

Fibonacci series from power series

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Abstract: We show how every power series gives rise to a Fibonacci series and a companion series involving Lucas numbers. For illustrative purposes, Fibonacci series arising from trigonometric functions, the gamma function and the digamma function are derived. Infinite series involving Fibonacci and Bernoulli numbers and Fibonacci and Euler numbers are also obtained.

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1 Introduction

The Fibonacci numbers, F_n , and the Lucas numbers, L_n , are defined, for all integers n by the Binet formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad (1)$$

where α and β are the zeros of the characteristic polynomial, $x^2 - x - 1$, of the Fibonacci sequence. Thus $\alpha + \beta = 1$ and $\alpha\beta = -1$; so that $\alpha = (1 + \sqrt{5})/2$ (the golden ratio) and $\beta = -1/\alpha = (1 - \sqrt{5})/2$. Koshy [10] and Vajda [13] have written excellent books dealing with Fibonacci and Lucas numbers.

Our task in this paper is to show how every power series gives rise to a Fibonacci series and a companion series involving Lucas numbers. For example, the power series

$$\sum_{j=2}^{\infty} \zeta(j) \frac{z^j}{j} \equiv \log \Gamma(1 - z) - \gamma z, \quad |z| < 1,$$

where γ is the Euler–Mascheroni constant, $\zeta(n)$ is the Riemann zeta function and $\Gamma(x)$ is the gamma function, leads to the following series involving Lucas numbers and the zeta function (Theorem 6.8, section 6):

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{j} L_{rj} z^j = \log \Gamma(1 - \alpha^r z) \Gamma(1 - \beta^r z) - \gamma z L_r ;$$

particular instances of which, for non-zero even integers r , are (Example 8)

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{L_r^j j} L_{rj} = \log (\pi \csc(\pi \alpha^r / L_r)) - \gamma ,$$

and

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{L_r^j j} L_{rj} = -2 \log |L_r| + \log (\pi \csc(\pi \alpha^r / L_r)) + \gamma .$$

In Section 2, we prove the theorem regarding how to obtain Fibonacci series from power series. Illustrative examples are then presented in Sections 3–6.

2 Fibonacci series from power series

Theorem. For real or complex z , let a given well-behaved function $h(z)$ have, in its domain, the representation $h(z) = \sum_{j=c_1}^{c_2} g(j) z^{f(j)}$ where $f(j)$ and $g(j)$ are given real sequences and $c_1, c_2 \in [-\infty, \infty]$. Let r and s be integers. Then,

$$\sum_{j=c_1}^{c_2} g(j) z^{f(j)} F_{r f(j)+s} = \frac{F_s}{2} (h(\alpha^r z) + h(\beta^r z)) + \frac{L_s}{2\sqrt{5}} (h(\alpha^r z) - h(\beta^r z)) , \quad (\text{F})$$

$$\sum_{j=c_1}^{c_2} g(j) z^{f(j)} L_{r f(j)+s} = \frac{L_s}{2} (h(\alpha^r z) + h(\beta^r z)) + \frac{F_s \sqrt{5}}{2} (h(\alpha^r z) - h(\beta^r z)) , \quad (\text{L})$$

whenever the series on the left-hand side of each of (F) and (L) converges.

Proof. We have

$$\sum_{j=c_1}^{c_2} g(j) z^{f(j)} = h(z) . \quad (2)$$

Writing $\alpha^r z$ for z in (2) and multiplying both sides by α^s , we obtain

$$\sum_{j=c_1}^{c_2} g(j) \alpha^{r f(j)+s} z^{f(j)} = \alpha^s h(\alpha^r z) . \quad (3)$$

Similarly, writing $\beta^r z$ for z in (2) and multiplying both sides by β^s , we obtain

$$\sum_{j=c_1}^{c_2} g(j) \beta^{r f(j)+s} z^{f(j)} = \beta^s h(\beta^r z) . \quad (4)$$

From (3) and (4), we have

$$\frac{1}{2} \sum_{j=c_1}^{c_2} g(j) L_{rf(j)+s} z^{f(j)} + \frac{\sqrt{5}}{2} \sum_{j=c_1}^{c_2} g(j) F_{rf(j)+s} z^{f(j)} = \alpha^s h(\alpha^r z), \quad (5)$$

and

$$\frac{1}{2} \sum_{j=c_1}^{c_2} g(j) L_{rf(j)+s} z^{f(j)} - \frac{\sqrt{5}}{2} \sum_{j=c_1}^{c_2} g(j) F_{rf(j)+s} z^{f(j)} = \beta^s h(\beta^r z), \quad (6)$$

where we have used the fact that, for any integer m ,

$$\alpha^m = \frac{L_m + F_m \sqrt{5}}{2} \text{ and } \beta^m = \frac{L_m - F_m \sqrt{5}}{2}. \quad (7)$$

Subtraction of (6) from (5) while making use of (7) again to resolve α^s and β^s produces identity (F). Addition of (5) and (6) gives identity (L). \square

Setting $s = 0$ in (F) and (L), we have the particular cases,

$$\sum_{j=c_1}^{c_2} g(j) z^{f(j)} F_{rf(j)} = \frac{1}{\sqrt{5}} (h(\alpha^r z) - h(\beta^r z)), \quad (F1)$$

$$\sum_{j=c_1}^{c_2} g(j) z^{f(j)} L_{rf(j)} = h(\alpha^r z) + h(\beta^r z). \quad (L1)$$

In Sections 3–6 we will apply identities (F) and (L) to derive Fibonacci series from certain power series.

3 Fibonacci series from trigonometric functions

Theorem 3.1. *If r and s are integers and z is a real or complex variable, then,*

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} F_{2rj+s} z^{2j+1} &= F_{s-r} \sin\left(\frac{zL_r}{2}\right) \cos\left(\frac{F_r z \sqrt{5}}{2}\right) \\ &+ \frac{L_{s-r}}{\sqrt{5}} \sin\left(\frac{F_r z \sqrt{5}}{2}\right) \cos\left(\frac{zL_r}{2}\right), \end{aligned} \quad (8)$$

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} L_{2rj+s} z^{2j+1} &= L_{s-r} \sin\left(\frac{zL_r}{2}\right) \cos\left(\frac{F_r z \sqrt{5}}{2}\right) \\ &+ F_{s-r} \sqrt{5} \sin\left(\frac{F_r z \sqrt{5}}{2}\right) \cos\left(\frac{zL_r}{2}\right). \end{aligned} \quad (9)$$

Proof. Consider the Maclaurin series expansion of $\sin z$:

$$\sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!} = \sin z.$$

Use $g(j) = (-1)^j / ((2j+1)!)$, $f(j) = 2j+1$, $c_1 = 0$, $c_2 = \infty$ and $h(z) = \sin z$ in identities (F) and (L); noting the identities

$$\sin x \pm \sin y = 2 \sin\left(\frac{x \pm y}{2}\right) \cos\left(\frac{x \mp y}{2}\right). \quad \square$$

In particular,

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} F_{r(2j+1)} z^{2j+1} = \frac{2}{\sqrt{5}} \sin\left(\frac{F_r z \sqrt{5}}{2}\right) \cos\left(\frac{z L_r}{2}\right), \quad (10)$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} L_{r(2j+1)} z^{2j+1} = 2 \sin\left(\frac{z L_r}{2}\right) \cos\left(\frac{F_r z \sqrt{5}}{2}\right). \quad (11)$$

Example 1. If r is an integer, then,

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{(2j+1)B_{2j}} \frac{F_{r(2j+1)}}{F_r^{2j+1} 5^j} = \frac{1}{2}, \quad (12)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{2^{2j+1}(2j+1)B_{2j}} \frac{F_{r(2j+1)}}{L_r^{2j+1}} = \frac{F_r}{4L_r}, \quad (13)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{(2j+1)B_{2j}} \frac{L_{r(2j+1)}}{L_r^{2j+1}} = \frac{1}{2}, \quad (14)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{2^{2j+1}(2j+1)B_{2j}} \frac{L_{r(2j+1)}}{F_r^{2j+1} 5^j} = \frac{L_r}{4F_r}. \quad (15)$$

Proof. To prove (12) and (13), set $z = 2\pi/(F_r \sqrt{5})$, $z = \pi/L_r$, in (10), in turn. To prove (14) and (15), set $z = 2\pi/L_r$, $z = \pi/(F_r \sqrt{5})$, in (11), in turn. Note the use of (38). \square

Theorem 3.2. If r and s are integers and z is a real or complex variable, then,

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} F_{2rj+s} z^{2j} &= F_s \cos\left(\frac{z L_r}{2}\right) \cos\left(\frac{F_r z \sqrt{5}}{2}\right) \\ &\quad - \frac{L_s}{\sqrt{5}} \sin\left(\frac{z L_r}{2}\right) \sin\left(\frac{F_r z \sqrt{5}}{2}\right), \end{aligned} \quad (16)$$

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} L_{2rj+s} z^{2j} &= L_s \cos\left(\frac{z L_r}{2}\right) \cos\left(\frac{F_r z \sqrt{5}}{2}\right) \\ &\quad - F_s \sqrt{5} \sin\left(\frac{z L_r}{2}\right) \sin\left(\frac{F_r z \sqrt{5}}{2}\right). \end{aligned} \quad (17)$$

Proof. Consider the Maclaurin series for $\cos z$:

$$\sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{2j!} = \cos z.$$

Use $f(j) = 2j$, $g(j) = (-1)^j/(2j!)$, $c_1 = 0$, $c_2 = \infty$ and $h(z) = \cos z$ in (F) and (L), noting the identities

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right),$$

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right). \quad \square$$

In particular,

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j)!} F_{2rj} z^{2j} = \frac{2}{\sqrt{5}} \sin\left(\frac{zL_r}{2}\right) \sin\left(\frac{F_r z \sqrt{5}}{2}\right), \quad (18)$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} L_{2rj} z^{2j} = 2 \cos\left(\frac{zL_r}{2}\right) \cos\left(\frac{F_r z \sqrt{5}}{2}\right). \quad (19)$$

Example 2. If r is an integer, then,

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{B_{2j}} \frac{F_{2rj}}{L_r^{2j}} = 0, \quad (20)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{B_{2j}} \frac{F_{2rj}}{F_r^{2j} 5^j} = 0, \quad (21)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{2^{2j} B_{2j}} \frac{L_{2rj}}{L_r^{2j}} = 1, \quad (22)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{2^{2j} B_{2j}} \frac{L_{2rj}}{F_r^{2j} 5^j} = 1. \quad (23)$$

Proof. Set $z = 2\pi/L_r$, $z = 2\pi/(F_r \sqrt{5})$ in (18) to prove (20) and (21). Set $z = \pi/L_r$, $z = \pi/(F_r \sqrt{5})$ in (19) to prove (22) and (23). Note the use of (38) \square

4 Infinite series involving Fibonacci numbers and Bernoulli numbers

The Bernoulli numbers, B_j , are defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!}, \quad z < 2\pi. \quad (24)$$

The first few Bernoulli numbers are

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, \dots \quad (25)$$

Basic properties of the Bernoulli polynomials are highlighted in recent articles by Frontczak [3] and by Frontczak and Goy [8] where new identities involving Fibonacci and Bernoulli numbers, and Lucas and Euler numbers are presented. Additional information on Bernoulli polynomials can be found in Erdélyi et al [2, §1.13].

Theorem 4.1. Let r and s be integers and z be any real or complex variable such that $z < 2\pi\alpha^{-r}$. Then,

$$\begin{aligned} \sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} F_{2rj+s} z^{2j} &= \frac{z F_{r+s}}{4} \left(\cot\left(\frac{\alpha^r z}{2}\right) + \cot\left(\frac{\beta^r z}{2}\right) \right) - F_s \\ &+ \frac{z L_{r+s}}{4\sqrt{5}} \left(\cot\left(\frac{\alpha^r z}{2}\right) - \cot\left(\frac{\beta^r z}{2}\right) \right), \end{aligned} \quad (26)$$

$$\begin{aligned} \sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} L_{2rj+s} z^{2j} &= \frac{z L_{r+s}}{4} \left(\cot \left(\frac{\alpha^r z}{2} \right) + \cot \left(\frac{\beta^r z}{2} \right) \right) - L_s \\ &+ \frac{z F_{r+s} \sqrt{5}}{4} \left(\cot \left(\frac{\alpha^r z}{2} \right) - \cot \left(\frac{\beta^r z}{2} \right) \right). \end{aligned} \quad (27)$$

Proof. Setting $x = iz$, z real, in the following identity [9, Formula 1.213]:

$$\sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} x^{2j} = \frac{x}{e^x - 1} + \frac{x}{2} - 1, \quad |x| < 2\pi,$$

and taking the real part, we have

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} z^{2j} = \frac{z}{2} \cot \left(\frac{z}{2} \right) - 1.$$

Now use $f(j) = 2j$, $g(j) = B_{2j}/(2j)!$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = z/2 \cot(z/2) - 1$ in (F) and (L). The identities of Theorem 4.1 follow after some algebra, including also the use of identities (7). Note that in the final simplification, we used

$$F_r L_s + F_s L_r = 2F_{r+s}, \quad \text{Vajda [13, (16a)]},$$

and

$$L_r L_s + 5F_r F_s = 2L_{r+s}, \quad \text{Vajda [13, (17a) and (17b)]}. \quad \square$$

In particular, for integer r and $z < 2\pi\alpha^{-r}$, we have

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} F_{r(2j-1)} z^{2j} = \frac{z}{2\sqrt{5}} \left(\cot \left(\frac{\alpha^r z}{2} \right) - \cot \left(\frac{\beta^r z}{2} \right) \right) + (-1)^r F_r, \quad (28)$$

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} L_{r(2j-1)} z^{2j} = \frac{z}{2} \left(\cot \left(\frac{\alpha^r z}{2} \right) + \cot \left(\frac{\beta^r z}{2} \right) \right) - (-1)^r L_r. \quad (29)$$

Example 3. Let r be an integer. Then,

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} \frac{F_{r(2j-1)}}{F_r^{2j} 5^j} (2\pi)^{2j} = (-1)^r F_r, \quad r \neq 0, \quad (30)$$

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} \frac{L_{r(2j-1)}}{L_r^{2j}} (2\pi)^{2j} = (-1)^{r-1} L_r. \quad (31)$$

Proof. Set $z = 2\pi/(F_r \sqrt{5})$ in (28) and $z = 2\pi/L_r$ in (29). □

Note that, in view of identity (38), identities (30) and (31) can also be written as

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{F_r^{2j-1} 5^j} F_{r(2j-1)} = \frac{(-1)^{r-1}}{2} F_r^2, \quad (32)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{L_r^{2j-1}} L_{r(2j-1)} = \frac{(-1)^r}{2} L_r^2, \quad (33)$$

which are the same identities (96) and (99) of Example 5.

5 Infinite series involving Fibonacci numbers and Euler numbers

The Euler numbers E_j are defined by the exponential generating function:

$$\frac{2}{e^z + e^{-z}} = \sum_{j=0}^{\infty} \frac{E_j}{j!} z^j.$$

The first few Euler numbers are

$$E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, \dots, \text{ with } E_{2j+1} = 0 \text{ for } j \geq 0.$$

Theorem 5.1. *If r and s are integers and z is a real or complex variable, then,*

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} F_{2rj+s} z^{2j} &= \frac{F_s}{2} (\sec(\alpha^r z) + \sec(\beta^r z)) \\ &+ \frac{L_s}{2\sqrt{5}} (\sec(\alpha^r z) - \sec(\beta^r z)), \end{aligned} \quad (34)$$

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} L_{2rj+s} z^{2j} &= \frac{L_s}{2} (\sec(\alpha^r z) + \sec(\beta^r z)) \\ &+ \frac{F_s \sqrt{5}}{2} (\sec(\alpha^r z) - \sec(\beta^r z)). \end{aligned} \quad (35)$$

Proof. Consider the identity [9, Formula 1.411 9.]

$$\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} z^{2j} = \sec z, \quad z^2 < \pi^2/4,$$

Use $f(j) = 2j$, $g(j) = |E_{2j}|/(2j)!$, $c_1 = 0$, $c_2 = \infty$ and $h(z) = \sec z$ in (F) and (L). □

In particular,

$$\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} F_{2rj} z^{2j} = \frac{1}{\sqrt{5}} (\sec(\alpha^r z) - \sec(\beta^r z)), \quad (36)$$

$$\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} L_{2rj} z^{2j} = \sec(\alpha^r z) + \sec(\beta^r z). \quad (37)$$

6 Infinite series involving Fibonacci numbers and the Riemann zeta function

As noted by Frontczak and Goy [7], studies in infinite series involving Fibonacci numbers and Riemann zeta numbers have not been previously documented. The narrative has changed, however, following research results by the aforementioned authors, as contained in their recent papers: Frontczak [4–6] and Frontczak and Goy [7]. In this section, we explore more infinite series involving the Fibonacci numbers and the Riemann zeta numbers.

The Riemann zeta function, $\zeta(n)$, $n \in \mathbb{C}$, defined by

$$\zeta(n) = \sum_{j=1}^{\infty} \frac{1}{j^n}, \quad \Re(n) > 1,$$

is analytically continued to all $n \in \mathbb{C}$ with $\Re(n) > 0$, $n \neq 1$ through

$$\zeta(n) = \frac{1}{1 - 2^{1-n}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^n}.$$

For positive even arguments, the numbers $\zeta(2n)$ are directly related to the Bernoulli numbers, B_{2n} :

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}. \quad (38)$$

No such simple formula is known for the zeta function at odd integer arguments.

More information on the Riemann zeta function can be found in the books by Edwards [1] and Srivastava and Choi [12].

The zeta number generating functions, found in Srivastava and Choi [12, p. 270–271, p. 280–281], also Erdélyi et al [2, §1.7.1], which we require to establish the infinite series here, are expressed in terms of the Gamma function and the digamma function.

The Gamma function is defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt = \int_0^{\infty} (\log(1/t))^{z-1} dt,$$

and is extended to the rest of the complex plane, excluding the non-positive integers, by analytic continuation. The Gamma function has a simple pole at each of the points $z = \dots, -3, -2, -1, 0$. The Gamma function extends the classical factorial function to the complex plane: $\Gamma(z) = (z-1)!$.

The digamma function, $\psi(z)$, is the logarithmic derivative of the Gamma function:

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

6.1 Functional equations for the gamma and the digamma function

Here is a list of basic functional equations for the gamma function (see Erdélyi et al [2, §1.2]):

$$\Gamma(z+1) = z\Gamma(z), \quad (39)$$

$$\Gamma(z)\Gamma(-z) = -\pi \csc(\pi z)/z, \quad (40)$$

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z), \quad (41)$$

$$\Gamma(1/2+z)\Gamma(1/2-z) = \pi \sec(\pi z), \quad (42)$$

$$\Gamma(1+z)\Gamma(1-z) = \pi z \csc(\pi z). \quad (43)$$

Writing $-x$ for z and $-y$ for z , in turn, in (39), we find

$$\Gamma(1-x)\Gamma(1-y) = xy\Gamma(-x)\Gamma(-y). \quad (44)$$

More functional equations that are required for our discussion will now be derived.

A consequence of (40) is

$$\Gamma(-x)\Gamma(-y) = \frac{2\pi^2}{xy} \frac{1}{\Gamma(x)\Gamma(y)} \frac{1}{\cos(x-y) - \cos(x+y)}, \quad (45)$$

so that

$$\begin{aligned} \log(\Gamma(-x)\Gamma(-y)) &= \log(2\pi^2/(xy)) - \log(\Gamma(x)\Gamma(y)) \\ &\quad + \log(\cos(x+y) - \cos(x-y)). \end{aligned} \quad (46)$$

From (43), it follows that if $x+y=1$, then,

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+y\right) = \left(\frac{1}{2}-x\right)\pi \sec(\pi x). \quad (47)$$

We require the following basic properties of the digamma function:

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \quad (48)$$

$$\psi(z) - \psi(-z) = -\pi \cot \pi z - \frac{1}{z}, \quad (49)$$

$$\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot \pi z, \quad (50)$$

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z, \quad (51)$$

$$\psi\left(\frac{1}{2}+z\right) - \psi\left(\frac{1}{2}-z\right) = \pi \tan \pi z, \quad (52)$$

$$\psi(z+n) = \psi(z) + \sum_{j=1}^n \frac{1}{z+j-1}, \quad (53)$$

$$\psi(mz) = \log m + \frac{1}{m} \sum_{j=0}^{m-1} \psi\left(z + \frac{j}{m}\right). \quad (54)$$

As a consequence of identity (49) we have the following useful identity:

$$\psi(-x) - \psi(-y) = \psi(x) - \psi(y) + \pi(\cot(\pi x) - \cot(\pi y)) - \frac{x-y}{xy}. \quad (55)$$

We observe from relation (48) that if $x-y=1$, then,

$$\psi(x) - \psi(y) = \frac{1}{y}, \quad y \neq 0, \quad (56)$$

while relation (51) is equivalent to saying that if $x+y=1$, then,

$$\psi(x) - \psi(y) = -\pi \cot(\pi x) \quad (57)$$

From relation (50), we have

$$\begin{aligned}\psi(1+x) - \psi(1+y) &= \psi(1-x) - \psi(1-y) \\ &+ \frac{1}{x} - \frac{1}{y} - \pi(\cot(\pi x) - \cot(\pi y)).\end{aligned}\quad (58)$$

Thus, if $x + y = 1$, $x \notin \mathbb{Z}$, then,

$$\psi(1+x) - \psi(1+y) = -\pi \cot(\pi x) + \frac{1}{x} - \frac{1}{1-x}, \quad x \notin \{0, 1\}, \quad (59)$$

while if $x + y = 2$, $x \notin \mathbb{Z}$, we have

$$\psi(1+x) - \psi(1+y) = -\frac{1}{1-x} + \frac{1}{x} - \frac{1}{2-x} - \pi \cot(\pi x), \quad x \notin \{0, 1, 2\}. \quad (60)$$

If $x + y = 1$, identity (52) also implies

$$\psi(1/2+x) - \psi(1/2+y) = -\pi \tan(\pi x) + \frac{2}{2x-1}, \quad x \neq 1/2, \quad (61)$$

while if $x - y = 1$, we have

$$\psi(1/2+x) - \psi(1/2+y) = \frac{2}{2x-1}, \quad x \neq 1/2. \quad (62)$$

Writing $z - 1$ for z in (50) gives

$$\psi(2-z) = \psi(z) + \pi \cot(\pi(z-1)) - \frac{1}{z-1}, \quad (63)$$

and hence, also

$$\psi(2+z) = \psi(-z) - \pi \cot \pi(z+1) + \frac{1}{z+1}. \quad (64)$$

From (63) we get

$$\begin{aligned}\psi(2-x) - \psi(2-y) &= \psi(x) - \psi(y) - \frac{2\pi \sin(\pi(x-y))}{\cos(\pi(x-y)) - \cos(\pi(x+y))} \\ &+ \frac{x-y}{1-x-y+xy},\end{aligned}\quad (65)$$

which, writing x for $-x$ and y for $-y$ and making use of (55), also gives

$$\psi(2+x) - \psi(2+y) = \psi(x) - \psi(y) - \frac{x-y}{xy} - \frac{x-y}{1+x+y+xy}. \quad (66)$$

Using $m = 2$ in (54) gives

$$\psi(2z) = \log 2 + \frac{1}{2}\psi(z) + \frac{1}{2}\psi\left(z + \frac{1}{2}\right), \quad (67)$$

which also means that,

$$\psi(2x) - \psi(2y) = \frac{1}{2}(\psi(x) - \psi(y)) + \frac{1}{2}\left(\psi\left(x + \frac{1}{2}\right) - \psi\left(y + \frac{1}{2}\right)\right). \quad (68)$$

If $x + y = 1$, then relation (68), in view of (51) and (52), gives

$$\psi(2x) - \psi(2y) = -\pi \cot(2\pi x) + \frac{1}{2x-1}; \quad (69)$$

while if $x - y = 1$, it produces

$$\psi(2x) - \psi(2y) = \frac{1}{2(x-1)} - \frac{1}{2x-1}. \quad (70)$$

6.2 Evaluations at the relevant arguments

Lemma 6.1. *We have*

$$\Gamma(\alpha)\Gamma(\beta) = \pi \csc(\pi\alpha) = \pi \csc(\pi\beta), \quad (71)$$

$$\Gamma(\alpha^2)\Gamma(\beta^2) = -\pi \csc(\pi\beta), \quad (72)$$

$$\Gamma(-\alpha)\Gamma(-\beta) = -\pi \csc(\pi\beta), \quad (73)$$

$$\Gamma(-\alpha^2)\Gamma(-\beta^2) = -\pi \csc(\pi\beta), \quad (74)$$

$$\Gamma(\alpha^3/2)\Gamma(\beta^3/2) = \frac{\pi\sqrt{5}}{2} \sec(\pi\beta), \quad (75)$$

$$\Gamma(\alpha^r/L_r)\Gamma(\beta^r/L_r) = \pi \csc(\pi\alpha^r/L_r), \quad (76)$$

$$\Gamma(-\alpha^r/L_r)\Gamma(-\beta^r/L_r) = (-1)^r \pi L_r^2 \csc(\pi\alpha^r/L_r). \quad (77)$$

Proof. Setting $z = \alpha$ in (41) gives (71). Use of $x = -\alpha$, $y = -\beta$ in (44) gives (72). In view of (44), we have (73) and (74). Identity (75) is proved by setting $x = \beta$ in identity (47). To prove (76), set $z = \alpha^r/L_r$ in (41). \square

Lemma 6.2. *We have*

$$\psi(\alpha) - \psi(\beta) = -\pi \cot \pi\alpha, \quad (78)$$

$$\psi(\alpha^2) - \psi(\beta^2) = -\pi \cot \pi\alpha + \sqrt{5}, \quad (79)$$

$$\psi(2\alpha) - \psi(2\beta) = \frac{1}{\sqrt{5}} - \pi \cot \pi\sqrt{5}, \quad (80)$$

$$\psi(\alpha^3) - \psi(\beta^3) = \frac{7\sqrt{5}}{10} - \pi \cot \pi\sqrt{5}, \quad (81)$$

$$\psi\left(\frac{\alpha^3}{2}\right) - \psi\left(\frac{\beta^3}{2}\right) = \frac{2}{\sqrt{5}} - \pi \cot \frac{\pi\sqrt{5}}{2}, \quad (82)$$

$$\psi(\alpha\sqrt{5}) - \psi(\beta\sqrt{5}) = -\frac{\alpha}{\sqrt{5}}, \quad (83)$$

$$\psi\left(\frac{\alpha^r}{L_r}\right) - \psi\left(\frac{\beta^r}{L_r}\right) = -\pi \cot\left(\frac{\pi\alpha^r}{L_r}\right) = \tan\left(\frac{\pi}{2} \frac{F_r\sqrt{5}}{L_r}\right), \quad (84)$$

$$\psi\left(\frac{\alpha^r}{F_r\sqrt{5}}\right) - \psi\left(\frac{\beta^r}{F_r\sqrt{5}}\right) = \frac{F_r\sqrt{5}}{\beta^r}, \quad (85)$$

$$\psi\left(\frac{-\alpha^r}{L_r}\right) - \psi\left(\frac{-\beta^r}{L_r}\right) = -\tan\left(\frac{\pi}{2} \frac{F_r\sqrt{5}}{L_r}\right) - (-1)^r F_{2r}\sqrt{5}, \quad (86)$$

$$\psi\left(-\frac{\alpha^r}{F_r\sqrt{5}}\right) - \psi\left(-\frac{\beta^r}{F_r\sqrt{5}}\right) = \frac{F_r\sqrt{5}}{\beta^r} - (-1)^r 5F_r^2. \quad (87)$$

Proof. Setting $z = \alpha$ in (51) proves (78). Identity (80) comes from choosing $x = \alpha$, $y = \beta$ in (69). Setting $x = \alpha$, $y = \beta$ in (59) proves (79). Use of $x = 2\alpha$, $y = 2\beta$ in (60) produces (81). Setting $(x, y) = (\alpha, \beta)$ in (61) gives (82). Identities (84) and (85) are proved by setting $(x, y) = (\alpha^r/L_r, \beta^r/L_r)$ in (56) and $(x, y) = (\alpha^r/(F_r\sqrt{5}), \beta^r/(F_r\sqrt{5}))$ in (57). Identities (86) and (87) are obtained from (84) and (85) with the aid of (55). \square

6.3 Fibonacci–Zeta infinite series

Theorem 6.1. Let r be an integer. Let z be a real or complex variable such that $|z| < \alpha^{-r}$. Then,

$$\sum_{j=1}^{\infty} \zeta(j+1) F_{rj} z^j = -\frac{1}{\sqrt{5}} (\psi(\alpha^r z) - \psi(\beta^r z)) - \frac{\pi}{\sqrt{5}} (\cot(\pi z \alpha^r) - \cot(\pi z \beta^r)).$$

Proof. Consider [12, p. 270, identity (13)]:

$$\sum_{j=1}^{\infty} \zeta(j+1) z^j = -\psi(1-z) - \gamma, \quad |z| < 1.$$

Use $f(j) = j$, $g(j) = \zeta(j+1)$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = -\psi(1-z) - \gamma$ in identity (F1). \square

Example 4. If r is an integer with $|r| > 1$, then

$$\sum_{j=1}^{\infty} \frac{\zeta(j+1)}{L_r^j} F_{rj} = \frac{\pi}{\sqrt{5}} \tan\left(\frac{\pi F_r \sqrt{5}}{2 L_r}\right), \quad (88)$$

$$\sum_{j=1}^{\infty} (-1)^{j-1} \frac{\zeta(j+1)}{L_r^j} F_{rj} = \frac{\pi}{\sqrt{5}} \tan\left(\frac{\pi F_r \sqrt{5}}{2 L_r}\right) - (-1)^r F_{2r}. \quad (89)$$

Proof. Set $z = \pm 1/L_r$ in the identity of Theorem 6.1 and use (84) and (86). \square

Theorem 6.2. Let r and s be integers. Let z be a real or complex variable such that $|z| < \alpha^{-r}$. Then,

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\zeta(2j)}{2j} F_{2rj+s} z^{2j} &= \frac{F_s}{4} \log\left(\frac{(-1)^r 2\pi^2 z^2}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)}\right) \\ &+ \frac{L_s}{4\sqrt{5}} \log\left((-1)^r \alpha^{2r} \frac{\sin \pi \beta^r z}{\sin \pi \alpha^r z}\right), \end{aligned} \quad (90)$$

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\zeta(2j)}{2j} L_{2rj+s} z^{2j} &= \frac{L_s}{4} \log\left(\frac{(-1)^r 2\pi^2 z^2}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)}\right) \\ &+ \frac{F_s \sqrt{5}}{4} \log\left((-1)^r \alpha^{2r} \frac{\sin \pi \beta^r z}{\sin \pi \alpha^r z}\right). \end{aligned} \quad (91)$$

Proof. Consider [12, p. 271, identity (17)]:

$$\sum_{j=1}^{\infty} \zeta(2j) \frac{z^{2j}}{j} = \log\left(\frac{\pi z}{\sin \pi z}\right), \quad |z| < 1.$$

Use $f(j) = 2j$, $g(j) = \zeta(2j)/j$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = \log(\pi z / \sin(\pi z))$ in identities (F) and (L). \square

Theorem 6.3. Let r and s be integers. Let z be a real or complex variable such that $|z| < \alpha^{-r}$. Then,

$$\begin{aligned} \sum_{j=1}^{\infty} \zeta(2j) F_{2rj+s} z^{2j-1} &= \frac{F_{s+r}}{2} \left(-\frac{\pi \sin(\pi z L_r)}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)} + \frac{(-1)^r L_r}{2z} \right) \\ &+ \frac{L_{s+r}}{2\sqrt{5}} \left(\frac{\pi \sin(\pi z F_r \sqrt{5})}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)} - \frac{(-1)^r F_r \sqrt{5}}{2z} \right), \end{aligned} \quad (92)$$

$$\begin{aligned} \sum_{j=1}^{\infty} \zeta(2j) L_{2rj+s} z^{2j-1} &= \frac{L_{s+r}}{2} \left(-\frac{\pi \sin(\pi z L_r)}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)} + \frac{(-1)^r L_r}{2z} \right) \\ &+ \frac{F_{s+r} \sqrt{5}}{2} \left(\frac{\pi \sin(\pi z F_r \sqrt{5})}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)} - \frac{(-1)^r F_r \sqrt{5}}{2z} \right). \end{aligned} \quad (93)$$

Proof. Consider [12, p. 271, identity (18)]:

$$\sum_{j=1}^{\infty} \zeta(2j) z^{2j-1} = -\frac{\pi}{2} \cot \pi z + \frac{1}{2z}, \quad |z| < 1.$$

Use $f(j) = 2j - 1$, $g(j) = \zeta(2j)$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = -(\pi/2) \cot(\pi z) + 1/(2z)$ in identities (F) and (L). \square

In particular, setting $s = -r$ in the identities of Theorem 6.3 yields

$$\sum_{j=1}^{\infty} \zeta(2j) F_{r(2j-1)} z^{2j-1} = \frac{1}{\sqrt{5}} \left(\frac{\pi \sin(\pi z F_r \sqrt{5})}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)} - \frac{(-1)^r F_r \sqrt{5}}{2z} \right), \quad (94)$$

$$\sum_{j=1}^{\infty} \zeta(2j) L_{r(2j-1)} z^{2j-1} = -\frac{\pi \sin(\pi z L_r)}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)} + \frac{(-1)^r L_r}{2z}. \quad (95)$$

Example 5. If r is an integer, then,

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{F_r^{2j-1} 5^j} F_{r(2j-1)} = \frac{(-1)^{r-1}}{2} F_r^2, \quad (96)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{F_r^{2j-1} 5^j} L_{r(2j-1)} = \frac{\pi}{\sqrt{5}} \tan \left(\frac{\pi}{2} \frac{L_r}{F_r \sqrt{5}} \right) + \frac{(-1)^r}{2} F_{2r}, \quad (97)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{L_r^{2j-1}} F_{r(2j-1)} = \frac{\pi}{\sqrt{5}} \tan \left(\frac{\pi}{2} \frac{F_r \sqrt{5}}{L_r} \right) - \frac{(-1)^r}{2} F_{2r}, \quad |r| > 1, \quad (98)$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{L_r^{2j-1}} L_{r(2j-1)} = \frac{(-1)^r}{2} L_r^2. \quad (99)$$

Proof. Set $z = 1/(F_r \sqrt{5})$ in (94) and in (95) to obtain (96) and (97). Set $z = 1/L_r$ in (94) and in (95) to obtain (98) and (99). \square

Frontczak [4, Theorem 2.1] also obtained the special case $r = 1$ of identities (96) and (97).

Theorem 6.4. Let r be an integer and z be any real or complex variable such that $|z| < 2\alpha^{-r}$. Then,

$$\sum_{j=1}^{\infty} (\zeta(j+1) - 1) F_{rj} z^j = -\frac{1}{\sqrt{5}} (\psi(\alpha^r z) - \psi(\beta^r z)) - \frac{\pi}{\sqrt{5}} (\cot(\pi z \alpha^r) - \cot(\pi z \beta^r)) - \frac{F_r z}{(-1)^r z^2 - L_r z + 1}.$$

Proof. Consider [12, p. 280, identity (146)]:

$$\sum_{j=1}^{\infty} (\zeta(j+1) - 1) z^j = -\psi(2-z) + 1 - \gamma, \quad |z| < 2.$$

Use $f(j) = j$, $g(j) = \zeta(j+1) - 1$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = -\psi(2-z) + 1 - \gamma$ in identity (F1). \square

Example 6. If r is an integer, then

$$\sum_{j=1}^{\infty} \frac{(\zeta(j+1) - 1)}{L_r^j} F_{rj} = \frac{\pi}{\sqrt{5}} \tan\left(\frac{\pi F_r \sqrt{5}}{2 L_r}\right) - (-1)^r F_{2r}.$$

Proof. Set $z = 1/L_r$ in the identity of Theorem 6.4 and use (84). \square

Frontczak [4, Theorem 2.2, identity (2.3)] obtained the $r = 1$ case of Example 6.

Theorem 6.5. Let r be an integer and z be any real or complex variable such that $|z| < 2\alpha^{-r}$. Then,

$$\sum_{j=1}^{\infty} (\zeta(2j+1) - 1) F_{2rj} z^{2j} = -\frac{1}{\sqrt{5}} (\psi(\alpha^r z) - \psi(\beta^r z)) - \frac{\pi}{2\sqrt{5}} (\cot(\pi z \alpha^r) - \cot(\pi z \beta^r)) - \frac{F_{2r} z^2}{(1 + (-1)^r z^2)^2 - L_r^2 z^2} + \frac{1}{2z} (-1)^r F_r.$$

Proof. Consider [12, p. 280, identity (149)]:

$$\sum_{j=1}^{\infty} (\zeta(2j+1) - 1) z^{2j} = -\frac{1}{2} (\psi(2+z) + \psi(2-z)) + 1 - \gamma, \quad |z| < 2.$$

Use $f(j) = 2j$, $g(j) = \zeta(2j+1) - 1$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = -\psi(2+z)/2 - \psi(2-z)/2 + 1 - \gamma$ in identity (F1).

Note that we used

$$\begin{aligned} & \psi(2-x) - \psi(2-y) + \psi(2+x) - \psi(2+y) \\ &= 2\psi(x) - 2\psi(y) - \frac{2\pi \sin(\pi(x-y))}{\cos(\pi(x-y)) - \cos(\pi(x+y))} \\ & \quad + \frac{2(x^2 - y^2)}{(1+xy)^2 - (x+y)^2} - \frac{x-y}{xy}, \end{aligned} \tag{100}$$

which follows from identities (65) and (66). \square

Example 7. If r is an integer, then,

$$\sum_{j=1}^{\infty} \frac{(\zeta(2j+1) - 1)}{L_r^{2j}} F_{2rj} = -\frac{L_r^2 F_{2r}}{2(-1)^r L_r^2 + 1} + \frac{(-1)^r}{2} F_{2r}, \quad (101)$$

$$\sum_{j=1}^{\infty} \frac{(\zeta(2j+1) - 1)}{5^j F_r^{2j}} F_{2rj} = -\frac{5F_r^2 F_{2r}}{10(-1)^{r-1} F_r^2 + 1} + \frac{(-1)^{r-1}}{2} F_{2r}. \quad (102)$$

Proof. Set $z = 1/L_r$, $z = 1/(F_r\sqrt{5})$ in the identity of Theorem 6.5 and use identities (84) and (85). We used

$$\cot\left(\frac{\pi\alpha^r}{L_r}\right) + \cot\left(\frac{\pi\beta^r}{L_r}\right) = 0$$

and

$$\cot\left(\frac{\pi\alpha^r}{F_r\sqrt{5}}\right) - \cot\left(\frac{\pi\beta^r}{F_r\sqrt{5}}\right) = 0. \quad \square$$

Theorem 6.6. Let r be an integer and z be any real or complex variable such that $|z| < 2\alpha^{-r}$. Then,

$$\begin{aligned} \sum_{j=1}^{\infty} (\zeta(2j) - 1) \frac{F_{2rj+s}}{j} z^{2j} &= \frac{F_s}{2} \log\left(\frac{(-1)^r 2\pi^2 z^2 (1 - L_{2r} z^2 + z^4)}{\cos(\pi F_r z \sqrt{5}) - \cos(\pi L_r z)}\right) \\ &+ \frac{L_s}{2\sqrt{5}} \log\left(\frac{\alpha^r - \alpha^{3r} z^2}{\beta^r - \beta^{3r} z^2} \frac{\sin(\pi\beta^r z)}{\sin(\pi\alpha^r z)}\right), \end{aligned} \quad (103)$$

$$\begin{aligned} \sum_{j=1}^{\infty} (\zeta(2j) - 1) \frac{L_{2rj+s}}{j} z^{2j} &= \frac{L_s}{2} \log\left(\frac{(-1)^r 2\pi^2 z^2 (1 - L_{2r} z^2 + z^4)}{\cos(\pi F_r z \sqrt{5}) - \cos(\pi L_r z)}\right) \\ &+ \frac{F_s \sqrt{5}}{2} \log\left(\frac{\alpha^r - \alpha^{3r} z^2}{\beta^r - \beta^{3r} z^2} \frac{\sin(\pi\beta^r z)}{\sin(\pi\alpha^r z)}\right). \end{aligned} \quad (104)$$

Proof. Consider [12, p. 281, identity (150)]:

$$\sum_{j=1}^{\infty} (\zeta(2j) - 1) \frac{z^{2j}}{j} = \log\left(\frac{\pi z(1 - z^2)}{\sin \pi z}\right), \quad |z| < 2.$$

Use $f(j) = 2j$, $g(j) = \zeta(2j) - 1$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = \log(\pi z(1 - z^2)/\sin(\pi z))$ in identities (F) and (L). \square

Theorem 6.7. Let r be an integer and z be any real or complex variable such that $|z| < 2\alpha^{-r}$. Then,

$$\begin{aligned} &\sum_{j=1}^{\infty} (\zeta(2j) - 1) F_{2rj+s} z^{2j-1} \\ &= \frac{F_{s+r}}{2} \left(-\frac{\pi \sin(\pi z L_r)}{\cos(\pi F_r z \sqrt{5}) - \cos(\pi L_r z)} + \frac{(-1)^r L_r}{2z} \frac{3z^4 - L_r^2 z^2 + 1}{z^4 - L_{2r} z^2 + 1} \right) \\ &+ \frac{L_{s+r}}{2\sqrt{5}} \left(\frac{\pi \sin(\pi z F_r \sqrt{5})}{\cos(\pi F_r z \sqrt{5}) - \cos(\pi L_r z)} - \frac{(-1)^r F_r \sqrt{5}}{2z} \frac{3z^4 - 5F_r^2 z^2 + 1}{z^4 - L_{2r} z^2 + 1} \right), \end{aligned} \quad (105)$$

$$\begin{aligned}
& \sum_{j=1}^{\infty} (\zeta(2j) - 1) L_{2rj+s} z^{2j-1} \\
&= \frac{L_{s+r}}{2} \left(-\frac{\pi \sin(\pi z L_r)}{\cos(\pi F_r z \sqrt{5}) - \cos(\pi L_r z)} + \frac{(-1)^r L_r}{2z} \frac{3z^4 - L_r^2 z^2 + 1}{z^4 - L_{2r} z^2 + 1} \right) \\
&+ \frac{F_{s+r} \sqrt{5}}{2} \left(\frac{\pi \sin(\pi z F_r \sqrt{5})}{\cos(\pi F_r z \sqrt{5}) - \cos(\pi L_r z)} - \frac{(-1)^r F_r \sqrt{5}}{2z} \frac{3z^4 - 5F_r^2 z^2 + 1}{z^4 - L_{2r} z^2 + 1} \right). \tag{106}
\end{aligned}$$

Proof. Consider [12, p. 281, identity (151)]:

$$\sum_{j=1}^{\infty} (\zeta(2j) - 1) z^{2j-1} = -\frac{\pi}{2} \cot \pi z + \frac{3z^2 - 1}{2z(z^2 - 1)}, \quad |z| < 2.$$

Use $f(j) = 2j - 1$, $g(j) = \zeta(2j) - 1$, $c_1 = 1$, $c_2 = \infty$ and

$$h(z) = -\pi \cot(\pi z)/2 + (3z^2 - 1)/(2z(z^2 - 1))$$

in identities (F) and (L).

Note that we used (Vajda [13, Formula (17c)])

$$L_{2r} + (-1)^r 2 = L_r^2 \tag{107}$$

and (Vajda [13, Formula (23)])

$$L_{2r} - (-1)^r 2 = 5F_r^2. \tag{108}$$

This completes the proof. \square

Theorem 6.8. *Let r be an integer and z be any real or complex variable such that $|z| < \alpha^{-r}$. Then,*

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{j} L_{rj} z^j = \log(\Gamma(1 - \alpha^r z) \Gamma(1 - \beta^r z)) - \gamma z L_r.$$

Proof. Consider [12, p. 270, identity (9)]:

$$\sum_{j=2}^{\infty} \zeta(j) \frac{z^j}{j} = \log \Gamma(1 - z) - \gamma z, \quad |z| < 1.$$

Use $f(j) = j$, $g(j) = \zeta(j)$, $c_1 = 2$, $c_2 = \infty$ and $h(z) = \log \Gamma(1 - z) - \gamma z$ in identity (L1). \square

Corollary 6.1. *If r is an even integer and z is a real variable such that $|z| < \alpha^{-r}$, then,*

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{j} L_{rj} z^j = 2 \log |z| + \log(\Gamma(-\alpha^r z) \Gamma(-\beta^r z)) - \gamma z L_r.$$

Example 8. *If r is an even integer, then*

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{L_r^j j} L_{rj} = \log(\pi \csc(\pi \alpha^r / L_r)) - \gamma, \tag{109}$$

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{L_r^j j} L_{rj} = -2 \log |L_r| + \log(\pi \csc(\pi \alpha^r / L_r)) + \gamma. \tag{110}$$

Proof. Set $z = \pm 1/L_r$ in the identity of Corollary 6.1 and use identities (76) and (77). \square

Theorem 6.9. Let r be an integer and z be any real or complex variable such that $|z| < \alpha^{-r}$. Then,

$$\sum_{j=1}^{\infty} \frac{\zeta(2j+1)}{2j+1} L_{r(2j+1)} z^{2j+1} = \log(\Gamma(1-\alpha^r z)\Gamma(1-\beta^r z)) - \gamma z L_r - \frac{1}{2} \log\left(\frac{(-1)^r 2\pi^2 z^2}{\cos(\pi z F_r \sqrt{5}) - \cos(\pi z L_r)}\right).$$

Proof. Let

$$t(j) = \frac{\zeta(j+1)}{j+1} L_{r(j+1)} z^{j+1}.$$

Then,

$$t(2j) = \frac{\zeta(2j+1)}{2j+1} L_{r(2j+1)} z^{2j+1}, \quad t(2j-1) = \frac{\zeta(2j)}{2j} L_{r(2j)} z^{2j}.$$

Use these in the summation formula

$$\sum_{j=1}^{\infty} t(2j) = \sum_{j=1}^{\infty} t(j) - \sum_{j=1}^{\infty} t(2j-1),$$

while taking note of Theorem 6.2 (identity (91)) and Theorem 6.8. □

7 Concluding comments

We have shown, through identities (F) and (L), how every power series, $h(z)$, gives rise to a Fibonacci–Lucas series. Examples were drawn from infinite series involving Bernoulli numbers, Euler numbers, the Zeta function as well as from infinite series representations of trigonometric functions. The identities (F) and (L) are quite general and apply to both finite and infinite series. To drive home this point we now derive linear binomial Fibonacci–Lucas identities associated with a variation on the Waring formula (in which case $h(z)$ is not an infinite series):

$$h(z) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} 2^{n-2j-1} z^j = \frac{(1 + \sqrt{1-z})^{n+1} - (1 - \sqrt{1-z})^{n+1}}{\sqrt{1-z}}.$$

Note that $h(z)$ is dual to the identity found in Riordan [11, p.57]. Proceeding as in the previous examples, we find

$$\begin{aligned} & \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} 2^{n-2j-1} z^j F_{rj+s} \\ &= \frac{F_s}{2} \left(\frac{(1 + \sqrt{1-\alpha^r z})^{n+1} - (1 - \sqrt{1-\alpha^r z})^{n+1}}{\sqrt{1-\alpha^r z}} \right. \\ & \quad \left. + \frac{(1 + \sqrt{1-\beta^r z})^{n+1} - (1 - \sqrt{1-\beta^r z})^{n+1}}{\sqrt{1-\beta^r z}} \right) \\ & \quad + \frac{L_s}{2\sqrt{5}} \left(\frac{(1 + \sqrt{1-\alpha^r z})^{n+1} - (1 - \sqrt{1-\alpha^r z})^{n+1}}{\sqrt{1-\alpha^r z}} \right. \\ & \quad \left. - \frac{(1 + \sqrt{1-\beta^r z})^{n+1} - (1 - \sqrt{1-\beta^r z})^{n+1}}{\sqrt{1-\beta^r z}} \right), \end{aligned}$$

$$\begin{aligned}
& \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} 2^{n-2j-1} z^j L_{rj+s} \\
&= \frac{L_s}{2} \left(\frac{(1 + \sqrt{1 - \alpha^r z})^{n+1} - (1 - \sqrt{1 - \alpha^r z})^{n+1}}{\sqrt{1 - \alpha^r z}} \right. \\
&\quad \left. + \frac{(1 + \sqrt{1 - \beta^r z})^{n+1} - (1 - \sqrt{1 - \beta^r z})^{n+1}}{\sqrt{1 - \beta^r z}} \right) \\
&+ \frac{F_s \sqrt{5}}{2} \left(\frac{(1 + \sqrt{1 - \alpha^r z})^{n+1} - (1 - \sqrt{1 - \alpha^r z})^{n+1}}{\sqrt{1 - \alpha^r z}} \right. \\
&\quad \left. - \frac{(1 + \sqrt{1 - \beta^r z})^{n+1} - (1 - \sqrt{1 - \beta^r z})^{n+1}}{\sqrt{1 - \beta^r z}} \right).
\end{aligned}$$

Choosing $r = 1$, $z = -1$ in the above identities and some algebra, we find the binomial Fibonacci–Lucas identities stated in the next theorem.

Theorem 7.1. *For a non-negative integer n and any integer s ,*

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} 2^{n-2j+1} F_{j+s} = F_{2n+s+1} - (-1)^s F_{n+2-s}, \quad (111)$$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} 2^{n-2j+1} L_{j+s} = L_{2n+s+1} + (-1)^s L_{n+2-s}. \quad (112)$$

Theorem 7.1 has several interesting consequences of which we can mention a couple.

Writing $2n - 1$ for n in (111) and (112) gives

Corollary 7.1. *For a non-negative integer n and any integer s ,*

$$\begin{aligned}
\sum_{j=0}^{n-1} \binom{2n-1-j}{j} 4^{n-j} F_{j+s} &= \begin{cases} L_{n+s-1} F_{3n}, & n \text{ even}; \\ F_{n+s-1} L_{3n}, & n \text{ odd}; \end{cases} \\
\sum_{j=0}^{n-1} \binom{2n-1-j}{j} 4^{n-j} L_{j+s} &= \begin{cases} 5F_{n+s-1} F_{3n}, & n \text{ even}; \\ L_{n+s-1} L_{3n}, & n \text{ odd}. \end{cases}
\end{aligned}$$

Setting $s = n + 2$ in (111) and (112), we obtain

Corollary 7.2. *For a non-negative integer n and any integer s ,*

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} 2^{n-2j+1} F_{j+n+2} = F_{3n+3}, \quad (113)$$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} 2^{n-2j+1} L_{j+n+2} = L_{3n+3} + (-1)^n 2. \quad (114)$$

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