}.\end{aligned}$$

Thus we get the following system of equations:

$$\begin{aligned} 5F_{m+1} &= 25F_{s+1}F_{n+1}+4L_{s+1}L_{n+2} &(14)\\ L_{m+1} &= 5F_{s+1}L_{n+1}-L_{s+1}L_{n+1} &(15)\\ L_{m+2} &= 5L_{s+2}F_{n+1}-L_{s+1}L_{n+2} &(16)\\ -L_{m+1} &= 4L_{s+2}L_{n+1}+L_{s+1}L_{n+1} &(17)\end{aligned}$$

If we sum the left-hand sides and the right-hand sides of Eq. (15) and Eq. (17), then we get

$$L_{n+1}\left(5F_{s+1}+4L_{s+2}\right)=L_{n+1}\left(13F_{s+1}+4F_{s+2}\right)=0.$$

Since $L_{n+1}\neq 0$ and $13F_{s+1}+4F_{s+2}\neq 0$ for $n,s\in\mathbb{Z}$ then

$$\varphi\left(LH_m\right)\neq\varphi\left(LH_s\right)\varphi\left(LH_n\right).$$

So there are no triples (m,n,s) of integers satisfying the Eq. (3). □

3 Conclusion

In this paper, we show that there are no triples (m, n, s) of integers satisfying Eqs. (2)–(7). In fact, since the multiplication of hybrid numbers is not commutative, we can consider the Eqs. (2)–(7) with two different versions. For example, for Eq. (2), we can investigate the isomorphisms $\varphi(FH_m) = \varphi(FH_s)\,\varphi(FH_n)$ and $\varphi(FH_m) = \varphi(FH_n)\,\varphi(FH_s)$ separately. Since calculations are similar, we omit the second versions of these equations in this paper.

References

[1] Catarino, P. (2019). On k-Pell hybrid numbers. *Journal of Discrete Mathematical Sciences and Cryptography*, 22(1), 83–89.

[2] Catarino, P., & Bilgici, G. (2020). A Note on Modified k-Pell Hybrid Numbers. *Konuralp Journal of Mathematics*, 8(2), 229–233.

[3] Cerda-Morales, G. (2018). *Investigation of Generalized Hybrid Fibonacci Numbers and Their Properties*. Preprint. arXiv:1806.02231v1 [math.RA].

[4] Irmak, N. (2020). More identities for Fibonacci and Lucas quaternions. *Communications Faculty of Sciences University of Ankara Series A1-Mathematics and Statistics*, 69(1), 369–375.

[5] Kızılateş, C. (2020). A new generalization of Fibonacci hybrid and Lucas hybrid numbers, *Chaos, Solitons & Fractals*, 130, 1–5. DOI: 10.1016/j.chaos.2019.109449.

[6] Kızılateş, C. (2020). A Note on Horadam Hybrinomials. *Preprints*, 2020010116, DOI: 10.20944/preprints202001.0116.v1.

[7] Liana, M., Szynal-Liana, A., & Włoch, I. (2019). On Pell Hybrinomials. *Miskolc Mathematical Notes*, 20(2), 1051–1062.

[8] Mangueira, M. C. S., Vieira, R. P. M., Alves, F. R. V., & Catarino, P. M. M. C. (2020). The Hybrid Numbers of Padovan and Some Identities. *Annales Mathematicae Silesianae*, 34(2), 256–267.

[9] Özdemir, M. (2018). Introduction to Hybrid Numbers, *Advances in Applied Clifford Algebras*, 28(11), DOI: 10.1007/s00006-018-0833-3.

[10] Polatlı, E. (2020). Hybrid Numbers with Fibonacci and Lucas Hybrid Number Coefficients, *Preprints*, 2020120349.

[11] Şentürk, T., Bilgici, G., Daşdemir, A., & Ünal, Z. (2020). A study on Horadam hybrid numbers. *Turkish Journal of Mathematics*, 44, 1212–1221.

[12] Soykan, Y., & Tasdemir, E. (2021). Generalized Tetranacci Hybrid Numbers. *Annales Mathematicae Silesianae*, 35(1), 113–130, DOI: 10.2478/amsil-2020-0021.

[13] Szynal-Liana, A. (2018). The Horadam Hybrid Numbers. *Discussiones Mathematicae General Algebra and Applications*, 38(1), 91–98.

[14] Szynal-Liana, A., & Włoch, I. (2018). On Pell and Pell–Lucas Hybrid Number. *Commentationes Mathematicae*, 58, 11–17.

[15] Szynal-Liana, A., & Włoch, I. (2019). On Jacobsthal and Jacobsthal–Lucas Hybrid Numbers. *Annales Mathematicae Silesianae*, 33, 276–283.

[16] Szynal-Liana, A., & Włoch, I. (2019). The Fibonacci hybrid numbers. *Utilitas Mathematica*, 110, 3–10.

[17] Szynal-Liana, A., & Włoch, I. (2020). Generalized Fibonacci-Pell hybrinomials. *Online Journal of Analytic Combinatorics*, 15, 1–12.

[18] Szynal-Liana, A., & Włoch, I. (2020). Introduction to Fibonacci and Lucas hybrinomials. *Complex Variables and Elliptic Equations*, 65, 1736–1747.

[19] Szynal-Liana, A. & Włoch, I. (2020). On Special Spacelike Hybrid Numbers, *Mathematics*, 8(10), Article No. 1671. DOI: 10.3390/math8101671.

[20] Tan, E., & Ait-Amrane, N. R. (2020). On a new generalization of Fibonacci hybrid numbers. Preprint. arXiv:2006.09727v1 [math.NT].

[21] Taşyurdu, Y. (2019). Tribonacci and Tribonacci–Lucas Hybrid Numbers. *International Journal of Contemporary Mathematical Sciences*, 14, 245–254.

[22] Yağmur, T. (2020). A Note on Generalized Hybrid Tribonacci Numbers. *Discussiones Mathematicae General Algebra and Applications*, 40, 187–199.